Birla Central Library
PILANI (Jaipur State)

Class No :- 624
Book No :- J 63T V. 2
Accession No :- 30074

Acc. No . ...

ISSUE LABEL

Not later than the latest date stamped below.

~~27Mar73~~

The Theory and Practice of
Modern Framed Structures

WORKS OF
THE LATE DEAN J. B. JOHNSON

PUBLISHED BY

JOHN WILEY & SONS, Inc.

JOHNSON'S MATERIALS OF CONSTRUCTION

Rewritten and Revised by M. O. WITHEY, Professor of Mechanics in the University of Wisconsin; and JAMES ASTON, Consulting Metallurgist with the A. M. Byers Co. of Pittsburgh. Eighth Edition. xxii + 867 pages, 6×9. Illustrated. Cloth.

BY THE LATE J. B. JOHNSON, THE LATE C. W. BRYAN, AND F. E. TURNEAURE

THEORY AND PRACTICE OF MODERN FRAMED STRUCTURES

Designed for the Use of Schools and for Engineers in Professional Practice. By the late Dean J. B. JOHNSON; the late C. W. BRYAN; and F. E. TURNEAURE. In Three Parts.

Part I. Stresses in Simple Structures. Tenth Edition, Partly Rewritten. xiv + 356 pages, 6×9. Illustrated. Cloth.

Part II. Statically Indeterminate Structures and Secondary Stresses. Tenth Edition, Revised and Rewritten by F. E. TURNEAURE and W. S. KINNE, Professor of Structural Engineering, University of Wisconsin. xviii + 590 pages, 6×9, 343 figures. Cloth.

Part III. Design. Ninth Edition, Rewritten by F. E. TURNEAURE and W. S. KINNE. xii + 486 pages, 6×9. Many figures and plates. Cloth.

The Theory and Practice of
Modern Framed Structures

Designed for the Use of Schools and for Engineers
in Professional Practice

BY

The Late J. B. JOHNSON, C.E.
The Late C. W. BRYAN, C.E., M.A.

AND

F. E. TURNEAURE, Dr. Eng.
DEAN EMERITUS OF THE COLLEGE OF ENGINEERING, UNIVERSITY OF WISCONSIN

IN THREE PARTS

PART II.—Statically Indeterminate Structures and
Secondary Stresses

TENTH EDITION

REVISED AND REWRITTEN BY F. E. TURNEAURE
AND W. S. KINNE
PROFESSOR OF STRUCTURAL ENGINEERING
UNIVERSITY OF WISCONSIN

NEW YORK
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED

IN THE REPRINTING OF THIS BOOK, THE RECOMMENDATIONS OF THE WAR PRODUCTION BOARD HAVE BEEN OBSERVED FOR THE CONSERVATION OF PAPER AND OTHER IMPORTANT WAR MATERIALS. THE CONTENT REMAINS COMPLETE AND UNABRIDGED.

Copyright, 1893, 1904
by
Estate of J. B. Johnson
F. E. Turneaure
C. W. Bryan

Ninth Edition, Copyright, 1911, by
Mrs. J. B. Johnson and F. E. Turneaure
1911 Copyright renewed in the names of
F. E. Turneaure, Paul Johnson, Robert Johnson, Marjorie Johnson
Laura Johnson, H. F. Ringo
Tenth Edition, Copyright, 1929, by
Phoebe Johnson and Frederick E. Turneaure

All Rights Reserved

*This book or any part thereof must not
be reproduced in any form without
the written permission of the publisher.*

(II)

TENTH EDITION

Fifth Printing, October, 1944

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE TO TENTH EDITION

Compared to the preceding edition of this work the present edition includes the following changes:

1. Additional material in Chapter I on continuous girders, including further illustrations of the application of the moment-area method of beam analysis and additional material on the effect of shear on deflection.

2. The elimination of a considerable amount of material on suspension bridges. Due to the availability of other literature on this subject it was thought unnecessary to retain some of the details of the development given in the former edition.

3. Addition of a chapter on the application of the slope-deflection method to the analysis of unbraced frames such as used in buildings, concrete trestles, portals, etc., and to the calculation of secondary stresses. This method of analysis is undoubtedly the most convenient for many of the problems relating to unbraced frames and has advantages in the calculations of secondary stresses. In the treatment of building frames an effort has been made to demonstrate methods of approximate solution which will give results of sufficient accuracy for all practical purposes and where an exact solution is quite impracticable.

F. E. T.
W. S. K.

Madison, Wis., September, 1929.

PREFACE TO FIRST EDITION

In the arrangement of material in the present edition of this work, Part I has included, for the most part, the analysis of such structures as are statically determinate, or which can be closely analyzed by the application of the usual methods of statics. In the last chapter of Part I there was also given a brief treatment of the calculation of deflections of ordinary trusses, and a general explanation of the method of determining stresses in redundant members based upon deflection formulas.

The present volume, Part II, treats, in the main, of structures which are statically indeterminate, but it also includes the analysis of cantilever bridges, which is generally a statically determinate problem. However, on account of other elements of analysis, such as the use of influence lines, and the relation of the cantilever bridge to the continuous girder, it seemed more convenient to include this subject in Part II than to place it in Part I. Furthermore, the cantilever without suspended span becomes a statically indeterminate structure.

As compared with the corresponding chapters of the former edition, the present work has been greatly extended in scope and entirely rewritten. The analysis of the different types of structures considered has been made much more complete, and two chapters have been added covering various problems in statically indeterminate structures and the subject of secondary stresses. The general method of influence lines has been freely used in many of the problems, both as an aid to clear treatment and for purposes of convenient calculation. Other graphical methods have also been used wherever they seemed to be advantageous.

Chapter I gives a brief general treatment of the continuous girder for both constant and variable moment of inertia. This is followed by Chapter II on swing bridges, where both approximate and exact methods are fully explained. In Chapter III the cantilever bridge is briefly discussed, and in Chapter IV the various types of arches are analyzed by the usual methods. Chapter V contains a relatively extended treatment of the suspension bridge. The space given to this subject is, perhaps, hardly warranted, inasmuch as such structures are of infrequent occurrence, but the relative inaccessibility of material on this subject, that is adequate for the treatment of important structures, led the author to develop this chapter to a considerable extent. The so-called "exact theory" (Art. 222) is based upon the work of Melan, "Eiserne Bogenbrücken und Haengebrücken." This theory was fully developed by Mr. L. S. Moisseiff of the Department of Bridges of the City of New York, and used in the calculations of the Manhattan suspension bridge, and later by the author, with various modifications, in the recalculations of this structure, for Mr. Ralph Modjeska and given in his report to the City of New York, 1909. (See *Engineering News*, Vol. 62, page 401, *Engineering Record*, Vol. 60, page 372.) In this treatise this theory has been further extended and a study made of the errors involved. The detailed development of the methods employed and the preparation of the same for publication have been almost entirely the work of Professor W. S. Kinne, to whom the author is also indebted for assistance in proof-reading and in many other valuable details.

Chapter VI includes several general problems of importance pertaining to statically indeterminate structures. These are grouped under five divisions, namely, multiple intersection trusses, lateral truss systems, quadrangular or open frames of four sides, trussed beams, and beams on multiple elastic supports. Under quadrangular frames are discussed several problems pertaining to portal bracing, viaduct bents, and towers. The last-named division includes the common problem of determining the maximum load on a single transverse element of a railroad bridge floor. Some of the problems included in this chapter are of relatively small importance and are introduced more to illustrate methods of analysis than for their direct practical value.

In Chapter VII the author has endeavored to present the analysis of secondary stresses in such detail as to enable it to be readily followed and adapted to any ordinary problem. Especial care has been taken to systematize the calculations so as to reduce the labor involved to a minimum, and it is believed that a little familiarity with the method will enable any good computor to completely analyze an ordinary single intersection truss for secondary stresses in less than two working days. The fundamental theory of the subject was developed many years ago by German writers, and the present work contains nothing new in that direction. In the lengthy calculations involved, however, the scheme of arrangement is very important and, in this respect, it is thought that the treatment herein given presents features of value which will aid in bringing this very important subject to a better working basis than has hitherto been the case. In addition to a full explanation of the methods of analysis employed, several illustrative examples are given, and also general calculations, which indicate, to a certain extent, the relative secondary stresses in various types of trusses. In addition to the secondary stresses in main trusses due to rigid joints, several interesting problems of more or less practical importance are discussed, and at the end of the chapter the subject of impact stresses is briefly treated on the basis of the experimental work done by the Impact Committee of the American Railway Engineering and Maintenance of Way Association.

The thanks of the author are due to many engineering friends for suggestions received with reference to practical problems of analysis arising in their practice. It is hoped that the present work will be of assistance to engineers as well as to students in acquiring a working knowledge of the methods of analysis applicable to statically indeterminate structures.

F. E. TURNEAURE.

MADISON, Dec., 1910.

CONTENTS

CHAPTER I

CONTINUOUS GIRDERS

SECTION I.—DEFLECTION OF STRAIGHT BEAMS

THE AREA—MOMENT METHOD OF ANALYSIS

PAGE
THE ELASTIC CURVE, ... 1
ANGULAR CHANGE AT ANY POINT, ... 3
DEFLECTION AT ANY POINT, ... 4
METHODS OF APPLICATION, ... 5
TABLE OF MOMENTS, STRESSES, AND DEFLECTIONS, ... 19
DEFLECTION DUE TO SHEAR, ... 19

SECTION II.—CONTINUOUS GIRDERS

DEFINITION, ... 29
SHEAR AND BENDING MOMENT IN ANY SPAN, ... 29
THEOREM OF THREE MOMENTS, ... 30
EFFECT OF MOVEMENT OF SUPPORTS, ... 32
MOMENTS AT SUPPORTS, ... 33
GENERAL FORMULAS FOR MOMENTS, ... 34
GENERAL FORMULAS FOR REACTIONS, ... 37
CONTINUOUS GIRDER OF TWO EQUAL SPANS, ... 38
CONTINUOUS GIRDER OF TWO UNEQUAL SPANS, ... 44
CONTINUOUS GIRDER OF THREE SPANS. *Uniform load on all spans*, ... 44
MAXIMUM MOMENTS AND SHEARS IN A CONTINUOUS GIRDER OF THREE EQUAL SPANS, . 48
CONTINUOUS GIRDERS OF SEVERAL SPANS, ... 48
CONTINUOUS GIRDERS WITH VARIABLE MOMENT OF INERTIA, ... 51
THE ANALYSIS OF CONTINUOUS TRUSSES, ... 54
THE MOMENT OF INERTIA OF A TRUSS, ... 55
USE OF CONTINUOUS GIRDERS, ... 55

CHAPTER II

SWING BRIDGES

SECTION I.—GENERAL CONSIDERATIONS

PAGE
General arrangement, ... 57
Arrangements at center supports, ... 57
Arrangements of end supports, ... 59
Application of continuous girder formulas, ... 59
Loads, ... 59

SECTION II.—THE CENTRE-BEARING SWING BRIDGE

Formulas for reactions, ... 60
End conditions assumed, ... 62
Plate-girder bridge of two equal spans, ... 64
Centre-bearing truss bridge of two equal spans, ... 72
Use of influence lines, ... 78
Use of equivalent uniform loads, ... 79
True reactions calculated from deflections, ... 80
Amount of uplift required, ... 85
Counterbalanced swing bridges, ... 86

SECTION III.—THE RIM-BEARING SWING BRIDGE

Truss continuous over four supports, ... 87
Truss continuous over three supports, ... 93
Truss partially continuous over four supports, ... 94
Rim-bearing turn-table; four supports, ... 101
Lift swing bridges, ... 102
Double swing bridges, ... 103
Stresses in lateral trusses, ... 104

CHAPTER III

CANTILEVER BRIDGES

General arrangement of spans, ... 106
Advantages of the cantilever bridge, ... 106
Analysis, ... 110
Divided supports at the piers, ... 113
Effect of temperature variations, ... 86
Deflection of cantilever bridges, ... 113
Cantilever bridge without suspended span, ... 114
Wind stresses, ... 115

CHAPTER IV

ARCH BRIDGES

SECTION I.—GENERAL CONSIDERATIONS

PAGE
Definition 116
Kinds of arches, ... 116
Loads and reactions, ... 118
Internal stresses, ... 120
Advantages of the arch bridge, ... 124
Deflection of curved beams, ... 126
Application of deflection formulas to arches, ... 131

SECTION II.—ARCHES OF THREE HINGES

Reactions and stresses for dead load, ... 133
Reactions for a single load, ... 134
Use of reaction lines, ... 135
Influence lines, ... 136
Equivalent uniform loads, ... 140
Deflection, ... 140
Stresses in lateral systems, ... 145

SECTION III.—ARCHES OF TWO HINGES

General formulas for reactions for arch ribs, ... 148
General method of application, ... 153
Parabolic arch with variable moment of inertia, ... 153
Circular arch of constant section, ... 156
Calculation of *H* for arch ribs, ... 158
The braced arch, ... 160
Stress calculation, ... 164
Examples, ... 169
Deflection of two-hinged arches, ... 179
Wind stresses, ... 180

SECTION IV.—ARCHES WITHOUT HINGES

General formulas for arched ribs, ... 183
Temperature stresses, ... 186
Stresses due to shortening of arch, ... 187
Parabolic arch with variable moment of inertia, ... 187
Deflection of an arch rib, ... 193
The braced arch, ... 193
Temperature stresses, ... 198
Wind stresses, ... 198
Relative advantages of the arch with fixed ends, ... 199

CHAPTER V

SUSPENSION BRIDGES

PAGE
INTRODUCTION, ... 200

SECTION I.—THE CABLE

FORM OF CABLE, ... 200
LENGTH OF CABLE, ... 206
STRESSES IN THE CABLE FOR UNIFORM LOADS, ... 208
DEFORMATION OF CABLE, ... 211

SECTION II.—UNSTIFFENED SUSPENSION BRIDGES

INTRODUCTION, ... 213
STRESSES IN THE CABLE, ... 213
DEFORMATION OF CABLE UNDER PARTIAL LOADING, ... 214

SECTION III.—STIFFENED SUSPENSION BRIDGES

INTRODUCTION, ... 219
METHOD OF PROCEDURE, ... 221

A. APPROXIMATE METHODS OF CALCULATION

GENERAL EQUATION FOR H AND FOR MOMENTS AND SHEARS, ... 222

(*a*) *Structure over a single opening. Truss hinged at ends*

VALUE OF H FOR VARIOUS CASES, ... 223
EFFECT OF TEMPERATURE CHANGES ON H, ... 229
MOMENTS AND SHEARS, ... 230
INFLUENCE LINES FOR MOMENTS, ... 231
MOMENTS FROM INFLUENCE LINES, ... 231
EFFECT OF TEMPERATURE ON MOMENTS, ... 233
INFLUENCE LINES FOR SHEARS, ... 234
SHEARS FROM INFLUENCE LINES, ... 235
EFFECT OF TEMPERATURE ON SHEARS, ... 235
DEFLECTION OF THE STIFFENING TRUSS, ... 235

(*b*) *Structure over three openings. Side spans suspended. Trusses hinged*

VALUE OF H, ... 236
INFLUENCE LINES FOR MOMENTS AND SHEARS, ... 238
DEFLECTION OF STIFFENING TRUSS, ... 239
DEFLECTION OF TOWER OR MOVEMENT OF SADDLE, ... 239
EXAMPLE, ... 241

B. Exact Methods of Calculation

(*a*) *Structure with trusses hinged or continuous at towers*

PAGE
Method of procedure, ... 250
Derivation of formulas for deflection, moment, and shear, ... 252
Constants of integration, ... 255
Formula for H, ... 257
Conditions of loading for maximum moments and shears, ... 263
Example, ... 268

SECTION IV.—STRESSES IN LATERAL TRUSSES, TOWERS, AND FLOORBEAMS

Wind stresses in structures with horizontal stiffening trusses, ... 273
Floorbeam stresses, ... 274
Tower stresses, ... 276

CHAPTER VI

MISCELLANEOUS PROBLEMS IN STATICALLY INDETERMINATE STRUCTURES

SECTION I.—MULTIPLE INTERSECTION TRUSSES

General formulas, ... 280
Forms of trusses and methods of analysis, ... 284
The double triangular truss with vertical end posts, ... 284
The double triangular truss with inclined end posts, ... 290
The Whipple truss, ... 293
Other multiple systems, ... 294
General conclusions regarding multiple system, ... 296
The double triangular truss with verticals, ... 297
General case of double diagonals in a Pratt system, ... 308
The counters of a Baltimore or a Pettit truss, ... 309

SECTION II.—LATERAL TRUSS SYSTEMS

Form of lateral trusses and general methods of analysis, ... 312
Stresses due to distortions of main members, ... 313
Lateral stresses in curved-chord trusses, ... 317
Lateral bracing necessary for rigidity of trusses, ... 317
Redundant reactions, ... 319
Stresses due to unequal settlement of supports, ... 320
Stresses in lateral trusses due to vertical loads, ... 325
Tranverse bracing, ... 326
Proportions of lateral bracing for maximum rigidity, ... 330

SECTION III.—THE QUADRANGULAR OR PORTAL FRAME

PAGE
THE QUADRANGULAR FRAME, ... 331
GENERAL SOLUTION, ... 332
SYMMETRICAL FRAMES, ... 334
QUADRANGULAR FRAMES WITH BRACKETS, ... 336
FRAME WITH POSTS FIXED AT ONE END, ... 337
FRAME WITH POSTS HINGED AT ONE END, ... 338
TEMPERATURE STRESSES, ... 340
EFFECT OF LATERAL FORCES, ... 342
PARTIAL TRUSSED PORTAL FRAMES, ... 344
FRAMES WITH INCLINED END POSTS, ... 346
PORTAL FRAMES OF MULTIPLE STORIES, ... 348
PORTAL-BRACED TOWERS WITH INCLINED POSTS, ... 351
DEFLECTION OF QUADRANGULAR FRAMES, ... 354

SECTION IV.—TRUSSED BEAMS

THE KING-POST TRUSS, ... 356
THE QUEEN-POST TRUSS WITHOUT DIAGONALS, ... 358
TRUSS OF SEVERAL PANELS WITH DIAGONALS OMITTED IN ONE PANEL, ... 360

SECTION V.—BEAMS ON MULTIPLE ELASTIC SUPPORTS

THE GENERAL PROBLEM, ... 360
GENERAL METHOD OF SOLUTION, ... 361
DEFLECTION OF THE TRANSVERSE BEAM, ... 362
THE LONGITUDINAL BEAM, ... 363
DIAGRAM OF VALUES OF REACTIONS, ... 367
STRINGER AND SLAB FLOORS FOR HIGHWAY BRIDGES, ... 371

CHAPTER VII

SECONDARY STRESSES

PAGE
PRIMARY AND SECONDARY STRESSES, ... 381
SECONDARY STRESSES DUE TO RIGIDITY OF JOINT, ... 381
CALCULATION OF CHANGES OF ANGLE IN ANY TRIANGLE, ... 383
THE DEFLECTION ANGLES OF A BEAM, ... 385
NOTATION AND CONVENTIONAL SIGNS, ... 386
VALUES OF DEFLECTION ANGLES, ... 387
SELECTION OF REFERENCE DEFLECTION ANGLES, ... 388
MOMENTS AT ANY JOINT IN TERMS OF DEFLECTION ANGLES, ... 389
MOMENT AND FIBRE STRESS, ... 390
ARRANGEMENT OF CALCULATIONS, ... 391
SECONDARY STRESSES IN A PRATT TRUSS, ... 398

PAGE
GENERAL REMARKS ON METHOD OF CALCULATION, ... 413
APPROXIMATE METHODS OF CALCULATION FOR A LIMITED NUMBER OF JOINTS, . . 415
EFFECT OF COLLISION STRUTS, ... 416
EFFECT OF ECCENTRIC JOINTS, ... 420
EFFECT OF PIN CONNECTIONS, ... 422
EFFECT OF BENDING MOMENTS DUE TO TRANSVERSE LOADS, ... 426
SECONDARY STRESSES IN TOP CHORD DUE TO WEIGHT OF MEMBER, ... 428
SECONDARY STRESSES DUE TO FIXED SUPPORTS, ... 431
CALCULATION OF CHANGES OF ANGLE IN FIGURES OF FOUR OR MORE SIDES, . . 434
RELATIVE AMOUNTS OF SECONDARY STRESSES IN STRUSSES OF DIFFERENT TYPES, . . 436
CALCULATED STRESSES IN SOME TYPICAL TRUSSES, ... 453
SECONDARY STRESSES DUE TO THE ACTION OF LATERAL TRUSSES, ... 456
EFFECT OF LACING ON SECONDARY STRESSES IN COMPRESSION MEMBERS, ... 459
EFFECT OF SECONDARY UPON PRIMARY STRESSES, ... 460
SECONDARY STRESSES IN TRANSVERSE FRAMES, ... 463
SECONDARY STRESSES DUE TO ACTION OF STRINGERS, ... 465
DEFLECTION OF MEMBERS DUE TO SECONDARY STRESSES, ... 468
EXACT METHOD OF CALCULATING SECONDARY STRESSES, ... 469
STRESSES IN RIVETED EXPANSION SUSPENDERS, ... 476
BENDING MOMENTS IN MEMBERS SUBJECTED TO TRANSVERSE LOADS AND DIRECT STRESSES, ... 477
IMPACT STRESSES, ... 484

CHAPTER VIII

ANALYSIS OF QUADRANGULAR FRAMES AND SECONDARY STRESSES BY THE METHOD OF SLOPE AND DEFLECTION

SECTION I.—GENERAL THEORY

INTRODUCTION, ... 493
BENDING MOMENTS IN TERMS OF ANGLES OF TWIST AND DEFLECTION, ... 495
SPECIAL CASES OF END RESTRAINT, ... 502
MOMENTS, SHEARS AND FIBER STRESSES IN MEMBERS, ... 502
GENERAL EQUATION OF EQUILIBRIUM AT A JOINT, ... 504
APPLICATIONS, ... 506

SECTION II.—RESTRAINED, CONTINUOUS AND PARTIALLY CONTINUOUS BEAMS

RESTRAINED BEAMS, ... 507
CONTINUOUS BEAMS, ... 508
PARTIALLY CONTINUOUS BEAMS, ... 510

SECTION III.—RECTANGULAR FRAMES

PAGE
General methods of analysis, ... 512
Derivation of general formulas, ... 515
Special loading conditions, ... 518
Formulas for single lateral force, ... 518

SECTION IV.—PORTAL FRAMES

General method of analysis, ... 522
Applications, ... 523

SECTION V.—BUILDING FRAMES UNDER VERTICAL LOADING

General methods of analysis, ... 530
Approximate methods, ... 531
Moments due to loads in a single panel, ... 533
Maximum moment at the center of a beam, ... 536
Maximum moment at the end of a beam, ... 538
Maximum moment in an interior column, ... 542
Maximum moment in an exterior column, ... 544

SECTION VI.—BUILDING FRAMES UNDER HORIZONTAL LOADING

General methods of analysis, ... 546
Calculation of wind stresses in a five-story frame, ... 549
Approximate solutions, ... 556

SECTION VII.—OPEN WEBBED GIRDERS

General methods of analysis, ... 565
Example, ... 565

SECTION VIII.—SECONDARY STRESSES

General methods of analysis, ... 569
Determination of α values by means of angle changes, ... 570
Determination of α values by means of Williot diagrams, ... 571
Determination of Secondary stresses in a Pratt truss, ... 572
Determination of true values of deflection and twist angles, ... 580
Comparison of methods of secondary stress analysis, ... 581
Secondary stresses in the top chord of a Pratt truss, ... 583

Index 587

THEORY AND PRACTICE

IN THE DESIGNING OF

MODERN FRAMED STRUCTURES

PART II

STATICALLY INDETERMINATE STRUCTURES
SECONDARY STRESSES

CHAPTER I

CONTINUOUS GIRDERS

SECTION I.—DEFLECTION OF STRAIGHT BEAMS

THE AREA-MOMENT METHOD OF ANALYSIS.

1. **The Elastic Curve.**—Let $A\ B$, Fig. 1, be any portion of a horizontal beam subjected to certain bending moments and shears due to the action of vertical forces. The curvature due to bending is only slight, and it is assumed that the length of any part may be taken equal to its horizontal projection. Assume the following notation:

M = bending moment at any section, assumed as positive when causing compression in the upper fibres;

V = shear at any section due to loads which cause M.

w = uniform load per unit of length.

x, y = coordinates of any point N referred to an origin at O, the axis of X being tangent to the beam at A (y is also the deflection of point N from its original position);

ϕ = inclination of tangent at N, = angular change in the beam at N due to stress;

$d\phi$ = angular change over the length dx;
I = moment of inertia at any section;
E = modulus of elasticity, assumed as constant;
ρ = radius of curvature at N.

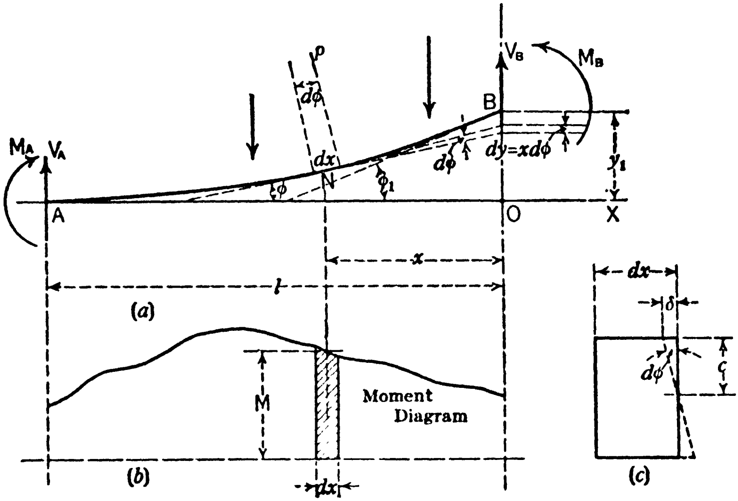


FIG. 1

It is shown in treatises on mechanics that, neglecting the distortion due to shear, the differential equation of the elastic curve is

$$\frac{d^2 y}{d x^2} = \frac{M}{E I} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

The radius of curvature is given by the formula

$$\frac{1}{\rho} = \frac{d^2 y}{d x^2} = \frac{M}{E I} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Since the rate of change of moment at any point is equal to the shear at that point, we have

$$\frac{d M}{d x} = V, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

or $dM = V\,dx$ and

$$M = \int V\,d x. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

The rate of change of shear for uniform loading is equal to the load per unit of length, that is,

$$\frac{dV}{dx} = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Then

$$V = \int w\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Placing V from eq. (6) in eq. (4)

$$M = \int\int w\,dx\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Combining eqs. (3) and (5), we have

$$w = \frac{dV}{dx} = \frac{d}{dx}\left(\frac{dM}{dx}\right) = \frac{d^2 M}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

On combining eqs. (2) and (3), we have

$$V = \frac{d}{dx}(M) = EI\frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right) = EI\frac{d^3 y}{dx^3}, \quad . \quad . \quad . \quad (9)$$

and likewise from eqs. (5) and (9)

$$w = \frac{d}{dx}(V) = \frac{d}{dx}\left(EI\frac{d^3 y}{dx^3}\right) = EI\frac{d^4 y}{dx^4} \quad . \quad . \quad . \quad (10)$$

The elastic relations expressed by eqs. (1) to (10) will be found useful in the analysis of problems in statically indeterminate structures.

2. Angular Change at Any Point.—Integrating eq. (11) between A and any point N of Fig. 1, we obtain the value of dy/dx or ϕ for this point. This angular change, referred to the tangent at A, is therefore expressed by the general equation

$$\phi = \frac{dy}{dx} = \int_A^N \frac{M}{EI}\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and for point B

$$\phi_1 = \int_A^B \frac{M}{EI}\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Equation (11) may be determined from Fig. 1 (*c*), which shows the beam element at N, Fig. 1 (*a*). The dotted lines in Fig. 1 (*c*) show the position of the face of the element due to the deformation caused by the bending moment. Then $d\phi = \delta/c$. But δ = deformation of an extreme fibre under bending moment. Hence $\delta = f\,dx/E$. With $f = M\,c/I$, we have

$$d\phi = \frac{M}{EI}\,dx,$$

and

$$\phi = \int \frac{M}{EI}\,dx.$$

From Fig. 1 (*b*) it can be seen that $M\,dx$ of eq. (11) is an element of the moment diagram area. Then $\int_N^B M\,dx$ is the area of the moment diagram from N to B. Hence we have the following important principle:

Rule 1.—The angle between tangents at any two points on the elastic curve of a beam is equal to the area of the moment diagram for the applied loading between these two points, divided by EI.

Problems involving the determination of the angles between tangents to the elastic curve of a beam may be solved by direct integration from eq. (11) or by means of a semi-graphical method based on Rule 1. Applications of eq. (11) and Rule 1 are given in Art. 4.

3. Deflection of Any Point with Respect to a Tangent at Another Point.—In Fig. 1 it can be seen that if the tangents to the elastic curve at the ends of the element at N are produced to an intersection with a vertical axis Y through point B, the intercept on axis Y is

$$dy = x\,d\phi,$$

and for tangents at A and B

$$y_1 = \int_A^B x\,d\phi.$$

Substituting the value of $d\phi$ from eq. (12), we have

$$y_1 = \int_A^B \frac{M}{EI} x\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

When the tangent at A is known to be horizontal, as shown in Fig. 1 (*a*), y_1 of eq. (13) gives directly the deflection of point B with respect to A.

The term $\int_A^B \frac{M}{EI} x\,dx$ of eq. (13) may be written in the form $\frac{1}{EI}\int_A^B (M\,dx)\,x$. As shown in Fig. 1 (*b*), $M\,dx$ is an element of the moment diagram area and x is the distance from the center of this element of area to the axis on which the intercept y_1 is measured. We then have a second important principle, which is

Rule 2.—The intercept y on any vertical axis cut by the tangents at any two points on the elastic curve is equal to $1/EI$ times the statical moment about the given axis of the moment diagram area between the two tangent points. Applications of eq. (13) and Rule 2 are given in Art. 4.

Equation (13) can also be readily derived from eq. (26), Art. 217, of Part I, which expresses the deflection in terms of work. For point B this becomes $y_1 = \int_A^B \frac{M\,dx}{EI} \cdot m$, in which m is the bending moment at any point due to a load unity acting at point B. In this case $m = x$, hence we have $y_1 = \int_A^B \frac{M}{EI} x\,dx$, as in eq. (13).

4. Methods of Application.—Many problems in the determination of the deflection of beams and the slope of the tangent to the elastic curve may readily be solved by the principles stated in the preceding articles. Applications of these principles give a convenient method of analysis for continuous girders, arches, and many other problems involving the deflection of beams.

Two general methods of solution may be used in the determination of deflections and slopes based on the principles stated in the preceding articles. One method, which will be called the Algebraic Method, consists in direct substitution in eqs. (11) to (13). The other method, which will be called the Semi-graphical Method, makes use of Rules 1 and 2. Moment diagrams are constructed for the given conditions and the area or statical moment of the moment diagram is determined.

In general the second, or Semi-graphical Method, is probably best adapted to the needs of the average engineer, for the desired results may be determined by dividing the moment diagram into simple figures whose areas and statical moments may readily be determined. The first, or Algebraic Method, requires an application of integral calculus, which, although not difficult, is avoided wherever possible by the average engineer.

Problems illustrating both methods are given in the following articles.

5. Algebraic Application of the Area Moment Method.—The application of eqs. (11) and (13) will be illustrated by the solution of several typical problems.

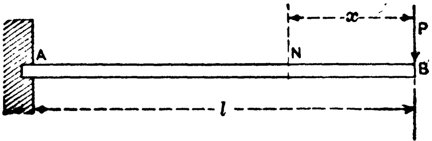


FIG. 2.

1. Cantilever, supporting a load P, Fig. 2. Required the deflection of point B and the inclination at this point. Assume E and I constant.

Here $M = P\,x$ and the deflection is

$$y_1 = \int_A^B \frac{M\,x\,d\,x}{E\,I} = \frac{P}{E\,I}\int_0^l x^2\,d\,x = \frac{P\,l^3}{3\,E\,I}.$$

The angular change at B is

$$\phi_1 = \int_A^B \frac{M}{E\,I}\,d\,x = \frac{P}{E\,I}\int_0^l x\,d\,x = \frac{P\,l^2}{2\,E\,I}.$$

2. Required the deflection and inclination at B, Fig. 3, the load P being applied at a distance a from B. Measuring x from B we have $M = P\,(x - a)$ and

$$y_1 = \frac{P}{EI}\int_a^l (x - a)\, x\, d\, x = \frac{P}{6\,EI}(2\,l^3 - 3\,a\,l^2 + a^3).$$

Also $$\phi_1 = \frac{P}{EI}\int_a^l (x - a)\, d\, x = \frac{P}{2\,EI}(l - a)^2.$$

This is the same as the value of ϕ at point C, the beam being straight from B to C.

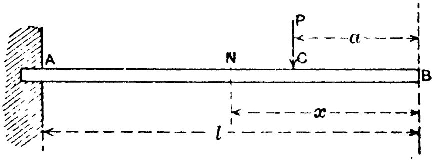


FIG. 3.

3. Beam loaded as shown in Fig. 4. Required the deflection of the centre point C. As the tangent at C remains horizontal the desired deflection will be found by calculating the upward movement of B with respect to C, Fig. (a).

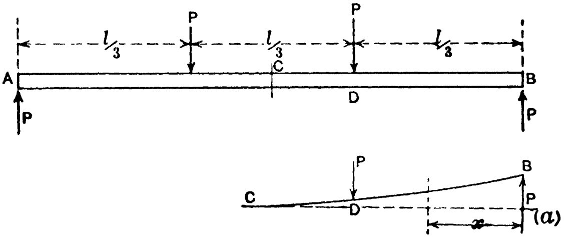


FIG. 4.

From B to D the value of M of eq. (13) is $P\,x$ and $\int \frac{M\,x\,d\,x}{EI} = \frac{P}{EI}\int_0^{l/3} x^2\,d\,x = \frac{1}{81}\frac{P\,l^3}{EI}$. From D to C, $M = \frac{P\,l}{3}$, and $\int\frac{M\,x\,d\,x}{EI} = \frac{P\,l}{3\,EI}\int_{l/3}^{l/2} x\,d\,x = \frac{5}{216}\,\frac{P\,l^3}{EI}$. The total deflection is therefore,

$$y_1 = \left(\frac{1}{81} + \frac{5}{216}\right)\frac{P\,l^3}{EI} = \frac{23}{648}\,\frac{P\,l^3}{EI}.$$

In a similar manner we find for the inclination at B,

$$\varphi_1 = \frac{1}{9}\,\frac{P\,l^2}{EI}.$$

Essentially the same calculations are required by the use of the general expression $y = \int \frac{M\,d\,x}{EI}\cdot m$, noted in Art. 3. Applying this method to problem (3) the calculations are as follows:

From B to D: $M = P\,x$; $m = \frac{1}{2}\,x$; $\int_B^D \frac{M\,d\,x}{E\,I}\cdot m = \frac{P}{2}\int_0^{l/3}\frac{x^2\,d\,x}{E\,I} = \frac{1}{162}\,\frac{P\,l^3}{E\,I}$.

From D to C: $M = \frac{P\,l}{3}$; $m = \frac{1}{2}\,x$; $\int_D^C \frac{M\,d\,x}{E\,I}\cdot m = \frac{P\,l}{6}\int_{l/3}^{l/2} x\,d\,x = \frac{5}{432}\,\frac{P\,l^3}{E\,I}$.

Adding these and multiplying by two, to take account of the left half of the beam, we have $y_1 = 2\left(\frac{1}{162} + \frac{5}{432}\right)\frac{P\,l^3}{E\,I} = \frac{23}{648}\,\frac{P\,l^3}{E\,I}$, as found before.

Except for comparatively simple cases, as here illustrated, the general formula for deflection, $y = \int\frac{M\,d\,x}{E\,I}\cdot m$, will be the easier of application.

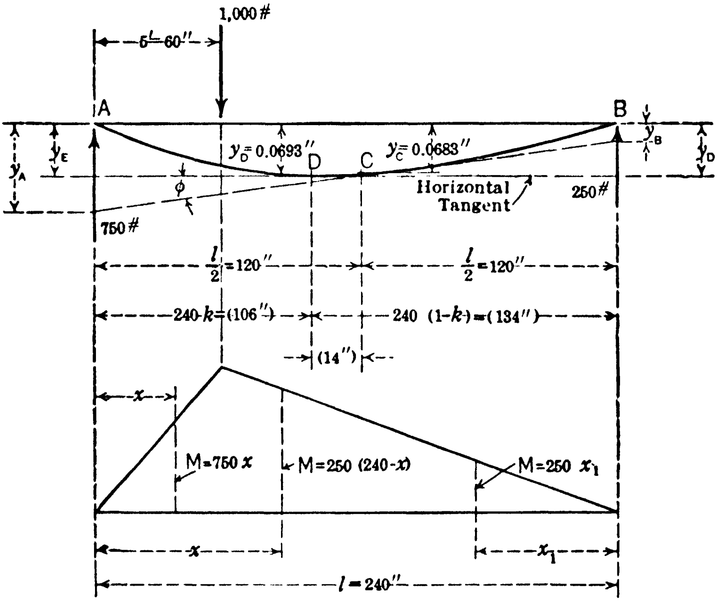


FIG. 5.

4. Beam loaded as shown in Fig. 5. Required the deflection of the center point C. $E = 29{,}000{,}000$ and $I = 100$ in.[4]

To determine the deflection at C, Fig. 5, we calculate first the intercepts y_A and y_B between the tangent at C and the vertical axes at A and B. Then y_C, the required deflection, may readily be determined from similar triangles.

From eq. (13), we may write

$$y_A = \int \frac{M}{EI} x dx = \frac{1}{EI} \left\{ \int_0^{60} (750x)\, x\, dx + \int_{60}^{120} [250\,(240 - x)]\, x\, dx \right\}$$

$$= \frac{252{,}000{,}000}{EI},$$

and

$$y_B = \int \frac{M(240 - x)}{EI}\, dx = \frac{1}{EI} \left\{ \int_{120}^{240} 250\,(240 - x)^2\, dx \right\} = \frac{144{,}000{,}000}{EI}.$$

Then

$$y_C = \tfrac{1}{2}\,(y_A + y_B) = \frac{288{,}000{,}000}{EI} = \frac{288{,}000{,}000}{2{,}900{,}000{,}000} = 0.0683 \text{ in.}$$

In this case the solution for y_B may be shortened somewhat by transferring the origin for M to the right end of the beam. Then $M = 250\,x_1$ and

$$\int_0^{120} M\, x_1\, d\,x = \int_0^{120} 250\, x_1^2\, d\,x = 144{,}000{,}000.$$

5. Determine the point of maximum deflection in the beam of Fig. 5. At the point of maximum deflection, the tangent to the elastic curve is horizontal. Then y_E and y_D, the intercepts of this tangent on vertical axes through A and B, Fig. 5, are equal. Assume D of Fig. 5 to represent the location of the point of tangency and assume its distance from A to be $240\,k$, where k, the unknown to be determined, represents some fractional portion of the span length. Values of y_E and y_D may be determined from eq. (13) in terms of k.

Thus

$$y_E = \int \frac{M}{EI}\, x\, d\,x = \frac{1}{EI} \left\{ \int_0^{60} (750\,x)\, x\, d\,x + \int_{60}^{240k} 250(240 - x)]\, x\, dx \right\}$$

$$y_E = \frac{250}{EI}\,(-\,4{,}608{,}000\, k^3 + 6{,}912{,}000\, k^2 - 144{,}000),$$

and

$$y_D = \int \frac{M}{EI}\, x\, d\,x = \frac{1}{EI} \int_0^{240(1-k)} (250\, x_1)\, x\, d\,x$$

$$= \frac{250}{EI}\,(4{,}608{,}000 - 13{,}824{,}000\, k + 13{,}824\,000\, k^2 - 4{,}608\,000\, k^3)$$

On equating these values of y_E and y_D, collecting terms, and solving for k, we have $k^2 - 2\,k + 0.6875 = 0$, from which $k = 0.441$ and therefore

$240k = 106$ ins. Hence the maximum deflection occurs at a point 106 ins. from the left support. The value of this deflection, found by substituting k in the expressions for y_E or y_D, is 0.0693 in.

When values of y_A and y_B, as calculated in example 4, are available, the solution may be made by the following method. Let ϕ = angle between the tangents at C and D, Fig. 5. Then, since the tangent at D is horizontal, ϕ may be determined from the values given in example 4, and we have

$$\phi = \frac{y_A - y_B}{240} = \frac{252{,}000{,}000 - 144{,}000{,}000}{240\,E\,I} = \frac{450{,}000}{E\,I}.$$

Also, since ϕ is equal to the moment diagram area from C to D divided by EI (see Rule 1), we may write from eq. (11), Art. 2,

$$\phi = \int_D^C \frac{M}{E\,I}\,d\,x = \frac{1}{E\,I}\int_{120}^{240\,(1-k)} 250\,x_1 d\,x = \frac{125}{E\,I}\,[43{,}200 + 57{,}600(k^2 - 2k)].$$

Equating these values of ϕ, and solving for k, we find $k = 0.441$, which checks the value given above.

6. Semi-Graphical Application of Area-Moment Method.—Problems similar to those given in Art. 5 will now be analyzed by the semi-graphical method.

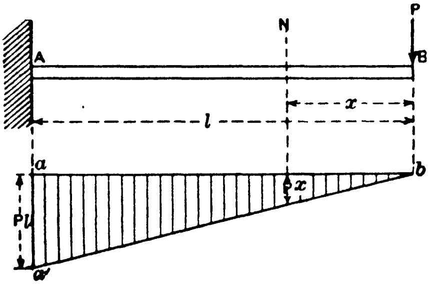


FIG. 6.

1. Fig. 6 represents the beam and its moment diagram. Determine the maximum deflection and the slope of the tangent to the elastic curve at point B.

From Rule 1, Art. 2,

$$\phi_B = \text{Area}\,\frac{a\,a'\,b}{E\,I} = \frac{1}{2}\,\frac{P\,l^2}{E\,I}.$$

The deflection at B, from Rule 2, Art. 3, is

$$y_B = \frac{\text{Moment of area } a\,a'\,b \text{ about } B}{E\,I} = \frac{1}{2} P\,l^2 \times \frac{2}{3} l \div E\,I.$$

$$y_B = \frac{1}{3} \frac{P\,l^3}{E\,I}.$$

These results check those found in Art. 5.

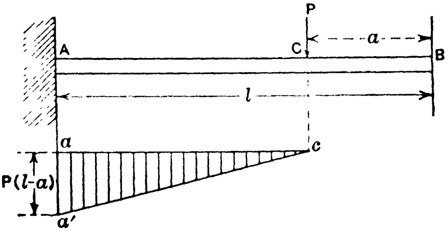


FIG. 7.

2. (Fig. 7.) The angular change at C or B = (Area $a\,a'\,c$) $\div E\,I = \dfrac{P\,(l-a)^2}{2\,E\,I}$. The deflection at B = (Moment of area $a\,a'\,c$ about B) $\div E\,I$ $= [\frac{1}{2} P\,(l-a)^2 \times (a + \frac{2}{3}(l-a))] \div E\,I = \dfrac{P}{6\,E\,I}(2\,l^3 - 3\,a\,l^2 + a^3)$.

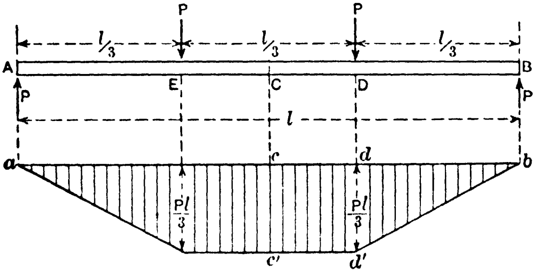


FIG. 8.

3. (Fig. 8.) The moment diagram has ordinates at D and E equal to $\dfrac{P\,l}{3}$. The inclination at B = (Area $c\,c'\,b$) $\div E\,I = \dfrac{1}{9}\dfrac{P\,l^2}{E\,I}$. The deflection is equal to the sum of the moments about B of the areas $c\,c'\,d'\,d$ and $d\,d'\,b$, divided by $E\,I$. Mom. of area $c\,c'\,d'\,d = \dfrac{1}{18} P\,l^2 \left(\dfrac{l}{3} + \dfrac{l}{12}\right) = \dfrac{5}{216} P\,l^3$, and mom. of area $d\,d'\,b = \dfrac{1}{18} P\,l^2 \times \dfrac{2}{9} l = \dfrac{1}{81} P\,l^3$. Hence the deflection

$$= \left(\frac{5}{216} + \frac{1}{81}\right)\frac{P\,l^3}{E\,I} = \frac{23}{648}\,\frac{P\,l^3}{E\,I}.$$

4. **Beam loaded as in Fig. 9. Determine the deflection of the center point** C. $E = 29{,}000{,}000$ and $I = 100$ in^4.

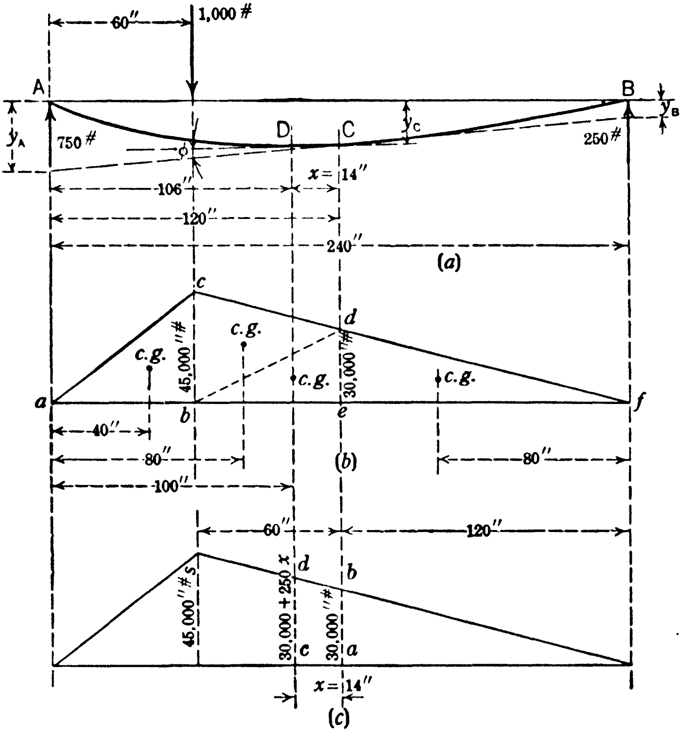


FIG. 9.

As in Example 4, Art. 5, from Rule 2, Art. 3,

$$y_A = \frac{\text{Moment of area } a\ c\ d\ e, \text{ Fig. 9 } (b), \text{ about } A}{E\,I}$$

and

$$y_B = \frac{\text{Moment of area } d\ e\ f, \text{ Fig. 9 } (b), \text{ about } B}{E\,I}.$$

The area-moment required for the determination of y_A may be obtained by dividing area *a c d e* into three triangles, as shown in Fig. (*b*). Distances from *A* to the centers of gravity of these triangles are shown on Fig. 9(*b*).

Then

$$y_A = \frac{(\text{Area } a\,b\,c)\ 40 + (\text{Area } c\,b\,d)\ 80 + (\text{area } b\,d\,e)\ 100}{E\,I}$$

$$y_A = \frac{\frac{1}{2}\,(45{,}000)\,(60)\,(40) + \frac{1}{2}\,(45{,}000)\,(60)\,(80) + \frac{1}{2}\,(30{,}000)\,(60)\,(100)}{E\,I}$$

$$y_A = \frac{252{,}000{,}000}{.\ E\,I}$$

$$y_B = \frac{(\text{Area } d\,e\,f)\ 80}{E\,I} = \frac{\frac{1}{2}\,(30{,}000)\,(120)\,(80)}{E\,I} = \frac{144{,}000{,}000}{E\,I}$$

Hence,

$$y_C = \tfrac{1}{2}\,(y_A + y_B) = \frac{252{,}000{,}000 + 144{,}000{,}000}{(29{,}000{,}000)\,(100)} = 0.0683 \text{ in.}$$

5. Determine the position of the point of maximum deflection in the beam of Fig. 9. Let D, at a distance x from C, represent the point of maximum deflection, and let ϕ = angle between the horizontal tangent at D and the tangent at C, the center point of the beam

From Fig. 9 (a) and Example 4,

$$\phi = \frac{y_A - y_B}{240} = \frac{252{,}000{,}000 - 144{,}000{,}000}{240\,E\,I} = \frac{450{,}000}{E\,I} \quad . \quad . \quad (a)$$

From Rule 1, Art. 3,

$$\phi = \frac{\text{Area } a\,b\,c\,d, \text{ Fig. 9 } (c)}{E\,I}$$

From Fig. (c),

$$cd = 30{,}000 + \frac{45{,}000}{180}\,x = 30{,}000 + 250\,x.$$

Then

$$\phi = \frac{\frac{1}{2}\,[(30{,}000 + 250\,x) + 30{,}000]\,x}{E\,I} = \frac{30{,}000\,x + 125\,x^2}{E\,I} \quad . \quad . \quad (b)$$

Equating eqs. (a) and (b), and reducing, we have $x^2 + 240\,x = 3{,}600$, from which, $x = 14$ ins.

The point of maximum deflection is then 120 − 14 = 106 ins. from the left support, which checks the result given in example 5, Art. 5.

6. Determine the maximum deflection of the girder shown in Fig. 10. Fig. (c) shows the moment diagram for the applied loads and Fig. (d) shows the I-diagram, which represents the variation in moment of inertia of the girder section. Fig. (e) gives values of M/I at critical points. These values were obtained by dividing the M-values of Fig. (c) by the corresponding I-values of Fig. (d).

To determine the maximum deflection for the beam in question, we first determine the values of the intercepts y_A and y_B of Fig. (b) with respect to a tangent at some convenient point, which in this case will be taken as the center point C. Then as in the latter part of example 5, the maximum deflection may readily be determined.

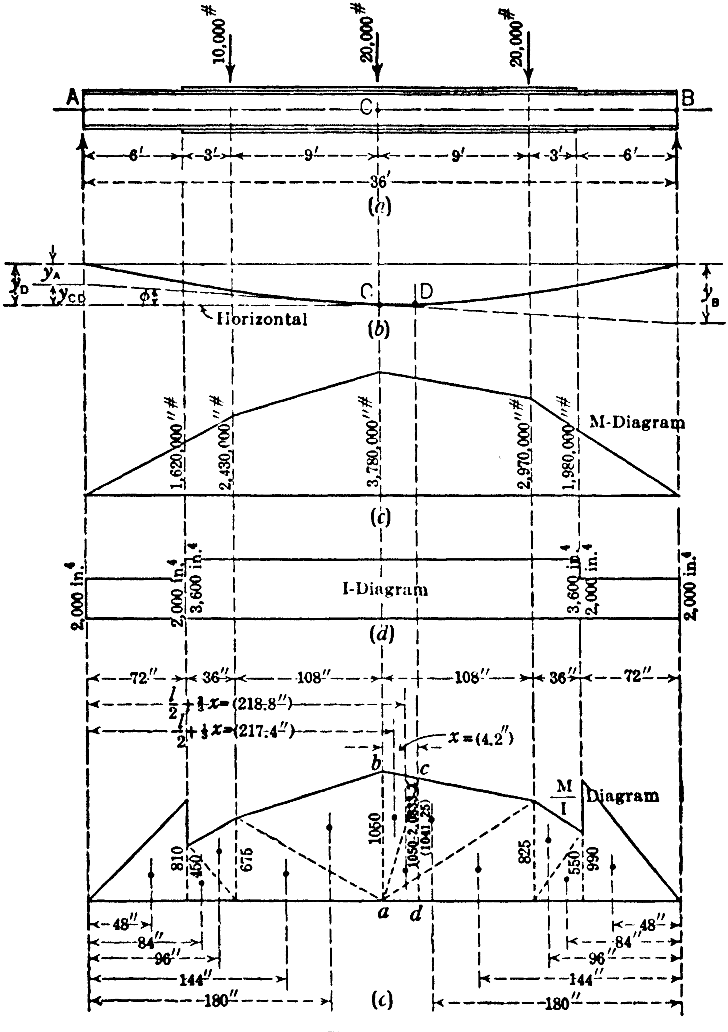


FIG. 10.

Values of y_A and y_B may be determined from Rule 2 of Art. 3. To aid in determining these values of y_A and y_B, Fig. (e) has been divided into triangles. The location of the center of gravity of each triangle has been indicated on the figure.

Then from Rule 2,

$$y_A = \frac{1}{E}[\tfrac{1}{2}(810)(72)(48) + \tfrac{1}{2}(450)(36)(84) + \tfrac{1}{2}(675)(36)(96) + \tfrac{1}{2}(75)(108)(144) + \tfrac{1}{2}(1050)(108)(180)]$$

$$y_A = \frac{18{,}701{,}280}{E},$$

and

$$y_B = \frac{1}{E}[\tfrac{1}{2}(990)(72)(48) + \tfrac{1}{2}(550)(36)(84) + \tfrac{1}{2}(825)(36)(96) + \tfrac{1}{2}(825)(108)(144) + \tfrac{1}{2}(1050)(108)(180)]$$

$$y_B = \frac{20{,}589{,}120}{E}.$$

From Fig. (b),

$$\phi = \frac{20{,}589{,}120 - 18{,}701{,}280}{(36)(12)E} = \frac{4{,}370}{E}. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

In a beam unsymmetrically loaded, the maximum deflection will occur at a point between the beam center and the position of the resultant of the applied loads. For the loading conditions shown in Fig. (*a*), the resultant is to the right of the center of the beam. Assume the point of maximum deflection to be at *D* distance *x* to the right of the beam center.

From Fig. (*e*) and Rule 1,

$$\phi = \frac{\text{Area } a\,b\,c\,d}{E} = \frac{(2{,}100 - 20{,}833\,x)\,x}{2\,E}. \quad . \quad . \quad . \quad . \quad (b)$$

Equating the two values of ϕ and solving, we find $x = 4.2$ ins.

The maximum deflection at *D*, which is y_D of Fig. (*b*), may be determined by adding to y_A, as determined above, the intercept y_{CD}, which is equal to the moment of area *a b c d* of Fig. (*e*) taken about point *A*. Values of the ordinates and lever arms, determined for $x = 4.2$ ins., are shown on Fig. (*e*). Then

$$y_{CD} = \frac{\tfrac{1}{2}(1050)(4.2)(217.4) + \tfrac{1}{2}(1041.25)(4.2)(218.8)}{E}$$

$$y_{CD} = \frac{957{,}801}{E}.$$

Hence

$$y_D = \frac{18{,}701{,}280 + 957{,}801}{29{,}000{,}000} = 0.678 \text{ in.}$$

7. Beam fixed at the ends. Consider the beam *A B*, Fig. 11, fixed at *A* and *B* and supporting a load *P* applied at a distance *kl* from *A*. Required the values of the end moments M_A and M_B.

The unknowns are M_A, M_B, V_A, and V_B, of Fig. (*b*). Four condition equations are required for the determination of these unknowns. Two equations based on static equilibrium may be obtained by writing $\Sigma M = 0$ and $\Sigma V = 0$ for the beam considered as a free body. Two additional equations may be obtained by calculating the total angular change from A to B and the deflection of B with respect to the tangent at A. Since the beam is fixed at the ends, it is evident that both of these quantities must be equal to zero. Hence four condition equations are available and the unknowns may be determined.

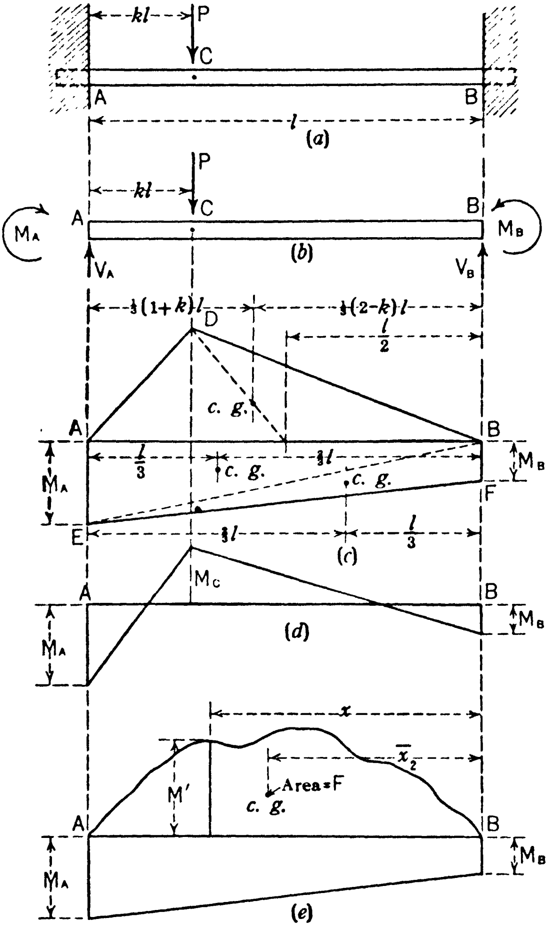


FIG. 11.

Fig. 11 (*d*) shows the moment diagram for the given conditions. For the purposes of this analysis it is convenient to represent the moment diagram in the form shown in Fig. (*c*). The triangle $A\,D\,B$ represents the effect of the applied load considered as acting on a simple beam of span l, and the trapezoid $A\ B\ F\ E$ represents the effect of the end moments M_A and M_B. For the present all moments are considered as positive.

As stated above, the total angular change from A to B is zero. From Rule 1, Art. 2,

$$\phi_{AB} = 0 = \frac{1}{EI}\,(\text{area of moment diagram, Fig. 11 }(c))$$

$$\Sigma\,(\text{moment areas}) = \frac{1}{EI}\left[\tfrac{1}{2} P\,(1-k)\,k\,l^2 + \tfrac{1}{2}\,(M_A + M_B)\,l\right] = 0. \qquad (a)$$

Again, as stated above, the deflection of B with respect to the tangent at A is zero. From Rule 2, Art. 3,

$$y_B = 0 = \frac{1}{EI}\begin{pmatrix}\text{Moment of moment diagram area,}\\ \text{Fig. 11 }(c)\text{, taken about point }B\end{pmatrix}$$

$$\Sigma\,(\text{area moments}) = \frac{1}{EI}\begin{bmatrix}\frac{1}{2} P\,(1-k)\,k\,l^2 \times \frac{l}{3}\,(2-k) \\ + \frac{1}{2} M_A\,l \times \frac{2}{3}\,l + \frac{1}{2} M_B\,l \times \frac{l}{3}\end{bmatrix} = 0. \;. \qquad (b)$$

Eqs. (a) and (b) form a pair of simultaneous equations in terms of M_A and M_B. On solving these equations:

$$\left.\begin{aligned} M_A &= -\,P\,(1-k)^2\,k\,l \\ \text{and} \qquad M_B &= -\,P\,(1-k)\,k^2\,l \end{aligned}\right\} \quad .\ .\ .\ .\ .\ .\ . \quad (14)$$

If $M' = P\,(1-k)\,k\,l$ = moment in a simple beam, we may write

$$\left.\begin{aligned} M_A &= -\,M'\,(1-k) \\ \text{and} \qquad M_B &= -\,M'\,k \end{aligned}\right\} \quad .\ .\ .\ .\ .\ .\ . \quad (15)$$

That is, the bending moments at A and B may be considered as equal to the moment at C in a simple beam, divided between A and B inversely as the distances $k\,l$ and $(1-k)\,l$, or as the reactions of a simple beam. Finally, Fig. 11 (d) shows the true moment diagram.

Values of V_A and V_B may readily be determined by means of the two conditions of static equilibrium mentioned above.

If more general expressions for the end moments of the beam of Fig. 11 are desired, we may draw the moment diagram of Fig. 11 (*c*) in the manner shown in Fig. 11 (*e*). Here the portion of the moment diagram above line $A\,B$ is intended to represent simple beam moment due to any loading condition instead of the single load shown in Fig. 11 (*a*).

Equation (*a*) then becomes

$$\phi_{AB} = 0 = \frac{1}{E\,I}\left[\int_0^l M' d\,x + \frac{1}{2}(M_A + M_B)\,l\right] = 0,$$

and eq. (*b*) becomes

$$y_B = 0 = \frac{1}{E\,I}\left[\int_0^l M' x\,d\,x + \frac{1}{3} M_A\,l^2 + \frac{1}{6} M_B\,l^2\right] = 0.$$

On solving these equations as before, we have

$$\left.\begin{aligned} M_A &= -\frac{2}{l^2}\int_0^l M'(3\,x - l)\,d\,x \\ \text{and}\qquad M_B &= -\frac{2}{l^2}\int_0^l M'(2\,l - 3\,x)\,d\,x \end{aligned}\right\} \quad . \; . \; . \quad (16)$$

In eqs. (16) the value of M_A may be written in the form

$$M_A = -\frac{2}{l^2}\left[\int_0^l 3\,M'\,x\,d\,x - l\int_0^l M'\,d\,x\right]$$

$$= -\frac{2}{l^2}\int_0^l M'\,d\,x\left[3\,\frac{\int_0^l M'\,x\,d\,x}{\int_0^l M'\,d\,x} - l\right]$$

Now $\int_0^l M' \, dx = F =$ area of the moment diagram and

$$\frac{\int_0^l M' x \, dx}{\int_0^l M' dx} = \bar{x}_2 = \text{distance from } B, \text{ Fig. } (e), \text{ to the center of gravity}$$

of the moment diagram area. Hence eqs. (16) may be written in the form

$$\left.\begin{aligned} M_A &= -\frac{2F}{l^2}(3\bar{x}_2 - l) \\ \text{and} \qquad M_B &= -\frac{2F}{l^2}(2l - 3\bar{x}_2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

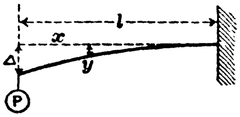
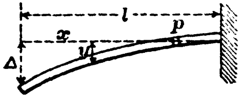
Equations for the end moments in a fixed beam expressed in the form of eqs. (14) to (17) will be found useful in the work to follow. When the variation in moment M' is readily expressed in the form of an equation, eqs. (16) may be used and values of M_A and M_B obtained by direct integration. When the value of M' is not readily expressed by an equation, or where several equations must be used because the law of variation of M' may change, as for a concentrated load system, then eqs. (17) will be found useful and the desired results may be obtained by the semi-graphical process described above.

7. Table of Moments, Stresses, and Deflections.—The table on pp. 20–23 gives values of moments, stresses, and deflections for various commonly occurring cases of beams, both simply supported and continuous. The values of deflection in terms of maximum fibre stress are very useful in problems in which the deflection may be the known factor.

8. Deflection Due to Shear.—In the theory of flexure presented in the preceding articles the effect of shearing strains has been neglected, the equations of the elastic curves being derived from a consideration of compressive and tensile strains only. In the case of beams of the usual proportions the deflection due to shear is small and generally negligible, but for very short beams it requires consideration. It is

TABLE NO. I

MOMENTS, STRESSES, AND DEFLECTION OF BEAMS

The Beam and its Load with the Moment and Shear Diagrams.	Moment Equation and Maximum Moment. M_x and $M_{max.}$	Equation of Elastic Line, and Maximum Deflection in Terms of the Loading. y and Δ.	Maximum Deflection in Terms of Stress on Extreme Fibre of Symmetrical Sections. Δ.	Maximum Stress on Extreme Fibre in Terms of the Loading, Symmetrical Sections. f.
	$M_x = -Px$ $M_{max.} = -Pl$	$y = \frac{P}{6EI}[2l^3 - 3l^2x + x^3]$ $\Delta = \frac{Pl^3}{3EI}$	$\frac{2fl^2}{3Eh}$	$\frac{Plh}{2I}$
	$M_x = -\frac{px^2}{2}$ $M_{max.} = -\frac{pl^2}{2}$	$y = \frac{p}{24EI}[x^4 - 4l^3x + 3l^4]$ $\Delta = \frac{pl^4}{8EI}$	$\frac{fl^2}{2Eh}$	$\frac{pl^2h}{4I}$

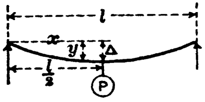
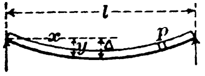
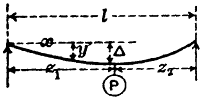
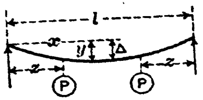
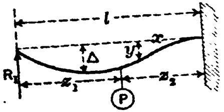
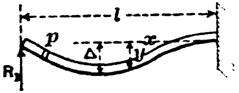
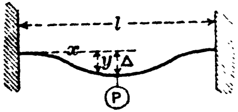
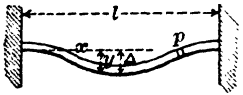
	$M_x = \frac{P}{2}x$ $M_{max.} = \frac{Pl}{4}$	$y = \frac{Px}{48EI}[3l^2 - 4x^2]$ $\Delta = \frac{Pl^3}{48EI}$	$\frac{fl^2}{6Eh}$	$\frac{Plh}{8I}$
	$M_x = \frac{px}{2}(l - x)$ $M_{max.} = \frac{pl^2}{8}$	$y = \frac{px}{24EI}[l^3 - 2lx^2 + x^3]$ $\Delta = \frac{5pl^4}{384EI}$	$\frac{5}{24}\frac{fl^2}{Eh}$	$\frac{pl^2h}{16I}$
	$x < z_1$ $M_x = \frac{Pz_2x}{l}$ $x > z_1$ $M_x = \frac{Pz_2x}{l} - P(x - z_1)$ $M_{max.} = \frac{Pz_1z_2}{l}$	$x < z_1$ $y = \frac{Pz_2x}{6EIl}[2lz_1 - z_1^2 - x^2]$ $x > z_1$ $y = \frac{Pz_1(l-x)}{6EIl}[2lx - x^2 - z_1^2]$ $\Delta = \frac{Pz_2}{27EIl}\sqrt{3[z_1(2z_2+z_1)]^3}$ for $x = \frac{1}{3}\sqrt{3[z_1(2z_2+z_1)]}$	$\frac{2f}{27Eh}\sqrt{3z_1(2z_2+z_1)^3}$	$\frac{Pz_1z_2h}{2Il}$

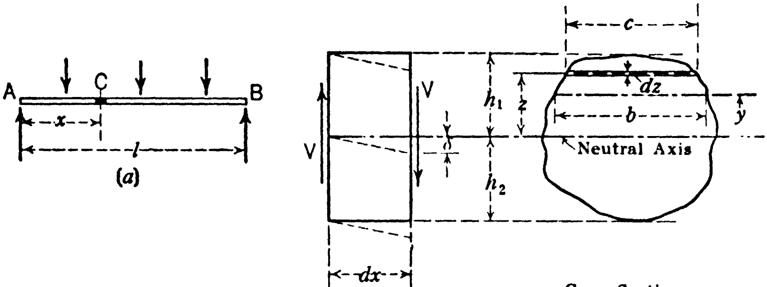
TABLE NO. 1—(*Continued*)

MOMENTS, STRESSES, AND DEFLECTION OF BEAMS.

The Beam and its Load with the Moment and Shear Diagrams.	Moment Equation and Maximum Moment. M_x and $M_{max.}$	Equation of Elastic Line, and Maximum Deflection in Terms of the Loading. y and Δ.	Maximum Deflection in Terms of Stress on Extreme Fibre of Symmetrical Sections. Δ.	Maximum Stress on Extreme Fibre in Terms of the Loading, Symmetrical Sections. f.
	$x < z$ $M_x = P\,x$ $x > z$ $M = P\,z = M_{max.}$	$x < z$ $y = \frac{P\,x}{6\,E\,I}\,[3\,l\,z - 3\,z^2 - x^2]$ $x > z$ $y = \frac{P\,z}{6\,E\,I}\,[3\,l\,x - 3\,x^2 - z^2]$ $\Delta = \frac{P\,z}{6\,E\,I}\left[\frac{3}{4}\,l^2 - z^2\right]$	$\frac{f}{3\,E\,h}\left[\frac{3}{4}\,l^2 - z^2\right]$	$\frac{P\,z\,h}{2\,I}$
	$R_1 = \frac{P}{2\,l^3}\,[3\,l\,z_2^2 - z_2^3]$ $x < z_2$ $M_x = R_1\,(l - x) - P\,(z_2 - x)$ $x > z_2$ $M_x = R_1\,(l - x)$ $M_{max.} = R_1\,(l - z_2)$ for $x = z_2$	$x < z_2$ $y = \frac{1}{6\,E\,I}\,[R_1\,x^3 - 3\,R_1\,l\,x_2 + 3\,P\,z_2\,x^2 - P\,x^3]$ $x > z_2$ $y = \frac{1}{6\,E\,I}\,R_1\,x_1^3 - 3\,R_1\,l\,x^2 + 3\,P\,z_2^2\,x - P\,z_2^3]$ $\Delta = \frac{P\,z_2^2}{6\,E\,I}\,(l - z_2)\sqrt{\frac{l - z_2}{3\,l - z_2}}$ for $x = l\left(1 - \sqrt{\frac{l - z_2}{3\,l - z_2}}\right)$	$\frac{2\,f}{3\,E\,h}\sqrt{(3\,l - z_2)\,(l - z_2)}$	$\frac{P\,h}{4\,l^3\,I}\,(3\,l\,z_2^2 - z_2^3)\,(l - x)$

	$R_1 = \frac{3}{8}\,p\,l$ $M_x = \frac{p}{8}(4x - l)(l - x)$ $M_{max.} = -\frac{p\,l^2}{8}$ for $x = 0$	$y = \frac{p\,x^2}{48\,E\,I}(l - x)(3\,l - 2\,x)$ $\Delta = 0.0054\,\frac{p\,l^4}{E\,I}$ for $x = 0.578\,l$	$\frac{0.0864\,f\,l^2}{E\,h}$	$\frac{p\,l^2\,h}{16\,I}$
	$M_x = \frac{P\,x}{2} - \frac{P\,l}{8}$ $M_{max.} = \frac{P\,l}{8}$ for $x = 0$ and $x = \frac{l}{2}$	$y = \frac{P}{8\,E\,I}\left(\frac{2\,x^3}{3} - \frac{l\,x^2}{2}\right)$ $\Delta = \frac{P\,l^3}{192\,E\,I}$	$\frac{f\,l^2}{12\,E\,h}$	$\frac{P\,l\,h}{16\,I}$
	$M_x = \frac{p}{2}(l\,x - x^2) - \frac{p\,l^2}{12}$ $M_{max.} = \frac{p\,l^2}{12}$ for $x = 0$	$y = \frac{p}{12\,E\,I}\left(l\,x^3 - \frac{x^4}{2} - \frac{l^2\,x^2}{2}\right)$ $\Delta = \frac{p\,l^4}{384\,E\,I}$	$\frac{f\,l^2}{8\,E\,h}$ (f at centre)	$\frac{p\,l^2\,h}{24\,I}$ (f at end)

also important that the influence of this factor be recognized in dealing with trusses by means of the formulas developed for solid beams.



Element at C
(b)

Cross Section
of Element at C
(c)

FIG. 12.

The deflection of any point C of the beam of Fig. 12 (*a*) due to shear is equal to the sum of the deflection δ of Fig. 12 (*b*) for all beam elements from A to C, Fig. (*a*), or between the limits x and 0. Then, if y_s = deflection at C due to shear, $y_s = \int_0^x \delta$.

To determine δ, the internal work due to shearing strains may be equated to the external work done by the shear V during the deflection δ. From Mechanics, the internal work of the shearing strain is $\frac{1}{2}\frac{v^2}{E_s}$ per unit of volume, where v = unit shearing stress and E_s = shearing modulus. If dA = element of area of section on which shearing stress = v, and dx = length of a beam element, Fig. (*b*), we have

$$\text{Internal Work} = \int_{h_2}^{h_1} \frac{1}{2}\frac{v^2}{E_s}\,dA\,dx.$$

If V = external shear on each face of the beam element of Fig. (*b*) and the deflection across the element is δ

$$\text{External Work} = \frac{1}{2}V\delta.$$

Equating these values of internal and external work and solving for δ, we have

$$\delta = \int_{h_2}^{h_1} \frac{v^2}{VE_s}\, dA\, dx.$$

Therefore

$$y_s = \int_0^x \delta = \int_0^x \int_{h_2}^{h_1} \frac{v^2}{VE_s}\, dA\, dx \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Two cases of shearing stress variation will be considered: (*a*) shearing stress uniform across the section, (*b*) shearing stress variable across the section. The first case includes plate girder and I-beam sections, for, as shown in Chapter VII, Part III, the shearing stresses in these sections are practically uniform across the section. The second case covers rectangular and circular sections.

(*a*) *Uniform Shearing Stress.*—In this case $v = V/A$, where A = area of the section. Also $\int_{h_2}^{h_1} dA = A$. Then

$$y_s = \frac{1}{A\,E_s} \int_0^x V\, dx. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

From eq. (4), Art. 1, $\int_0^x V\, dx = M_C$, where M_C = moment at C, Fig. 12 (*a*) due to the applied loads, hence,

$$y_s = \frac{M_C}{A\,E_s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

Therefore the shearing deflection at any point, with respect to the point where $M = 0$, is equal to $1/A\,E_s$ times the bending moment at that point due to the applied loading. This relation offers a very simple method for the calculation of shearing deflection in simple beams.

(*b*) *Variable Shearing Stress.*—From Mechanics, the shearing stress v at any plane distance y from the neutral axis of the beam section, Fig. 12 (*c*), is

$$v = \frac{V\,m}{I\,b}.$$

in which V = external shear, m = statical moment, about the neutral axis, of the portion of the section outside the shear plane, b = width of section at shear plane, and I = moment of inertia of section. Then

$$v = \frac{V}{I\,b}\int_y^{h_1} z\,c\,d\,z$$

and from eq. (18), noting that $d\,A = b\,d\,y$

$$y_s = \int_0^x \int_{h_2}^{h_1} \frac{1}{E_s\,V}\left[\frac{V}{I\,b}\int_y^{h_1} z\,c\,d\,z\right]^2 b\,d\,y\,d\,x$$

For a beam of uniform cross-section this may be written in the form

$$y_s = \frac{1}{E_s}\left[\int_0 V\,d\,x\right]\left[\frac{1}{I^2}\int_{h_2}^{h_1}\frac{1}{b}\left(\int_y^{h_1} c\,z\,d\,z\right)^2 d\,y\right]. \quad . \quad . \quad . \qquad (21)$$

Eq. (21) is divided into two parts. The first part $\int_0^x V\,d\,x$, is a function of the loading conditions which, as given above, is equal to M_c, the moment at C due to the applied loads. The second part of eq. (21) is a function of the cross section of the member. Let

$$N = \frac{1}{I^2}\int_{h_2}^{h_1}\frac{1}{b}\left(\int_y^{h_1} c\,z\,d\,z\right)^2 d\,y. \quad . \quad . \quad . \quad . \qquad (22)$$

Equation (21) may then be written

$$y_s = \frac{M_c}{E_s}N. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

As soon as the form of the cross section of the member is known, the value of N, eq. (23), may be determined. Thus for a rectangle of width b and depth h, we have $c = b$, $h_1 = + h/2$, $h_2 = - h/2$, $A = b\,h$, and $I = b\,h^3/12$. Then

$$N = \frac{144}{b^2 h^6} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{1}{b} \left(\int_y^{\frac{h}{2}} b\,z\,d\,z \right)^2 d\,y = \frac{1.2}{A}.$$

Thus the shearing deflection for a rectangular section under variable shearing stress is 20 per cent greater than for the same section under uniform shearing stress. For a circular section, $N = 10/9\,A$.

For restrained and continuous beams, where the end moments are not zero, the value of y_s of eq. (20) or (23) is the deflection of any point with reference to a straight line joining the points of inflection on either side of the given point. At the supports this deflection is upwards and to get the deflection of the given point with respect to the supports due allowance for this condition must be made. The resulting total deflection will be found to be the same as in a simple beam.

The following examples illustrate the application of the methods given above to the calculation of shearing deflection for a few typical cases.

1. Determine the deflection at point D, Fig. 5, due to shearing stress. Assume $E_s = 12{,}000{,}000$ and $A = 9.0$ sq. in. The shearing stress is assumed to be uniformly distributed across the beam section.

From Fig. 5 (*b*), $M_D = 250 \times 134 = 33{,}500$ in. lb. Then from eq. (20),

$$y_D = \frac{M_D}{A\,E_s} = \frac{33{,}500}{9 \times 12{,}000{,}000} = 0.0003102 \text{ in.}$$

Note that the deflection due to shear is 0.45 per cent of the deflection due to moment as calculated in Example 4, Art. 5.

2. Determine the deflection due to shear in the cantilever beam loaded as shown in Fig. 6. $M_A = P\,l$ and

$$y_A = \frac{P\,l}{A\,E_s}.$$

3. Determine the maximum deflection due to shear in a simple beam carrying a concentrated load at the center.

Since the bending moment is a maximum at the beam center, the shearing deflection will also have its maximum value at this same point. Then

$$M_c = \frac{Pl}{4}.$$

and

$$y_c = \frac{Pl}{4AE_s}.$$

4. Determine the maximum deflection due to shear in a simple beam carrying a uniform load.

$$M_c = \frac{pl^2}{8}.$$

and

$$y_c = \frac{pl^2}{8AE_s}.$$

5. Determine the shear deflection for point C of Fig. 11. Fig. (*d*) is the moment diagram, hence $M_C N/E_s$ is the downward deflection of C with respect to a line joining the points of inflection, and $M_A\,N/E_s$ and $M_B\,N/E_s$ are the upward deflections at the supports. The downward deflection of C with respect to the supports is therefore

$$y_s = [M_C + (M_A - M_B)k]N/E_s.$$

This will be found equal to $M'N/E_s$, where M' = moment at C for a simple beam.

The deflection curve due to shear is a straight line where V is constant and a parabolic curve where V varies uniformly.

The relative effects of shear and moment may be illustrated by comparing the results for the two cases of a single concentrated load, and a uniformly distributed load. If y_m = deflection due to moment and y_s = deflection due to shear we have:

For a concentrated load at the centre

$$\frac{y_s}{y_m} = \frac{12\,EI}{E_s A l^2}.$$

For a uniform load

$$\frac{y_s}{y_m} = \frac{48\,EI}{5\,E_s A l^2}.$$

For steel E_s is approximately 0.4 E, and $I/A = r^2$. Hence for the two cases $y_s/y_m = 30\frac{r^2}{l^2}$ and $24\frac{r^2}{l^2}$ respectively. For plate girders r = one-half the depth, and for a ratio of depth to span of 1:10 the values of y_s/y_m become equal to 7.5% and 6% respectively; that is, the deflection due to shear is in these cases 7.5% and 6% of that due to moment. In deeper beams the ratio is larger, and in ordinary trusses it reaches a proportion too large to be neglected.

SECTION II.—CONTINUOUS GIRDERS

9. Definition.—A continuous girder is a beam or truss which is supported at more than two points. In such a case the reactions depend upon the form of the structure as well as upon the loads and cannot be determined from the principles of statics alone; it is necessary to take into account the dimensions and material of the structure itself. All supports in excess of two are redundant, in the sense employed in Chap. VII of Part I, and methods of analysis must be applied similar to those used in that chapter.

In the analysis and design of continuous girders it is necessary to assume certain proportions or dimensions before the stresses can be determined. If the girder is to be made of uniform section the problem is relatively simple, but if it is to vary from point to point an exact solution is very laborious and generally not fully accomplished. In such cases an approximate solution is made to suffice. In the following analysis, certain formulas will first be derived for beams of solid section; the application of these formulas to trusses will be considered subsequently.

10. Shear and Bending Moment in Any Span.—Let $A\ B\ C\ D$, Fig. 13, be a continuous girder of any number of spans and loaded in any manner. Consider any span, as $A\ B$, passing sections p and q, close to the reactions. Fig. 14 shows this span separated and all forces indicated. M_1 and M_2 are the bending moments at the two supports, V_1 the shear on the right of A and V_2' the shear on the left of B. P is any load distant $k\ l$ from A. Taking moments about A we have

$$V_2' = \frac{M_1 - M_2 + P\,k\,l}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Taking moments about B we have

$$V_1 = \frac{M_2 - M_1 + P\,(l - k\,l)}{l}. \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Again, the bending moment at any point N distant x from A is

$$\left.\begin{array}{ll} \text{for } x < k\,l, & M_x = M_1 + V_1\,x \\ \text{for } x > k\,l, & M_x = M_1 + V_1\,x - P\,(x - k\,l) \end{array}\right\} \quad . \quad . \quad . \qquad (3)$$

and the shear at N is

$$\left.\begin{array}{ll} \text{for } x < k\,l, & V_x = + V_1 \\ \text{for } x > k\,l, & V_x = + V_1' + P \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Equations (1) to (4) show that the bending moments and shears at any section of a continuous girder are easily found when the moments at the supports are known. Similar formulas can readily be written out for uniform loads.

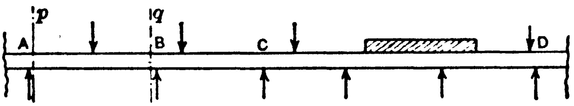


FIG. 13.

11. The Theorem of Three Moments.—Referring again to Fig. 13, which represents a continuous girder of any number of spans and loaded in any manner (loads vertical), let us consider any two consecutive spans as A–B–C. The theorem of "three moments," which will now be derived, is an equation expressing the relation between the bending moments at the three supports, A, B and C, and the loads on the two included spans, a relation which is independent of the loads on

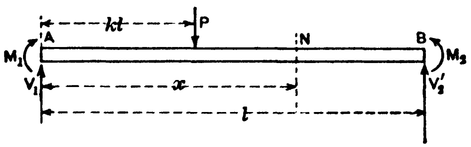


FIG. 14.

any other part of the structure. This theorem will be derived, first, for concentrated loads, and secondly, for uniform loads.

12. *Concentrated Loads.*—Fig. 15 shows the two spans in question, separated by sections passed close to the supports at A and C. The supports are numbered 1, 2, and 3 and the span lengths are l_1 and l_2.

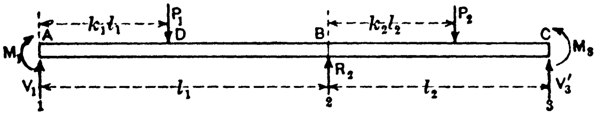


FIG. 15.

The centre reaction is R_2 and the end moments are M_1 and M_3. These are shown as positive in order that the signs will be correctly given in applying any resulting formulas.

Consider first a single load P on each span. Let $k_1 l_1$ = distance of P_1 from A, and $k_2 l_2$ = distance of P_2 from B, the coefficient k representing the ratio of $k\,l/l$ in general for any load. Fig. 16 represents the curved form of the elastic line, and $A'C'$ the tangent at B. The deflections at A and C with respect to the tangent are y_1 and y_3.

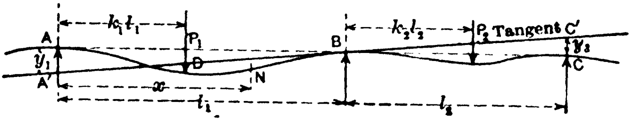


FIG. 16.

From eq. (5), Art. 3, $y_1 = \int_A^B \frac{M\,x\,d\,x}{E\,I}$, in which M is the bending moment at any point N. Substituting the value of M from eq. (3), integrating from A to D and from D to B, and finally replacing V_1 by its value given in eq. (2), we derive

$$y_1 = \frac{M_1 l_1^2 + 2\,M_2 l_1^2 + P_1 l_1^3\,(k - k^3)}{6\,E\,I}. \quad . \quad . \quad . \quad (5)$$

In a similar manner we find for span $B\,C$, origin at C,

$$y_3 = \frac{M_3 l_2^2 + 2\,M_2 l_2^2 + P_2 l_2^3\,(2\,k - 3\,k^2 + k^3)}{6\,E\,I}. \quad . \quad . \quad (6)$$

If the supports at A, B and C remain at a fixed level then, with due attention paid to sign, we have $\frac{y_1}{l_1} = -\frac{y_3}{l_2}$, whence we derive from (5) and (6) the relation

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -P_1l_1^2(k - k^3) - P_2l_2^2(2k - 3k^2 + k^3) \quad . \quad (7)$$

If there are several loads on each span a similar equation may be written out for each load and, by addition, we have the general *equation of three moments* for any number of concentrated loads.

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -\Sigma P_1l_1^2(k - k^3) - \Sigma P_2l_2^2(2k - 3k^2 + k^3) \quad . \quad (8)$$

13. **Uniform Loads.**—If the spans are loaded with a uniformly distributed load of p_1 and p_2 per unit length, respectively, then in eq. (8) P may, in general, be replaced by $p\,d\,x$ and k by $\frac{x}{l}$. Then integrating each of the terms in the second member from 0 to l we have, for *uniform loads*

$$M_1\,l_1 + 2\,M_2\,(l_1 + l_2) + M_3\,l_2 = -\tfrac{1}{4}\,p_1\,l_1^3 - \tfrac{1}{4}\,p_2\,l_2^3. \quad . \quad (9)$$

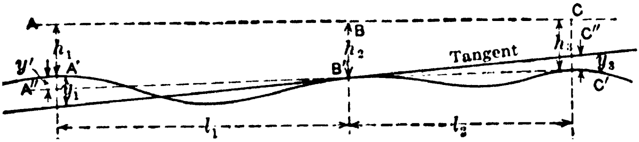


FIG. 17.

14. **Effect of Movement of Supports.**—In Fig. 16 the elevation of the supports was assumed as fixed. If, however, their relative elevation changes, due to unequal settlement or imperfect adjustment, the values y_1 and y_3 will be modified accordingly. In Fig. 17 let A, B and C represent the initial position of the supports. Suppose that the support at A settles a distance h_1, the support at B a distance h_2 and that at C a distance h_3. Unless these movements are equal or proportional the beam will be bent out of its original form by such settlement. This effect

may be measured by the ordinate y' at A', from the support A' to the straight line $A'' B' C'$ drawn through B' and C'. By similar triangles

$$\frac{y' + h_1 - h_2}{l_1} = \frac{h_2 - h_3}{l_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

whence

$$y' = (h_2 - h_3)\frac{l_1}{l_2} + h_2 - h_1. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Then, as in Art. 12, $\frac{y_1 - y'}{l_1} = -\frac{y_3}{l_2}$. Substituting the values of y_1 and y_3 from (5) and (6) we derive, finally, for any number of loads

$$\begin{aligned} &M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 = \\ &- \Sigma P_1 l_1^2 (k - k^3) - \Sigma P_2 l_2^2 (2k - 3k^2 + k^3) \\ &- 6EI\left(\frac{h_1 - h_2}{l_1} + \frac{h_3 - h_2}{l_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12) \end{aligned}$$

as the equation of three moments which takes account of a settlement of supports.

Generally we speak of the supports as being *level* or *out of level*, but if the supports are fixed and the beam is built to conform to their elevation, even though they be at different levels, there will be no stress resulting from such variation. It is only a variation in level from the original or normal position that is significant. The fact that such a variation or settlement of supports causes large stresses in a continuous girder constitutes a serious objection to its use in practice.

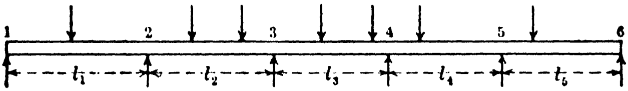


FIG. 18.

15. Calculation of Moments at Supports.—The theorem of three moments enables the bending moments due to any given loading to be calculated in any continuous girder of any number of spans. This is accomplished by the successive application of the theorem to each group of two consecutive spans, this process furnishing as many equations as there are unknown moments. Thus, for a girder of five spans, Fig. 18, the theorem written out for spans 1 and 2, 2 and 3, 3 and 4, 4 and 5, gives four equations. The moments at 1 and 6 being zero, there are but four unknown moments, which are readily found by solving these four equations. The moments at the supports being determined the moments and shears at intermediate sections are derived as in Art. 10.

In the case of fixed ends, the theorem may still be applied by considering the fixed end as equivalent to an additional span of length zero. Thus let A B, Fig. 19, be a beam fixed at the ends and supporting a single load P. Fig. 20 shows this as a three-span continuous girder, the values of l_1 and l_3 being indefinitely small. Applying eq. (8) we have, for spans l_1 and l_2 (noting that $M_1 = 0$, and $l_1 = 0$),

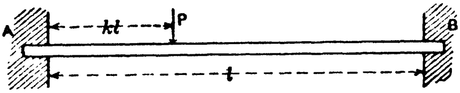


FIG. 19.

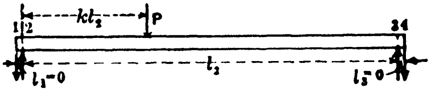


FIG. 20.

$$2\,M_2\,l_2 + M_3\,l_2 = -\,P\,l_2^2\,(2\,k - 3\,k^2 + k^3). \quad . \quad . \quad (a)$$

And applying to spans 2 and 3 (noting that $M_4 = 0$ and $l_3 = 0$),

$$M_2\,l_2 + 2\,M_3\,l_2 = -\,P\,l_2^2\,(k - k^3). \quad . \quad . \quad . \quad (b)$$

Solving (a) and (b) for M_2 and M_3 we have

$$\left.\begin{aligned} M_2 &= -\,P\,l\,(k - 2\,k^2 + k^3) \\ \text{and } M_3 &= -\,P\,l\,(k^2 - k^3) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

These results are the same as given in eq. (10), Art. 6, the value of M' of that article being equal to $P\,l\,(k - k^2)$.

16. General Formulas for Moments at Supports.—The result of the application of the theorem of three moments in the manner explained above, to a girder of any number of spans, may be expressed in a general formula for the moment at any given support. Such a formula will be here given for both uniform and concentrated loads, spans all equal.

Let n = whole number of spans;
m = number of support in question, counting from the left;
r = number of any span in which a load occurs;
c = a coefficient, tabulated below.

(*a*) *Uniform Load on Any Span.*—For the case where one or more spans are uniformly loaded with a load p per unit length, the bending moment at the mth support is

$$M_m = -\frac{pl^2}{4(c_{n-1}+4c_n)}[\underbrace{\Sigma(c_r+c_{r+1})c_{n-m+2}}_{\text{(For loaded spans on left.)}} + \underbrace{\Sigma(c_{n-r+2}+c_{n-r+1})c_m}_{\text{(For loaded spans on right.)}}]. \quad (14)$$

The following values are to be used for the c coefficients:

$$\begin{array}{llllll}
c_1 = & 0 & c_5 = - & 56 & c_9 = - & 10{,}864 \\
c_2 = + & 1 & c_6 = + & 209 & c_{10} = + & 40{,}545 \\
c_3 = - & 4 & c_7 = - & 780 & c_{11} = - & 151{,}316 \\
c_4 = + & 15 & c_8 = + & 2{,}911 & c_{12} = + & 564{,}719
\end{array}$$

following the law that $c_m = -4c_{m-1} - c_{m-2}$.

As an example, let it be required to find the value of M_2 for a girder of four equal spans, loaded as shown in Fig. 21. Here $m = 2$, $n = 4$. For

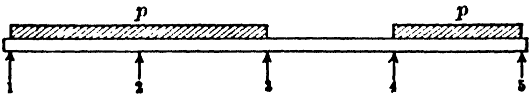


FIG. 21.

loads on the left of the 2nd support, $r = 1$, and for loads on the right, $r = 2$ and 4. Eq. (14) then becomes

$$M_2 = -\frac{pl^2}{4(c_3+4c_4)}[(c_1+c_2)c_4 + (c_4+c_3)c_2 + (c_2+c_1)c_2].$$

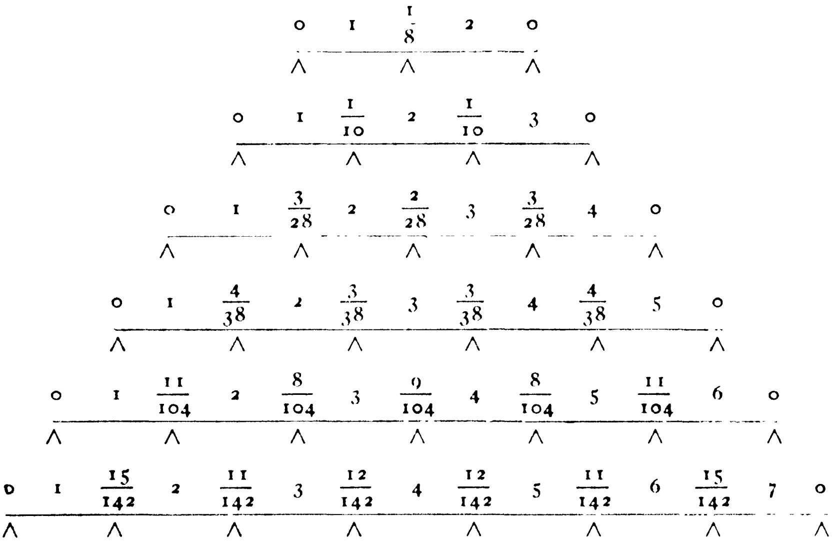
Substituting the values of c we have $M_2 = -\frac{27}{224}pl^2$.

(*b*) ***Uniform Load on All Spans.***—For this case eq. (14) reduces to

$$M_m = \left[\frac{c_{n+1}(c_{m+1} - 1) - c_m(c_{n+2} - 1)}{c_{n+1}}\right]\frac{pl^2}{12}. \quad . \quad (15)$$

This formula is evaluated in the following diagram for all supports for girders of seven spans or less.

MOMENTS AT SUPPORTS; TOTAL UNIFORM LOAD; SPANS ALL EQUAL.
COEFFICIENTS OF $(-pl^2)$.



(*c*) ***Concentrated Loads on any Span.***—For the case where one or more spans are loaded with concentrated loads,

$$M_m = -\frac{l}{c_{n-1} + 4c_n}\Big[\Sigma\left\{\Sigma P(2k - 3k^2 + k^3)c_r + \Sigma P(k - k^3)c_{r+1}\right\}c_{n-m+2}$$

(For loaded spans on left of *m*th support.)

$$+ \Sigma\left\{\Sigma P(2k - 3k^2 + k^3)c_{n-r+2} + \Sigma P(k - k^3)c_{n-r+1}\right\}c_m\Big]. \quad . \quad (16)$$

(For loaded spans on right of *m*th support.)

If both uniform and concentrated loads are found upon the same span, then both (14) and (16) must be used. When more than one span is loaded, the data must be worked out for each, and the sum taken, as indicated by the primary signs of summation. The secondary summation signs for concentrated loads signify the summation for the several concentrated loads in any one span.

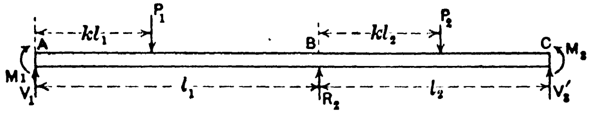


FIG. 22.

17. General Formula for Reactions.—It is convenient to find the reactions directly from the moments at the supports. Fig. 22 represents any two consecutive spans. Let R_2 = reaction at the middle support. In Fig. 23, $R_2 = V_2' + V_2$. By eq. (1), $V_2' = \frac{M_1 - M_2}{l_1} + P_1 k$, and by eq. (2), $V_2 = \frac{M_3 - M_2}{l_2} + P_2(1 - k)$, whence

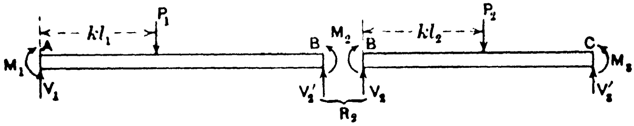


FIG. 23.

$$R_2 = \frac{M_1 - M_2}{l_1} + \frac{M_3 - M_2}{l_2} + P_1 k + P_2 (1 - k). \quad (17)$$

Note that the last two terms of (17) give the total reaction at B for two simple spans. For uniform loads these terms become $\frac{p_1 l_1}{2}$ and $\frac{p_2 l_2}{2}$.

The following gives the shears on each side of the supports for the case of uniform load over the entire girder. The supporting force is the sum of the two shears at that support.

SHEARS AT SUPPORTS; TOTAL UNIFORM LOAD; SPANS ALL EQUAL
COEFFICIENTS OF (pl).

1

$$\frac{0 \mid 1}{2} \qquad \frac{1 \mid 0}{2}$$

1 2

$$\frac{0 \mid 3}{8} \qquad \frac{5 \mid 5}{8} \qquad \frac{3 \mid 0}{8}$$

1 2 3

$$\frac{0 \mid 4}{10} \qquad \frac{6 \mid 5}{10} \qquad \frac{5 \mid 6}{10} \qquad \frac{4 \mid 0}{10}$$

1 2 3 4

$$\frac{0 \mid 11}{28} \qquad \frac{17 \mid 15}{28} \qquad \frac{13 \mid 13}{28} \qquad \frac{15 \mid 17}{28} \qquad \frac{11 \mid 0}{28}$$

1 2 3 4 5

$$\frac{0 \mid 15}{38} \qquad \frac{23 \mid 20}{38} \qquad \frac{18 \mid 19}{38} \qquad \frac{19 \mid 18}{38} \qquad \frac{20 \mid 23}{38} \qquad \frac{15 \mid 0}{38}$$

1 2 3 4 5 6

$$\frac{0 \mid 41}{104} \qquad \frac{63 \mid 55}{104} \qquad \frac{49 \mid 51}{104} \qquad \frac{53 \mid 53}{104} \qquad \frac{51 \mid 49}{104} \qquad \frac{55 \mid 63}{104} \qquad \frac{41 \mid 0}{104}$$

1 2 3 4 5 6 7

$$\frac{0 \mid 56}{142} \qquad \frac{86 \mid 75}{142} \qquad \frac{67 \mid 70}{142} \qquad \frac{72 \mid 71}{142} \qquad \frac{71 \mid 72}{142} \qquad \frac{70 \mid 67}{142} \qquad \frac{75 \mid 86}{142} \qquad \frac{56 \mid 0}{142}$$

18. Continuous Girder of Two Equal Spans.—*Uniform Load on Both Spans.*—(Fig. 24.) For a uniform load of p per unit length on both spans we have from eq. (9), noting that $M_1 = 0$, $M_3 = 0$ and $p_1 = p_2 = p$,

$$M_2 = - \tfrac{1}{8}\, p\, l^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Then we readily find

$$\left.\begin{aligned} R_1 &= R_3 = \tfrac{3}{8}\, p\, l, \\ R_2 &= \tfrac{5}{4}\, p\, l, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The bending moment at any distance x from A is

$$M_x = R_1 x - \frac{p x^2}{2} = \frac{p}{8} (3\, l\, x - 4\, x^2). \quad . \quad . \quad . \quad (20)$$

The shear is,

$$V_x = R_1 - p\, x = \frac{p}{8} (3\, l - 8\, x). \quad . \quad . \quad . \quad . \quad (21)$$

The variation in moment and shear is shown in Figs. 24, (*b*) and (*c*) The maximum positive moment occurs for zero shear, or for $x = \frac{3}{8} l$, and zero moment for $x = \frac{3}{4} l$.

19. *Uniform Load on One Span.*—(Fig. 25.) Applying eq. (9),

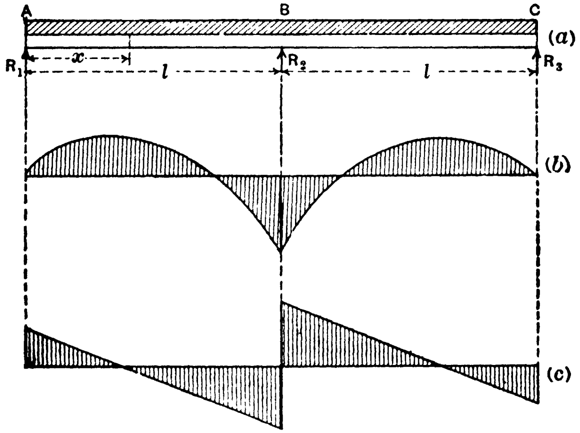


FIG. 24.

$p_2 = 0$, and the value of M_2 is one-half that given by (18), or

$$M_2 = -\frac{1}{16} p l^2. \qquad (22)$$

Then, as before,

$$\left.\begin{aligned} R_1 &= \frac{7}{16} p l;\ R_3 = -\frac{1}{16} p l, \\ R_2 &= +\frac{5}{8} p l. \end{aligned}\right\} \qquad (23)$$

The variation in moment and shear is represented in Fig. 25. Maximum moment occurs for $x = \frac{7}{16} l$, and zero moment for $x = \frac{7}{8} l$.

20. *Single Concentrated Load on One Span.—Reactions.*—(Fig. 26.) For a single load P on the first span we find, from eq. (7), noting that $M_1 = 0$, $M_3 = 0$, $P_2 = 0$, and $l_1 = l_2 = l$,

$$M_2 = -\frac{P l}{4} (k - k^3). \qquad (24)$$

and from this we derive

$$\left.\begin{aligned} R_1 &= \frac{P}{4}\,(4 - 5k + k^3), \\ R_2 &= \frac{P}{2}\,(3k - k^3), \\ R_3 &= -\frac{P}{4}\,(k - k^3). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Moments. The moments vary as shown in Fig. 26 (*b*). Between A and D, $M = R_1 x$, and is always positive.

From D to B, $M = R_1 x - P(x - kl) = P\left(kl - \frac{5}{4} kx + \frac{1}{4} k^3 x\right)$.

This value varies from positive to negative, becoming zero at E for a value of x given by the equation

$$x_0 = \frac{4l}{5 - k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In the second span the moment is negative throughout.

For larger values of k the point of inflection, E, lies farther to the right, being located always between the load and the centre support; for smaller values of k the point E lies farther to the left, the least value of x_0 of eq. (26) being 0.8 l.

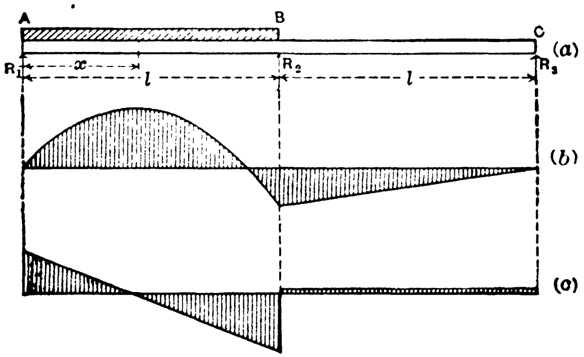


FIG. 25.

Shears. The shears are readily deduced from the value of R_1. They are represented in Fig. 26 (*c*).

21. *Influence Lines for Moments and Shears.* (*a*) *Moments.*—The position of moving uniform loads for maximum moments can readily be deduced from the preceding equations. It will be of assistance, however, to construct influence lines for two or three points. Such lines are almost indispensable if calculations are to be made for concentrated loads.

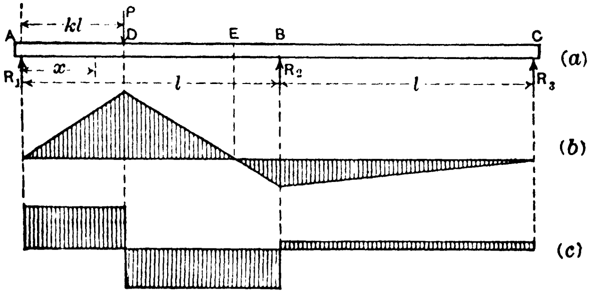


FIG. 26.

In Fig. 27 consider any section D in the first span, whose distance from A is less than 0.8 l. For a load on span $B\,C$, $M_D = R_1\,a$. The value of R_1 is given by the expression for R_3 in eq. (25), in which $k\,l$ is to be measured from C. Hence $M_D = -\frac{1}{4} P (k - k^3)\,a$. This

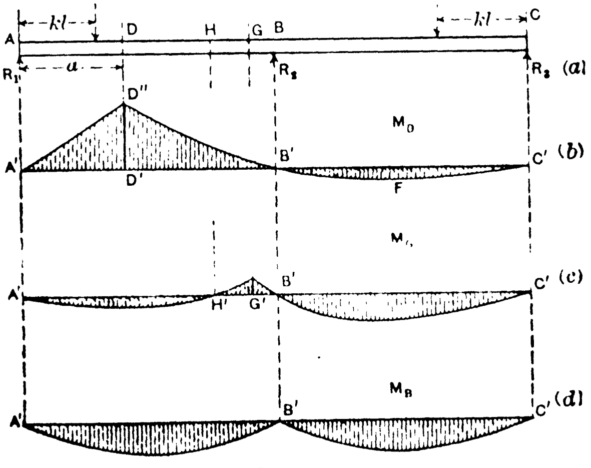


FIG. 27.

moment is represented by the curve $C' F B'$, Fig. (*b*). For a load between B and D, R_1 is directly given by eq. (25), and $M_D = R_1 a = \frac{P a}{4} (4 - 5 k + k^3)$. This moment is represented by the curve $B' D''$, a continuation of the curve $F B'$. On the left of D, $M_D = R_1 a - P (a - k l)$. Eq. (25) gives R_1. The curve $D'' A'$ represents this moment, thus giving the complete influence line $C' F B' D'' A'$. Note that since the reactions are functions of the third degree with respect to k the influence line is no longer a series of straight lines, as in the case of simple structures, but consists of arcs of curves.

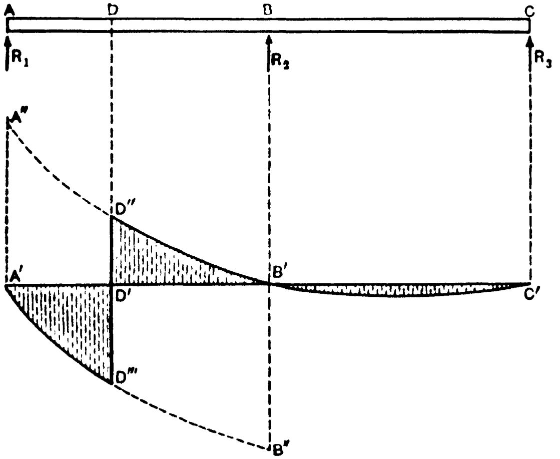


FIG. 28.

If the point D lies near B, so that $\frac{a}{l} > 0.8$, then, as shown by eq. (26) and Fig. 26, the moment at D will be zero for a certain position of the unit load. When the load is on the left of such position the moment at D will be negative and when the load is on the right it will be positive. Take a point G such that the distance $A G = 0.9 \, l$. The influence line is shown in Fig. 27 (*c*). For loads between G and A it will be found that at some point H the moment at G' will be zero, and for loads to the left it will be negative, giving the form of influence line as shown. The value of k, such that the moment at G will be zero, may be found by placing $x_0 = 0.9\,l$ in eq. (26) and solving for k. This gives $k = \sqrt{5/9} = .745$. Notice the relatively small effect of the load from A to H. The influence line for moment at B is given in Fig. 27 (*d*).

The relative values of the maximum bending moments are well shown by the areas of the influence diagrams. For uniform moving loads they give directly the correct positions and values; for concentrated loads the method of determining positions and maximum moments by trial is the most expeditious. This is illustrated in the next chapter.

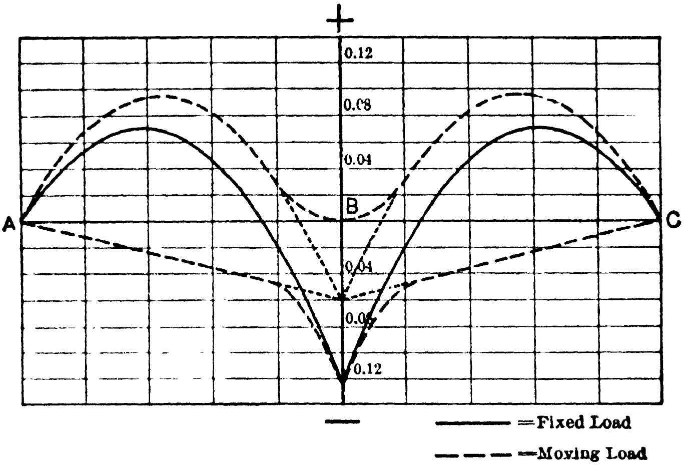


FIG. 29.—Moments in a Two-Span Continuous Girder.

(*b*) *Shears.*—Influence lines for shear are given in Fig. 28. The curve $A'' B' C'$ is the influence line for R_1, and the curve $A' D''' B''$ is drawn a unit distance below. The complete influence line for any point D is then given by the curves $A' D'''$ and $D'' B' C'$. The construction shows clearly the position of moving loads for maximum values.

22. ***Maximum Moments and Shears Due to Fixed and Moving Uniform Loads.***—Fig. 29 shows graphically the greatest positive and negative moments at all sections of a continuous girder of two equal spans, due to dead and live loads, considered as uniform loads. The numerical ordinates are the coefficients of $p\,l^2$. The dead-load curve is the same as given in Fig. 24. The positive live-load moments for all sections, excepting those that are nearer the centre than 0.8 l, are given by loading one span fully, as shown in Art. 19, and Fig. 27 (*b*). For sections near the centre a partial load gives the maximum positive moment as shown in Fig. 27 (*c*). For maximum negative moment at all sections to the left the 0.8 point, the second span only is loaded (Fig. 27 (*b*)); for sections near the centre a portion of the beam at the left end is also loaded (Fig. 27 (*c*)). For point *B* the entire span is loaded. If the partial loadings for sections near the centre are neglected, either one or both spans being fully loaded, the resulting curves will be as shown by the dotted lines in Fig. 29.

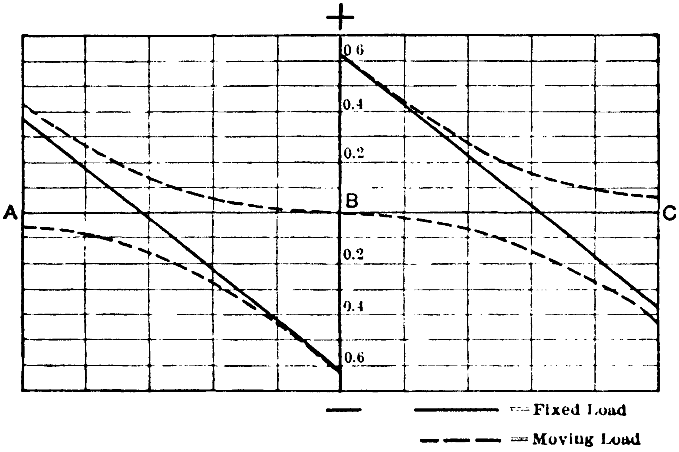


FIG. 30.—Shears in a Two-Span Continuous Girder.

Fig. 30 shows the maximum shears at all sections due to a fixed and a moving uniform load, the values being coefficients of $p\,l$.

23. Continuous Girder of Two Spans, Spans Unequal.—If the span lengths are unequal the calculations of reactions, moments, and shears will be but slightly modified. The same general position of moving loads will be required for maximum values, but the point of inflection noted in Art. 20 will be shifted somewhat.

24. Continuous Girder of Three Spans.—*Uniform Load on All Spans.*—Assume a girder having equal end spans, but a centre span of any length, Fig. 31. Let l = length of end spans and $n\,l$ = length of centre span. This type of girder is often used in swing-bridge construction and is occasionally met with elsewhere.

For a uniform load extending over the entire girder of p per unit length we have, from eq. (9), for spans 1 and 2, noting that $M_3 = M_2$,

$$2\,M_2\,(l + n\,l) + M_2\,n\,l = -\tfrac{1}{4}\,p\,l^3 - \tfrac{1}{4}\,p\,n^3\,l^3, \quad . \quad (27)$$

whence

$$M_2 = M_3 = -\frac{\tfrac{1}{4}\,p\,l^2\,(1 + n^3)}{2 + 3\,n}. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

From this is readily derived,

$$R_1 = R_4 = \frac{p\,l}{4}\left[\frac{3 + 6\,n - n^3}{2 + 3\,n}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$R_2 = R_3 = \frac{p\,l}{4}\left[\frac{5 + 10\,n + 6\,n^2 + n^3}{2 + 3\,n}\right]. \quad . \quad . \quad . \quad (30)$$

The moments and shears are found at intermediate points as explained before. Fig. 31 represents the variation in moments and shears throughout the girder.

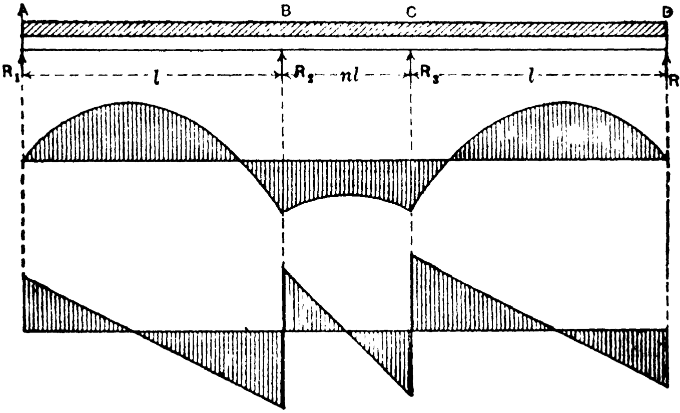


FIG. 31.

25. *Single Load on First Span.* (Fig. 32.) A load P is placed a distance $k\,l$ from A. Substituting in eq. (8) for spans 1 and 2 we have

$$2\,M_2\,(l + n\,l) + M_3\,n\,l = -P\,l^2\,(k - k^3), \quad . \quad . \quad (a)$$

and again for spans 2 and 3

$$M_2\,n\,l + 2\,M_3\,(n\,l + l) = 0. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

From these we derive

$$M_2 = -\frac{2 + 2n}{N} P l (k - k^3). \quad . \quad . \quad . \quad . \quad (31)$$

$$M_3 = \frac{n}{N} P l (k - k^3), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

in which $N = 4 + 8n + 3n^2$.

Thence the reactions are

$$\left.\begin{aligned} R_1 &= P(1 - k) - \frac{2 + 2n}{N} P (k - k^3), \\ R_2 &= P k + \frac{2 + 5n + 2n^2}{nN} P (k - k^3), \\ R_3 &= -\frac{2 + 3n + n^2}{nN} P (k - k^3), \\ R_4 &= \frac{n}{N} P (k - k^3). \end{aligned}\right\} \quad . \quad . \quad . \quad (33)$$

In a similar manner the moments and reactions for a single load on the second span may be found. The variation in moments and shears

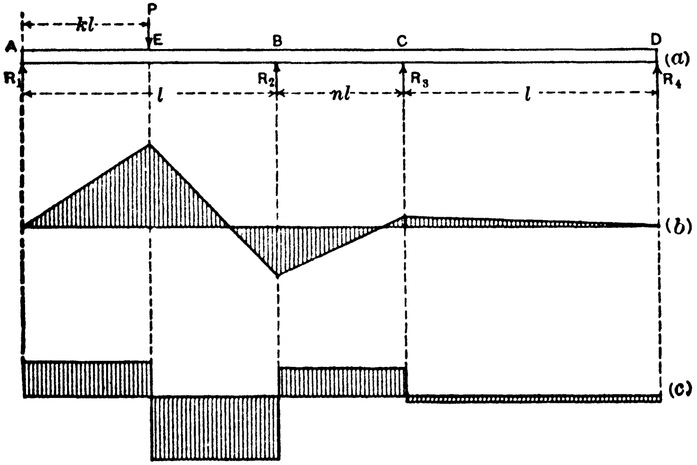


FIG. 32.

for a single load in the first span is shown in Fig. 32. It will be noted that there is a point of inflection in both the first and second spans showing, as in Art. 20, that for certain distances near the centre supports the moments may be either positive or negative, depending upon the position of the load.

26. *Influence Lines for Moments and Shears.*—In Fig. 33 are shown the influence lines for moments at the centre point E of the first span, ($k = \frac{1}{2}$), the point F ($k = 0.9$), and the centre point G of the second span. The length of span $B\,C$ is taken at one-half of $A\,B$ ($n = \frac{1}{2}$).

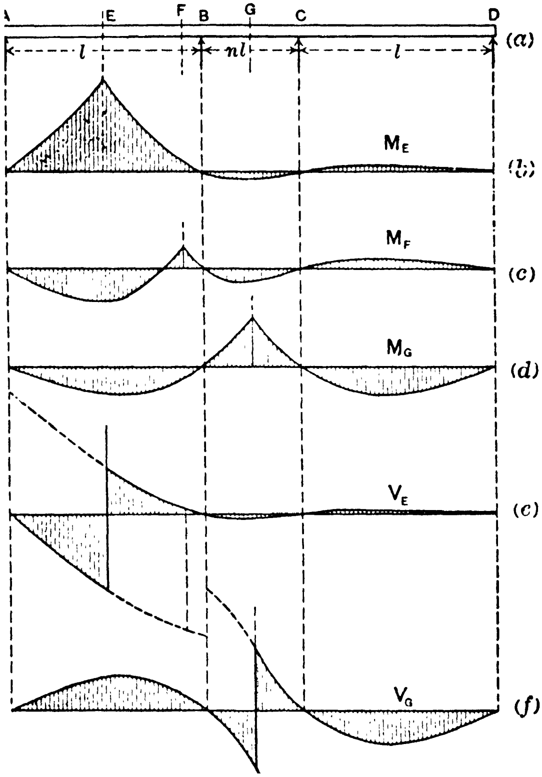


FIG. 33.

The influence lines for shears at E and G are also given. These figures show clearly the relative effect of loads on the several spans and particularly the small effect of loads in the third span upon the stresses in the first span. The influence line for M_F brings out the same conditions as to loading as shown in Fig. 27 for the two-span girder.

27. Maximum Moments and Shears in a Continuous Girder of Three Equal Spans.—Fig. 34 gives graphically the moments due to a fixed uniform load, and the maximum positive and negative moments

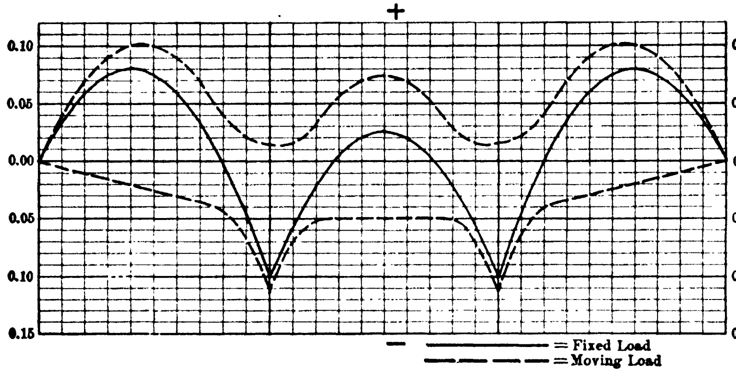


FIG. 34.—Moments in a Three-Span Continuous Girder.

due to a moving uniform load, the ordinates being the coefficients of $p\,l^2$, where p = load per unit length.

Fig. 35 gives the maximum shears for the same kind of loading. It is to be noted that the shears do not vary greatly from those in a simple span, the maximum being 0.6 $p\,l$ for fixed load and 0.62 $p\,l$ for moving load.

28. Continuous Girders of Several Spans.—It is seldom that a

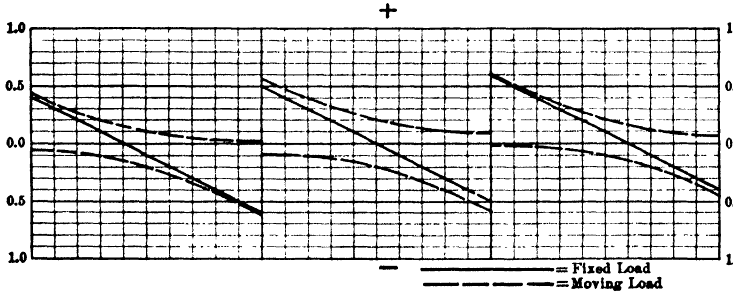


FIG. 35.—Shears in a Three-Span Continuous Girder.

continuous girder of more than three spans is employed for bridges, but in modern building construction, especially where reinforced concrete is used, continuous beams of numerous spans frequently occur.

While an exact solution of such cases is seldom necessary, it is important to know the possible maximum stresses which may be caused by moving loads. The determination of moments is of more importance in this case than that of shears, as the shears do not differ greatly from those in simple beams.

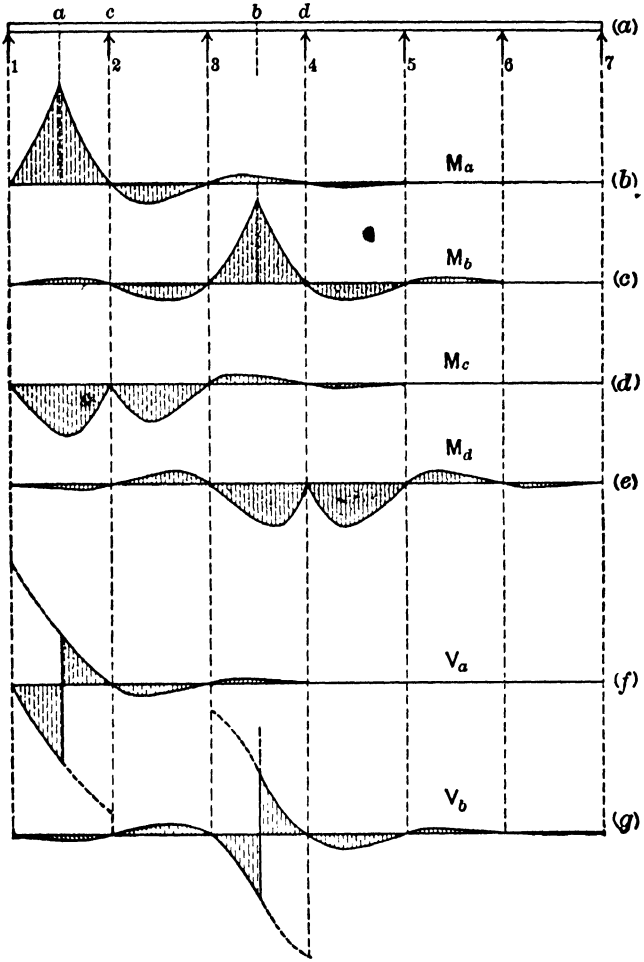


Fig. 36.

29. *Influence Lines for Moments and Shears.*—Assume a girder of six equal spans. To illustrate the effect of loads on various spans

upon the bending moments, influence lines have been drawn in Fig. 36 for moments at the centres of spans 1–2 and 3–4, and at supports 2 and 4. A maximum moment at the centre of a span requires each alternate span to be loaded, and a maximum moment at the support requires the two adjacent spans to be loaded and then each alternate span. The small effect of loads on remote spans is to be noted.

Fig. 36, (*f*) and (*g*), shows influence lines for shears in spans 1–2 and 3–4. The general rule of loads in alternate spans is seen to hold true as for moments, but the effect of remote loads upon maximum shears is relatively less than in the case of moments.

30. *Maximum Moments for Uniform Fixed and Moving Loads.*—To assist in estimating the probable maximum moments due to uniform loads, calculations have been made of the maximum moments at supports and at centres of spans, for continuous girders of from two to seven equal spans, and the results are given in the following table.

COEFFICIENTS OF $p\,l^2$, FOR MAXIMUM MOMENTS IN CONTINUOUS GIRDERS FOR UNIFORM FIXED AND MOVING LOADS.

No. of Spans.	INTERMEDIATE SPANS AND SUPPORTS.				END SPAN AND SECOND SUPPORT.			
	At Centre of Span. (+)		At Support. (−)		At Centre of Span. (+)		At Support. (−)	
	Fixed.	Moving.	Fixed.	Moving.	Fixed.	Moving.	Fixed.	Moving.
Two	......	......	......	......	.070	.095	.125	.125
Three	.025	.075	......	......	.080	.100	.100	.117
Four	.036	.081	.071	.107	.071	.098	.107	.120
Five	.046	.086	.079	.111	.072	.099	.105	.120
Six	.043	.084	.086	.116	.072	.099	.106	.120
Seven	.044	.084	.085	.114	.072	.099	.106	.120

It is found in general that for all spans, excepting the end span and the adjoining support, the maximum positive and negative moments do not vary greatly in the several spans. For the end spans and adjacent supports the maximum are considerably higher. The results are, accordingly, arranged in two groups in the table. For the intermediate spans the greatest value for the several spans is given in each case. The quantities are the coefficients of $p\,l^2$. It is to be noted that for girders of five or more spans the maximum moments are but little affected by the number of spans.

31. Continuous Girders with Variable Moment of Inertia.—If the moment of inertia varies, then in the expression for deflection, eq. (5), Art. 3, the quantity I must be left under the integral sign. It is then possible, as in Art. 11, to develop a relation between three consecutive moments and the loads in the two included spans, in which the various integrals containing I are solved by substituting in detail the values of this quantity. It is unnecessary, however, to derive such general formulas, as the cases arising in practice can more readily be solved by direct methods.

32. *Reactions for a Two-Span Continuous Girder with Variable Moment of Inertia.*—It will be convenient to derive at once the formula for one of the reactions, as R_1, for any load P on one span (Fig. 37).

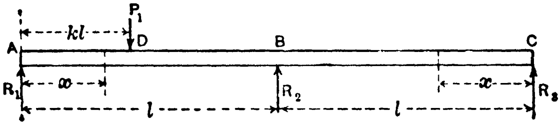


FIG. 37.

This may be done by the application of the theory of deflection or of redundant members, explained in Part I, Chapter VII. Considering the reaction R_1 as the redundant force, the conditions require that the total effect of this reaction, and of all other forces (P, R_2, and R_3), must be such as to cause zero deflection at A.

If M = bending moment at any section due to all the given forces, and m = bending moment due to a reaction $R_1 = 1$, then, as in Art. 222, Part I, for zero deflection at A,

$$\int_A^C \frac{M\,dx}{EI} \cdot m = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

As in the case of redundant members, the moment M may be considered in two parts: a part M' due to the given loads, with the support at A removed, and a part M'' due to R_1. Then we have $M'' = R_1 \times m$, and hence

$$\int_A^C \frac{M\,dx}{EI} \cdot m = \int_A^C \frac{M'\,m\,dx}{EI} + R_1 \int_A^C \frac{m^2\,dx}{EI} = 0. \quad . \quad (35)$$

From which (E being assumed constant),

$$R_1 = -\frac{\displaystyle\int_A^C \frac{M' m\,d x}{I}}{\displaystyle\int_A^C \frac{m^2 d x}{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

This expression is general for spans of any length and for any value of I.

To illustrate the use of eq. (36) consider the case of two equal spans. It will be necessary to perform some of the integrations indicated in three parts, namely, from A to D, D to B, and B to C. The values of M' and m for these sections are as follows:

Section	M'	m	$M' m$
A to D	0	x	0
D to B	$-P(x - kl)$	x	$-P(x^2 - klx)$
B to C (origin at C)	$-P(1 - k)x$	x	$-P(1 - k)x^2$

Therefore

$$\int_A^C \frac{M' m\,d x}{I} = -P\int_{kl}^{l} \frac{(x^2 - k\,l\,x)\,d x}{I} - P\,(1 - k)\int_0^l \frac{x^2 d x}{I},$$

and $\displaystyle\int_A^C \frac{m^2 d x}{I} = 2\int_0^l \frac{x^2 d x}{I}$; whence from (36),

$$R_1 = \frac{P}{2}\left[(1 - k) + \frac{\displaystyle\int_{kl}^{l} \frac{(x^2 - k\,l\,x)\,d x}{I}}{\displaystyle\int_0^l \frac{x^2 d x}{I}}\right]. \quad . \quad . \quad . \qquad (37)$$

For constant I, eq. (37) reduces to the value given in eq. (25). Note that in eq. (36) the numerator is the deflection of A for load P, the support being removed, and the denominator is the deflection of this point for a load unity acting at A. In applying eq. (37) the values of I would be taken as constant for short lengths along the girder and the integration performed in parts (or as a summation).

For a girder of three spans the redundant reactions are two in number and the two desired equations are obtained by placing equal to zero the deflections at two supports and expressing all moments M' in terms of the loads, with the two redundant reactions being removed, exactly as in the solution of two redundant members. For an example, see Art. 77.

33. EXAMPLE.—*Plate girder with variable Moment of Inertia.* Assume the following data: two equal spans of 60 ft.; dead load = 1,200 lbs. per ft. per girder; live load, including impact, =9,600 lbs. per ft. per girder. The moments will first be calculated on the assumption of uniform moment of inertia, using the result of Art. 22. The maximum dead- and live-load moments are given in Fig. 38, and the total maximum, irrespective of sign, by the upper full line

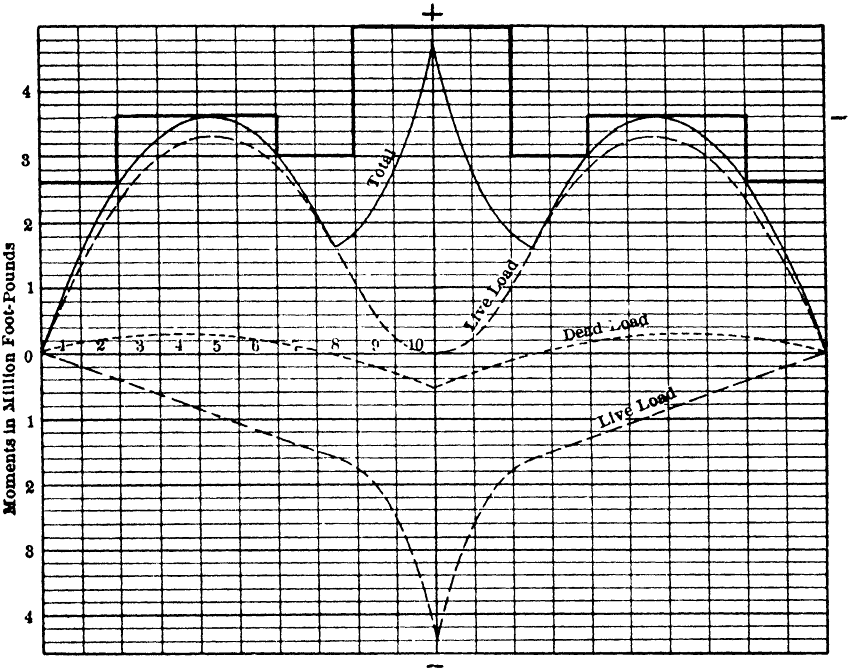


FIG. 38.

Suppose, now, the moments of inertia of the girder be made to fit this maximum curve in the manner shown by the heavy stepped line, varying the flanges in four sections.

We will now proceed to calculate the true reactions for a uniform load of one pound per foot over the entire structure. For this purpose each span will be divided into ten sections as numbered, and the integrations of eq. (36) performed as summations. Let R_2 be assumed as the redundant force. Then M' of eq. (36) will be the bending moment at any section in the 120-ft. girder under the load of one pound per foot and with R_2 removed; m will be the moment due to a one-pound load at the centre of the 120-ft. span. For the purpose of this calculation the values of I need be known only relatively The I for sections 1 and 2 will therefore be called unity, that for sections 3–6 will be 1.38, for 7 and 8, 1.15, and for 9 and 10, 1.92. The value of dx = 6 ft The complete calculations are as follows:

Section.	I	$\frac{dx}{I}$	M'	m	$\frac{M' m\,dx}{I}$	$\frac{m^2\,dx}{I}$
1	1.0	6.0	175	1.5	1,600	13
2	1.0	6.0	499	4.5	13,500	121
3	1.38	4.33	787	7.5	25,600	244
4	1.38	4.33	1,039	10.5	47,300	478
5	1.38	4.33	1,255	13.5	73,500	789
6	1.38	4.33	1,435	16.5	102,700	1,179
7	1.15	5.20	1,580	19.5	160,200	1,977
8	1.15	5.20	1,687	22.5	197,400	2,632
9	1.92	3.12	1,760	25.5	140,000	2,030
10	1.92	3.12	1,795	28.5	156,200	2,530
				Σ	918,000	11,993

$$R_2 = \frac{918,000}{11,993} = 76.4;\ R_1 = 60 - 76.4/2 = 21.8.$$

The value of R_2 for uniform moment of inertia is $5/4 \times 60 = 75.0$. The true bending moment at the centre for unit load $= 21.8 \times 60 - 60 \times 30 = -492$, whereas, for uniform I the value of $M_2 = -\frac{1}{8} P l^2 = -440$, a difference of about 10%. The maximum positive moment for $x = \frac{3}{8} l$ is $21.8 \times 22.5 - \frac{22.5^2}{2} = 237$, while for uniform I this moment $= 253$, a difference of about 2%.

The effect of a settlement of support will be also determined. If the girder is 8 ft. deep its moment of inertia will be about 200,000 in^4. The centre deflection for a concentrated load of 1,000 lbs. is given by the formula $\frac{1}{48}\frac{P l^3}{E I}$ and amounts to .00104 in. A settlement of the centre support of .001 in. would therefore decrease the centre reaction by 1,000 lbs.; a settlement of 0.4 in. would decrease it by 400,000 lbs., which is one-half the total dead and live load reaction.

34. **The Analysis of Continuous Trusses.**—The methods of analysis of this chapter have been based on the theory of the deflection of solid beams, in which the effect of shearing distortion is so small compared with that from flexural stresses as to be negligible. In the analysis of trusses the same formulas are generally applied, but in this case the errors due to shearing strains (web strains) are much greater and in many cases are too large to be neglected. Then, again, if the formulas for constant I be employed still further errors are introduced in case the moment of inertia is made variable. The usual formulas must there-

fore be considered as only roughly approximate and suitable only for a preliminary design. Such a design being made, an exact solution may be worked out by the principle of deflections, in a manner similar to that explained in Art. 32. The analysis of two- and three-span trusses is considered in detail in the next chapter.

35. The Moment of Inertia of a Truss. In applying to a truss various formulas derived from solid beams it becomes necessary to determine the value of the moment of inertia, or its equivalent, for the truss. This equivalent is found by considering the chord members only, as the bending moments concerned are fully resisted by the chord members. The web members offer no aid in resisting these moments, and hence they do not strengthen the truss against bending moment as here considered. In getting the moment of inertia for purposes of calculating reactions, etc., the gross-sections of the chords should be taken, as the stiffness of the structure is under consideration and this depends on gross rather than net sections. Due allowance should be made for long splice plates. If the two chords differ much in cross-section the centre of gravity of the two opposite segments may be found and the correct value of I determined with respect to this centre of gravity. The effect of web members is not to make the truss more rigid by adding to its moment of inertia, but less rigid by reason of the distortion due to shear or web stresses as noted in Art. 8. Variable moment of inertia can be taken account of by using corresponding formulas developed from the solid beam, but web distortion can be taken account of only by the method of deflection as explained later (Art. 63).

36. Use of Continuous Girders.—Continuous girders are now very rarely built excepting in the form of swing bridges, a type fully discussed in the next chapter. Compared with a series of simple spans a continuous girder shows some saving in material, as the average bending moment is less. The shears are, however, about the same. Thus, in the two-span girder, Fig. 29, the total positive moment area for one span, for fixed load, is $\int_0^{\frac{3}{4}l}(R_1 x - \frac{1}{2} p x^2)\, d x = \frac{9}{256} p l^3$, and the negative moment area $= \int_{\frac{3}{4}l}^{l} (R_1 x - \frac{1}{2} p x^2)\, d x = \frac{11}{768} p l^3$. The total area, neglecting sign, $= \frac{19}{384} p l^3$. For a simple span the moment area $= \frac{1}{12} p l^3$.

The total moment area for the continuous girder is therefore only 60% of that of the simple span. This represents approximately the relative chord sections required for a fixed load.

For moving loads the average moments are nearly as great in the continuous girder as in the simple span. In the two-span girder the average maximum live load moment = .070 $p\,l^2$ while for the simple span it is .083 $p\,l^2$. Considering the reversal of stress which occurs in the continuous girder and which calls for additional material, the saving is very little or nothing.

For the three-span girder the saving will be somewhat less in the end spans, but more in the centre span, than in the case of the two-span girder.

It will be seen from the foregoing that the economy of the continuous girder may well be considered where the fixed loads are relatively large. The continuous girder possesses also an advantage in the fact that it may be erected conveniently without the use of false work by building out from the portion already in place. This form of construction has, however, serious disadvantages which generally outweigh its advantages. Chief of these is the effect upon the stresses of a slight settlement of supports, or a variation of temperature among the different members of the structure. Convenience of erection, as well as economy of material in certain cases, is substantially secured by the cantilever bridge (Chapter III), which is constructed as a continuous structure, but is afterward modified so that hinged joints are introduced at suitable places, thus breaking up the continuous structure into a series of simple trusses. The stresses in the structure are then no longer affected by unequal settlement of supports.

CHAPTER II

SWING BRIDGES

SECTION I.—GENERAL CONSIDERATIONS

37. General Arrangement.—Swing bridges are built as continuous girders or trusses, arranged to turn on a centre pier. When closed, they are supported to a greater or less extent at the two ends, but designs differ considerably in the arrangement of details, both at the centre pier and at the end supports.

38. Arrangements at Centre Support.—With respect to the centre support there are two general arrangements, the *centre-bearing* and the *rim-bearing* structure. In the first, the entire weight of the bridge, when open, is carried by a centre pivot P (Fig. 1), the weight of the two trusses being transferred thereto by a cross-beam or beams, $c\,c'$. The centre bearing at P may consist of a pivot or a nest of conical rollers. In either case the bridge is known as a *centre-bearing* bridge. When the structure is open it is prevented from tipping on the pivot by means of a few guide or balance wheels, attached to the frame of the structure and bearing on a circular track placed on the pivot pier. These wheels are not intended to carry any considerable load and are neglected in the calculations of the stresses in the main trusses or girders. When the bridge is closed, wedges are generally inserted at c and c' in such a manner as to support the trusses or girders directly, and so relieve the pivot of most or all of the weight of the live load.

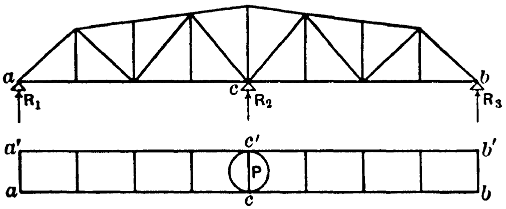


FIG. 1.

In the rim-bearing structure the trusses or girders are supported by a large circular girder or drum, which, in turn, rests and turns upon a series of closely spaced rollers moving on a circular track below. Fig. 2 illustrates this type of turn-table, and four methods of transferring the load from the trusses to the drum. In (*b*) the load is transferred to the drum at four points, while in (*c*) it is transferred at 8 points by the aid of the short beams, *e f*, *g h*, etc. In either case the supports of each truss at the centre pier are two in number, *c* and *d*, *c′* and *d′*. The panel or span *c d* is equal to the width, centre to centre of trusses, and may or may not be equal in length to the other panels of the structure. The arrangement here described is known as a ***rim-bearing*** turn-table.

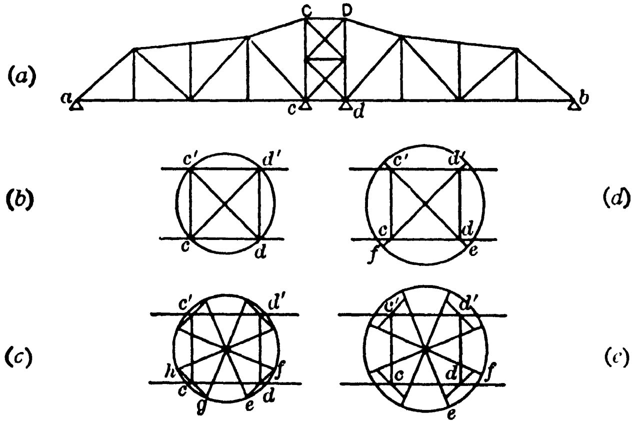


FIG. 2.

In the rim-bearing structure it is necessary to place a pivot at the centre to guide the structure when in motion and to resist the action of the wind, but in the forms shown in Figs. 2 (*b*) and (*c*) this pivot carries no vertical load. On some accounts it is advantageous to transfer a part of the load to this pivot. This may be done to any desired extent by the use of radial girders, as shown in Figs. (*d*) and (*e*), upon which the truss is supported. This arrangement is known as part rim and part centre bearing, but as regards the trusses the supports are at *c*, *d*, *c′*, and *d′*, exactly as in Figs. (*b*) and (*c*). In either case, therefore, the main girder or truss is to be considered as having four supports.

39. Arrangements of End Supports.—The arrangement of end supports must be such that when ready to turn the bridge will be free and clear from its bearings. When closed and ready for traffic the ends may (1) be left free, or (2) be latched down to prevent hammering, or (3) be raised by wedges or other mechanical device so that some of the dead load is carried by the end support. For ordinary highway bridges the first method is often used, but for railway structures, and preferably in all cases, either the second or third arrangement is employed in order to prevent hammering at the ends and to secure a better continuity of roadway. The third method is the one generally used. The various details of these arrangements are discussed in Part III, but their effect upon stress calculations is fully explained in the following articles.

40. Application of Continuous Girder Formulas in Stress Calculations.—In the analysis of swing bridges the formulas derived in the preceding chapter are, to a greater or less degree, applicable. When the structure is closed and the ends held in place, either by latching down or by raising sufficiently to prevent them from lifting, any live load coming upon the structure affects all the reactions; that is to say, the bridge, under live load, acts as a continuous structure over three or four supports, as the case may be. The resulting reactions must be determined on this basis and in their calculations use is made of the formulas of Chapter I. Generally the formulas for solid beams of constant moment of inertia are used for all cases, whether the structure is a plate girder or a truss bridge. The results are therefore only approximate but are usually sufficiently accurate for all practical purposes. After the design is completed more exact values of the stresses can be determined, if desired, by the use of the methods already mentioned in Art. 34 These methods will be illustrated in the following articles and the errors of the usual approximate method will be determined in certain cases.

41. Loads.—*The dead load* for railway swing bridges is approximately equal to the weight of a fixed span of the same length, less the weight of the turn-table, or the portion supported directly by the center pier.

The Live Load.—In the case of continuous bridges the inaccuracies in the calculations, due to approximations in assumptions and to temperature effects, are very considerable, so that the refinements of the wheel-load method of analysis are quite unwarranted. The use of an equivalent uniform load is to be preferred, the amount of which may be selected by a consideration of the forms of the influence lines, as explained later. If the wheel-load method of calculation is preferred, the use of influence lines is the most convenient method of procedure. This is illustrated in Art. 57.

Section II.—The Centre-Bearing Swing Bridge

42. Formulas for Reactions.—Applying the theorem of three moments, as in Art. 20, we derive the following values for the reactions, for a single load P on the first span:

$$R_1 = -\frac{P l_1}{2(l_1 + l_2)}(k - k^3) + P(1 - k), \quad . \quad . \quad . \quad (1)$$

$$R_3 = -\frac{P l_1^2}{2(l_1 + l_2)l_2}(k - k^3), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$R_2 = P - R_1 - R_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For equal spans

$$R_1 = \frac{P}{4}(4 - 5k + k^3), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$R_2 = \frac{P}{2}(3k - k^3), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$R_3 = -\frac{P}{4}(k - k^3). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To assist in the calculations, values of R_1 and R_3 are given in the following table, for $P = 1,000$ lbs., for various values of k, for trusses of equal spans and for various numbers of equal panels from two to ten in each span. For R_2 use the relation $R_1 + R_2 + R_3 = P$. The table may also be readily used for girders by dividing the span into a convenient number of equal divisions, such as 8 or 10, and treating the points of division as load points. From this table the total reactions for any series of joint loads or concentrations can readily be obtained.

TABLE No. 2

CONSTANTS FOR REACTIONS, P = 1,000 POUNDS, FOR BEAM CONTINUOUS OVER THREE SUPPORTS, WITH TWO EQUAL SPANS

(For loads on the left span only.)

No. of Equal Panels in Each Span.	Values of *k*.	R_1 +	R_3 —
2......	1 ÷ 2	+ 406.25	− 93.75
3.....	1 ÷ 3	592.6	74.1
	2 “	240.7	92.6
		$\Sigma R_1 = +$ 833.3	$\Sigma R_3 = -$ 166.7
4.....	1 ÷ 4	691.4	58.6
	2 “	406.3	93.8
	3 “	168.0	82.0
		$\Sigma R_1 = +$ 1265.7	$\Sigma R_3 = -$ 234.4
5.....	1 ÷ 5	752.0	48.0
	2 “	516.0	84.0
	3 “	304.0	96.0
	4 “	128.0	72.0
		$\Sigma R_1 = +$ 1700.0	$\Sigma R_3 = -$ 300.0
6.....	1 ÷ 6	792.8	40.5
	2 “	592.6	74.1
	3 “	406.25	93.75
	4 “	240.75	92.60
	5 “	103.0	63.70
		$\Sigma R_1 = +$ 2135.4	$\Sigma R_3 = -$ 364.65
7.....	1 ÷ 7	822.2	35.0
	2 “	648.7	65.6
	3 “	484.0	87.5
	4 “	332.4	96.2
	5 “	198.2	87.5
	6 “	86.0	50.9
		$\Sigma R_1 = +$ 2571.5	$R \Sigma_3 = -$ 428.7
8.....	1 ÷ 8	844.3	30.5
	2 “	691.4	58.6
	3 “	544.5	80.6
	4 “	406.3	93.8
	5 “	279.8	95.2
	6 “	168.0	82.0
	7 “	73.7	51.3
		$\Sigma R_1 = +$ 3008.0	$\Sigma R_3 = -$ 492.3

TABLE No. 2—(*Continued*)

No. of Equal Pannels in Each Span.	Values of k,	R_1 +	R_3 —
9.....	1 ÷ 9	861.5	27.5
	2 "	725.0	52.8
	3 "	592.6	74.1
	4 "	466.4	89.2
	5 "	348.4	96.0
	6 "	240.7	92.6
	7 "	145.4	76.8
	8 "	64.5	46.7
		$\Sigma R_1 = + 3444.5$	$\Sigma R_3 = - 555.7$
10.....	1 ÷ 10	875.25	24.75
	2 "	752.0	48.0
	3 "	631.75	68.25
	4 "	516.0	84.0
	5 "	406.25	93.75
	6 "	304.0	96.0
	7 "	210.75	89.25
	8 "	128.0	72.0
	9 "	57.25	42.75
		$\Sigma R_1 = + 3881.25$	$\Sigma R_3 = - 618.75$

43. End Conditions Assumed in Calculating Stresses.—Where the customary practice is followed of raising the ends after closing, so as to cause some dead-load reaction, the amount of this uplift is usually determined on the basis that it shall be sufficient to prevent hammering, or what amounts to the same thing, that the end reaction shall never become negative under partial live load. This requires the temperature variations to be considered as well as the deflections, questions which are discussed in Art. 65. The net result is that the ends are usually lifted an amount which, under uniform temperature conditions, will develop dead-load reactions of about one and one-half times the maximum negative live load reaction, without impact. Temperature differences among the various members of the truss will then cause considerable variations in reactions, and to provide for this in the design it is generally assumed that the reactions under dead load may vary anywhere between zero and a value equal to one and one half times the maximum negative live load reaction as mentioned above.

In calculating dead-load stresses, therefore, two cases are considered: Case I, bridge open, or bridge closed and ends just touching, and acting as a double cantilever; and Case II, bridge closed and ends raised, bridge acting as a continuous girder. The stresses resulting from either case are then to be combined with the live-load stresses to give maximum results.

In calculating live-load stresses, the conditions giving rise to maximum or minimum total stress should be kept in mind. In getting positive live-load moments and positive shears (considering the first or left-hand span), it is to be noted that the corresponding dead-load values of positive sign are a maximum with maximum uplift. For this case the structure is then to be assumed as lifted at the ends a maximum amount, and therefore as acting as a continuous girder under live load. For live-load negative moments and shears a minimum uplift is the most unfavorable condition and therefore for these calculations the bridge is generally assumed as just touching the supports under dead load. This being the case, when the live load extends over one arm only, the other arm is free or just touching, and the live-load stresses in the first arm are calculated as for a simple span. When both arms are loaded the ends come to a bearing and the structure is again a continuous girder so far as the live loads are concerned. In either case the dead-load stresses are the same as for bridge swinging free, or Case I. Considering the position of live loads producing maximum positive and negative moments and shears, as shown by the influence lines of Art. 26, it will be found that the analysis may be separated into the following cases.

Case I.—Dead load, bridge swinging, cantilever action.

Case II.—Dead load, bridge closed and ends raised, continuous-girder action.

Case III.—Live load on first span for maximum positive moments and positive shears, bridge acting as a continuous girder.

Case IV.—Live load on both arms for negative moments and negative shears, both ends bearing, bridge acting as a continuous girder.

The maximum results under Case IV, as shown by the influence lines of Art. 26, will generally require loads to be applied on two separate parts of the structure, or the use of "broken loads." This may or may not be a reasonable assumption, depending upon the location of the bridge. If broken loads are not to be considered then the negative shears, excepting for sections near the centre, will be a maximum with loads on one span only, which condition, with minimum dead-load uplift, will cause the opposite end to rise and the first span to act as a simple span. We then may include an additional case:

Case V.—Broken loads not considered. Live load on first span for negative shears, opposite end free, first span acting as a simple span.

For sections near the centre the maximum negative values will be caused by a fully loaded bridge, Case IV.

The maximum combined stresses are then found by combining Case II with Case III, and Case I with either Case IV or Case V.

If the ends are not raised but are simply latched down, the dead-load stresses are determined by Case I. These are then to be combined with those of either Case III or Case IV. In this case the effect of temperature variations should be separately allowed for.

44. Plate Girder Bridge of Two Equal Spans.—Assume the following data: $l = 80$ ft.; dead load $= 1{,}200$ lbs. per ft. per girder; live load = equivalent uniform for moment in an 80-ft. span for Cooper's *E*-60 loading, Fig. 3.

From Fig. 4 we find for this loading, and a span of 80 feet, an equivalent uniform load for the quarter point of 3,450 lbs. per foot, hence for *E*-60 the loading will be $3{,}450 \times 6/5 = 4{,}140$ lbs. per ft. per girder. Broken loads will be considered as possible. The loads will be assumed as concentrated at points 10 feet apart. The dead-load concentration will be 12,000 lbs., and the live-load concentration 41,400 lbs. The dead-load concentration at the end will be taken at 6,000 lbs., and the moments and shears will be determined for the first span. Fig. 5 (*a*) shows the load points in this span.

45. *Case I. Dead Load, Bridge Swinging, Cantilever Action.*—The calculation of moments and shears for this case requires no explanation. The results are shown by the curves marked *I* in Figs. 5 (*b*) and (*c*). In the curves of Fig. (*c*) the shears between consecutive load points are plotted as ordinates midway between loads and the points so found are connected by straight lines.

46. *Case II. Dead Load, Bridge Closed, Ends Raised, Continuous-girder Action.*—For this case the value of R_1 is readily determined from the table on p. 47, using the 8-panel span. The total value of R_1 for

Spacing		8	5	5	5	9	5	6	5	8	8	5	5	5	9	5	6	5	10	10
No.	∠①	②	③	④	⑤	⑥	⑦	⑧	⑨	∠⑩	⑪	⑫	⑬	⑭	⑮	⑯	⑰	⑱	19	20
Σ Dist.	0	8	13	18	23	32	37	43	48	56	64	69	74	79	88	93	99	104	114	124
P	12.5	25	25	25	25	16.25	16.25	16.25	16.25	12.5	25	25	25	25	16.25	16.25	16.25	16.25	25	25
Σ P	12.5	37.5	62.5	87.5	112.5	128.75	145	161.25	177.5	190	215	240	265	290	306.25	322.5	338.75	355	380	405
M	0	100	287.5	600	1,037.5	2,050	2,693.75	3,563.75	4,370	5,790	7,310	8,385	9,585	10,010	13,520	15,051.25	16,986.25	18,680	22,230	26,030
M′ (P_1 omitted)	0	0	125	375	750	1,650	2,231.25	3,026.25	3,770	5,000	6,510	7,522.5	8,660	9,922.5	12,420	13,888.75	15,748.75	17,380	20,805	24,480
ΣP′ (P_1 omitted)	0	25	50	75	100	116.25	132.50	148.75	165.	177.5	202.5	227.5	252.5	277.5	293.75	310.0	326.25	342.5	367.5	392.5

Spacing	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
No.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Σ Dist.	134	144	154	164	174	184	194	204	214	224	234	244	254	264	274	284	294	304	314	324
P	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25
Σ P	430	455	480	505	530	555	580	605	630	655	680	705	730	755	780	805	830	855	880	905
M	30,080	34,380	38,930	43,730	48,780	54,080	59,630	65,430	71,480	77,780	84,330	91,130	98,180	105,480	113,030	120,830	128,880	137,180	145,730	154,530
M′ (P_1 omitted)	28,405	82,580	87,005	41,680	46,605	51,780	57,205	62,880	68,805	74,980	81,405	88,080	95,005	102,180	109,605	117,280	125,205	133,380	141,805	150,480
ΣP′ (P_1 omitted)	417.5	442.5	467.5	492.5	517.5	542.5	567.5	592.5	617.5	642.5	667.5	692.5	717.5	742.5	767.5	792.5	817.5	842.5	867.5	892.5

FIG. 3.

a fully loaded bridge (1,000-pound joint loads) is + 3,008 − 492 = 2,516 lbs. Hence for Case II, R_1 = 2,516 × 12 = 30,200 lbs. The moments at the several points are then found as for a single span. For point *b*, M = 30,200 × 10 = 302,000 ft.-lbs.; for point *c*, M = 30,200 × 20 − 12,000 × 10 = 484,000 ft.-lbs., etc. The results are shown by curve *II*, Fig. 5 (*b*). The shear in section $a - b = R_1$ = 30,200 lbs.; shear in $b - c$ = 30,200 − 12,000 = 18,200 lbs., etc. These shears are given by curve *II*, Fig. 5 (*c*).

47. *Case III. Live Load Positive Moments and Shears, Continuous-girder Action.*—In accordance with the principles explained in Art. 21, Chapter I, the first span is fully loaded for all moment centres excepting in the fifth part of the span next to the centre support. This part will include only the load point *h*. Hence for all other points the first span is fully loaded, and the second span is not loaded. It will be assumed that the ends are fully raised before the live load comes on, as this condition gives greatest positive or least negative dead-load moments and shears. The combination of live- and dead-load effects under these conditions will therefore give maximum values.

For first span fully loaded, Table 2, p. 61, gives R_1 = 3,008 × 41.4 = 124,500 lbs. The moments are then found up to point *g*, inclusive, as in Case II, and are shown in Fig. 5 (*b*) by curve *III*. For point *h* some of the loads should be omitted, determined as follows: The position of a single load which will cause zero moment at *h* is found by substituting in the formula $\frac{x_0}{l} = \frac{4}{5 - k^2}$ (see Art. 20, eq. (26)). Here $\frac{x_0}{l} = 7/8$, whence *k* is found to be 0.65. Therefore, a load placed 0.65 *l* from *a* will give zero moment at *h*; loads placed on the right of 0.65 *l* will cause positive moments at *h* and loads placed on the left will cause negative moments. As the 0.65 point comes between *f* and *g*, then for maximum positive moment at *h* joints *g* and *h* should be loaded. Loading these joints, Table No. 2 gives R_1 = (168 + 74) × 41.4 = 10,000 lbs. M_h = 10,000 × 70 − 41,400 × 10 = 286,000 ft.-lbs.

For the first span *fully loaded* the moment at *h* is 0, and that at *i* is negative. The dotted line *III a* in Fig. 5 (*b*) shows the moments arising from a fully loaded condition.

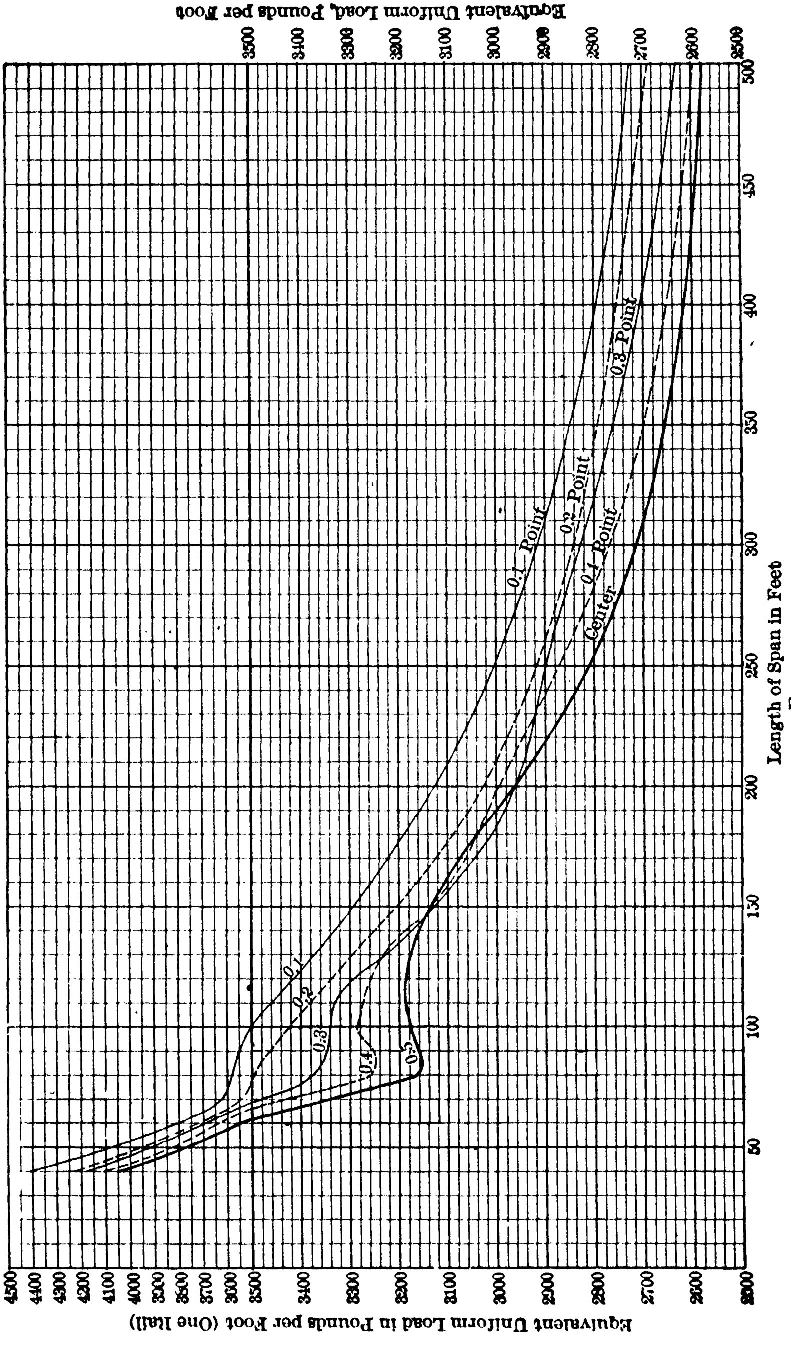

Equivalent Uniform Load, Pounds per Foot
Equivalent Uniform Load in Pounds per Foot (One Half)
Length of Span in Feet
50
100
150
200
250
300
350
400
450
500
4500
4400
4300
4200
4100
4000
3900
3800
3700
3600
3500
3400
3300
3200
3100
3000
2900
2800
2700
2600
2500
0.1 Point
0.2 Point
0.3 Point
0.4 Point
Center
0.1
0.2
0.3
0.4
0.5

FIG. 4

For maximum positive shears the first span is loaded from the section in question to the centre. The shear in section $a - b = R_1 =$ 124,500 lbs. For section $b - c$, load from c to h, inclusive. Table 3 gives $R_1 = (3{,}008 - 844) \times 41.4 = 89{,}500$ lbs. This is also the shear. The results are given in Fig. 5 (*c*), curve *III*.

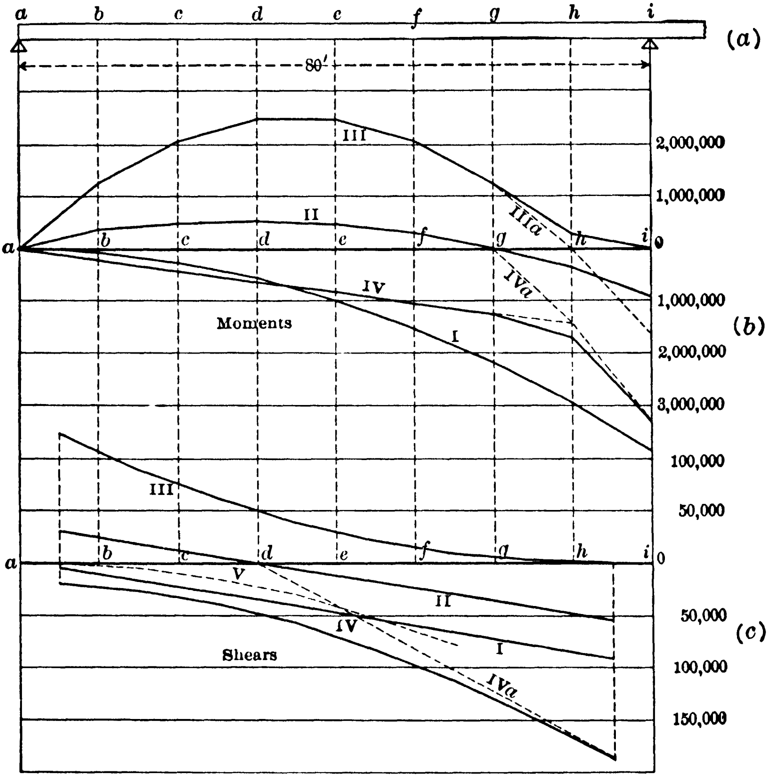


FIG. 5.—Dead and Live Load Moments and Shears.

48. *Case IV. Live Load Negative Moments and Shears, Continuous-girder Action.*—The maximum negative moments at all points except *h* will occur when the second span is fully loaded and the first span unloaded. Case I, dead load, will need to be combined with this case, but it is to be noted that for Case I the dead-load end reaction is zero. Hence a live load on the second span only will lift the end at *a* free from the support and the girder will not act as a continuous structure. It is assumed, however, in this analysis, that the live load may be divided into two or more separate portions ("broken loads"), if necessary to produce maximum stresses. In this case, therefore, it will be assumed that the point *a* is also loaded, which will be sufficient to hold this end down. The second span will be fully loaded.

The reaction is given in Table 2 (under R_3) and is $- 492 \times 41.4 = - 20{,}400$ lbs. From this value the negative moments from *b* to *g* are determined. For point *h* it has already been shown under Case III that loads from *b* to *f* cause negative moments at *h*. Hence these points, in addition to the second span, should be loaded. The value of R_1 is $[3{,}008 - (168 + 74) - 492] \times 41.4 = 2{,}274 \times 41.4 = 94{,}200$ lbs., and $M_h = 94{,}200 \times 70 - 41{,}400 \times 5 \times 40 = - 1{,}700{,}000$ ft.-lbs. For point *i* both spans are fully loaded and the value of M is $- 3{,}280{,}000$ ft.-lbs. These moments are shown by curve *IV*.

The value of M_h, if loads from *b* to *f* are omitted, is equal to $- 20{,}400 \times 70 = - 1{,}428{,}000$ ft.-lbs. This is shown by the dotted extension of curve *IV*. For *both* spans fully loaded the moments at *g*, *h*, and *i* are shown by the dotted line *IV a*. The moment at *h* is the same as for second span only loaded.

The diagram illustrates clearly the fact that the moments at *h* are but slightly increased by the use of the broken loads, over the values determined for one or both spans fully loaded.

For negative shears the second span is fully loaded and the first span from the left end up to the section in question. The results are given in Fig. 5 (*c*), curve *IV*.

49. *Maximum Total Moments and Shears.*—Assuming that the dead-load stresses when the bridge is closed may be either those of Case I or Case II, these stresses are to be combined with the live-load moments and shears of both kinds to arrive at the maximum positive and negative values. The results are shown by the full lines in Fig. 6.

Fig. (*a*) shows that for the loads assumed, positive moments cannot exist at points nearer than 10 ft. from the centre; negative moments may occur at any point. Except for this distance of 10 ft. near the centre all flange stresses are therefore subject to reversal.

In the case of shears no positive shears are possible nearer than about 30 feet from the centre; negative shears may occur at any point.

50. *Impact Allowance.*—In Fig. 6 the dead- and live-load moments and shears have been directly added, and the total stresses taken for the maximum or minimum. Under most modern specifications, however, the live-load stresses would first be increased by a certain percentage to allow for impact, or dynamic effect, before being combined with the dead-load stresses. The amount of such allowance is dis-

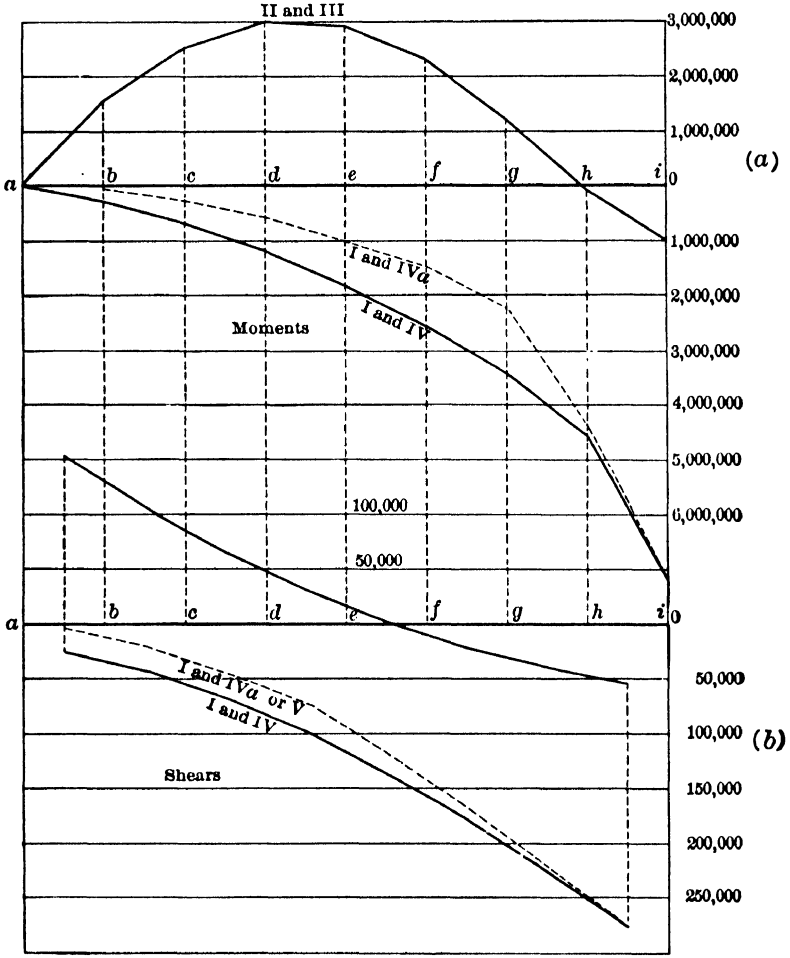


FIG. 6.—Maximum and Minimum Moments and Shears.

cussed in Chapter VII. Whatever this may be, it would be applied directly to the results for live load given in Fig. 5 and then combined with the dead-load curves. The resulting curves would vary considerably from those given in Fig. 6 and would cross the axis at different points, thus affecting not only the maximum stresses but the range over which a reversal is possible.

51. *Reversal of Stress.*—The reversal of stress indicated by the curves of Fig. 6 is based not only upon the varying effect of live load but also upon a changed condition of end supports, it being assumed in the one case that the ends are fully raised and, in the other case, that they are not lifted at all. The change due to live load may occur quite rapidly, but that due to a change in condition of the supports can hardly occur during the passage of a particular train. This circumstance is commonly recognized in specifications by providing that in determining the sections of members subject to reversal only such reversal need be considered as may occur during the passage of a train. During such passage, therefore, the condition of end supports is to be considered constant and may be either fully raised or not raised. The effect of this assumption is considered in detail in the example of Art. 54.

52. *Effect of "Unbroken Loads."*—In the analysis represented in Fig. 5 it has been assumed that the live load may be divided into two or more parts ("broken loads"). This assumption may or may not be legitimate, depending upon local conditions. Such a distribution of load would not occur except at very low speeds which would eliminate the question of impact.

Positive moments and positive shears are not affected by this question, as the loading assumed for these values has been unbroken. For negative moments, however, if unbroken loads are assumed, the left end cannot be assumed as held down, and the live-load moments caused in the first span by loading the second will be zero. If both spans are loaded the resulting moments are as given by the dotted curve *IV a*, Fig. 5 (*b*), and these are the maximum negative moments desired. These are then to be combined with Case I, dead load, giving the dotted curve of Fig. 6 (*a*), for the maximum total negative moments.

Negative shears will be caused by loading the first span from the left end to the section in question, the second span being unloaded.

With this loading it is assumed that the right reaction is zero and therefore that the first span acts as a simple beam. The shears for this condition are given by curve *V*, Fig. 5 (*c*). For sections near the centre, negative shear also exists when both spans are fully loaded, continuous girder action. These shears are given by curve *IV a*, Fig. 5 (*c*), and are seen to be greater than those of Case V for points on the right of *e*. Combining Case I, dead load, with Cases V and IV *a* gives the dotted line of Fig. 6 (*b*), as representing maximum negative shears with unbroken loads.

53. *Ends Latched Down.*—If the ends are not raised but are simply latched down, then in the use of unbroken loads negative reactions will occur. For negative moments the results will be the same as for broken loads, excepting for point *h*, which maximum will now be given by the dotted line *IV*, or *IV a*, Fig. 5 (*b*). For negative shears a full load on the second span will give a negative reaction at *a*, equal to $492 \times 41.4 =$ 20,400 lbs. This will be the greatest negative shear up to the point where the shear for a fully loaded bridge (curve *IV a*) gives a greater result, or to point *d* inclusive.

Where the ends are latched down the dead-load stresses are those of Case I, but temperature variations need to be taken into account by a separate allowance. This may be done by adding the stresses resulting from a certain positive or negative end reaction, calculated on the basis of assumed temperature variation, as explained in Art. 66. These stresses will be plus or minus and should be added to all combinations of dead- and live-load stresses. Where the ends are raised, as described in Art. 43, the combination of both Case I and Case II, dead load, with the live-load stresses, makes adequate allowance for these temperature variations. The possible amount of such variations is discussed in Art. 66.

54. **Centre-Bearing Truss Bridge of Two Equal Spans.**—Assume a truss of the dimensions given in Fig. 7. The dead load above the turntable is assumed to be 1,200 lbs. per ft. per truss, and all applied at the lower joints. The joint load at *a*, bridge open, is taken at 18,000 lbs. The live load assumed is the engine loading of Cooper's *E*-50 class. The method of influence lines will be used in getting live-load stresses. Unbroken loads will be assumed.

55. *Case I. Dead Load, Bridge Swinging.*—The stresses for this

condition are readily found, either graphically or algebraically. Fig. 7 (*d*) gives the stress diagram and in Table *A*, p. 63, the stresses are given in Col. (2).

56. *Case II. Dead Load, Ends Raised.*—In this case the load at *a* need not be considered. The value of R_1 is given in Table 2 on p. 47, for the 4-panel truss. It is equal to (1,266 − 234) × 36 = 37,150 lbs.

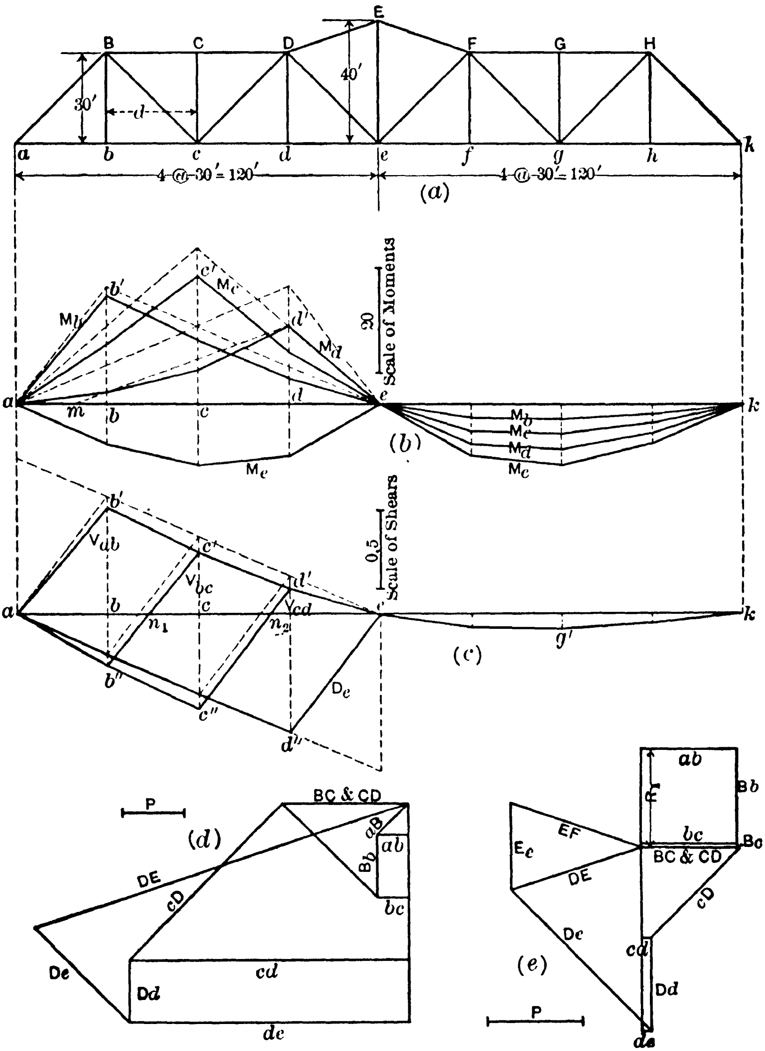


FIG. 7.

The diagram of Fig. 7 (*e*) is then readily drawn. The stresses are given in Col. (3) of the table of stresses.

57. *Case III. Live Load on First Span Only, Maximum Positive Moments and Positive Shears, Continuous-girder Action.*—This case gives maximum tension in lower chord, maximum compression in upper chord, and maximum stresses in the web members resulting from positive shears. For member $D\,e$ the vertical component will be considered instead of the shear, as the chord $D\,E$ takes part of the shear.

The influence lines for moments at *b*, *c*, *d*, and *e* are given in Fig. 7 (*b*). These are constructed for unit loads, using the reactions given in Table 2. Thus for joint *c* the bending moments due to load unity, placed successively at the several joints, are calculated as follows:

Position of Unit Load.	Value of R_1.	Moment at *c*.
b	.691	$.691 \times 2d - 1 \times d = 11.46$
c	.406	$.406 \times 2d = 24.36$
d	.168	$.168 \times 2d = 10.08$
f	$-.082$	$-.082 \times 2d = -4.92$
g	$-.094$	$-.094 \times 2d = -5.64$
h	$-.059$	$-.059 \times 2d = -3.54$

For joint *d* the work may be shortened by noting that the moments for unit load at *d*, *f*, *g*, and *h* are each equal to 3/2 times the corresponding moment at *c*, as the only change is in the lever arm of R_1. For point *b* the moments due to loads at *c*, *d*, *f*, *g*, and *h* are one-half those at joint *c*.

Having the influence lines drawn the maximum moments are found by trial for the given wheel-loads concentrations. For this purpose the influence lines may conveniently be drawn on profile or cross-section paper so that the ordinates can be read off directly. Then the wheel loads should be laid off on a strip of paper to the same scale. The locomotive may be headed in either direction but the second span is not to be loaded. On this strip should be drawn circles indicating relative weights of wheels, and the weight of each wheel written in the circle. To get any desired moment or web stress for any given position of the loads, place this strip in the given position on the base line of the influence line, read off the ordinate above each load, and sum the products of the loads times ordinates. Two or three trials will usually serve to determine the maximum value of a moment or stress. The principle stated in Art. 123, Part I, that maximum values are caused only when a load is placed under a convex point on the curve, will aid in selecting the proper position of loads. The similarity of the influence lines to those for simple bridges will also aid in finding this position. The calculations can be shortened by dividing the loads into convenient groups, then read off the ordinate above the centre of gravity of such group, and multiply by the total weight. Only those wheels under a single straight segment of the influence line can be so grouped.

The accuracy of this method of computation is much greater than that of ordinary graphical methods. Errors are compensating rather than cumulative. If the lines are drawn to such a scale that the larger ordinates are two or three inches in length, results may easily be obtained within one per cent of the correct values.

In Fig. 7 (*c*) are shown the influence lines for shears in the panels *a b*, *b c*, and *c d*, and for the vertical component of the stress in *D e*. In this case members *D E* and *d e*, produced, intersect at *a*, so that the stress in *D e* is the same as in a simple span. Continuous-girder action is here assumed. The maximum positive shears are determined by trial, train headed toward the left and the second span not loaded. This assumes a single locomotive or two locomotives present, as the case may be. The small amount of load which might necessarily come upon the second span is neglected, as the effect in any case is small.

From the results for positive moments and positive shears, the chord and web stresses are calculated and these are given in Col. (4) of the table of stresses. The compression in *c D* is small and much less than the dead-load tension.

58. *Case IV. Maximum Live-Load Negative Moments and Negative Shears, Continuous-girder Action.*—These moments and shears are determined from the influence lines of Figs. (*b*) and (*c*) in the manner already described. For negative moments the second span only is loaded, excepting for point *e*, which requires a load extending over both spans. For negative shears three conditions of loading may need to be considered, the first span loaded on the left of the panel in question, the second span fully loaded, and both spans loaded. For panel *b c* the area *a* b'' n_1 of the influence diagram, as compared to area *e* g' *k* shows the relative effect produced by the first two positions mentioned. The large positive area, n_1 c' *e* shows that the third position need not be tried in this case. For panel *c d*, a load extending from *a* to n_2 will evidently cause greater shear than a load on the second span, but the positive area n_2 d' *e* being small, it is possible that the greatest shear will occur for a load on both spans. For stress in *D e* both spans should be loaded. The several results are given in Cols. (5) and (6) of the table. For *B c* a load on the first span gives a stress of 36,700 lbs., while a load on the second span gives a stress of 32,300 lbs. For *c D* the maximum stress is caused by loading the first span only.

59. *Case V. First Span Fully or Partially Loaded, Bridge Acting as a Simple Span.*—When the ends of the truss are not fully lifted, a load on one arm is assumed to cause no reaction at the far end, in which case the loaded arm will act as a simple span. This condition will give maximum negative shears in some of the panels, as shown in the previous problem. It will also give information required in determining reversal of stress during the passage of a train. The stresses are determined exactly as for a simple span, but both kinds of web stress should be found for members *B c* and *c D*. Influence lines for these moments and shears are shown in Fig. 7 by dotted lines. Resulting stresses are given in Col. (7) for all members, but it will be found that many of these are not needed in the combinations.

60. *Maximum and Minimum Stresses.*—In Cols. (8) and (10) are given the maximum and minimum stresses as determined by combining Case II with Case III, for stresses caused by positive moments and shears, and Case I with IV or V for stresses caused by negative moments and shears. In the latter combination it is to be carefully noted whether Case IV or Case V is to be used. Case IV is to be used only when both arms are loaded (Col. (6)), for when only one arm is loaded, with ends not raised, the far end will be lifted and the loaded arm becomes a simple span. Case V is then to be used. Member *D E* is the only chord member having a tensile stress (negative moment at *e*) for Case IV, both spans loaded. This stress is given in Col. (6). In the case of the web members, *B c*, *c D*,

TABLE A

STRESSES IN A CENTRE-BEARING SWING BRIDGE

NOTE: + = tension, − = compression.

Member.	DEAD LOAD.		LIVE LOAD.				COMBINATIONS.			
				Case IV − *M* and − V, Cont. G.			Dead Load Case II with Live Load Case III or IV.		Dead Load Case I with Live Load Case IV or V.	
	Case I Cantilever.	Case II. Cont. Gird.	Case III + *M* and + *V* Cont. Gird.	One Arm Loaded.	Both Arms Loaded.	Case V + M and ± V Simple Span.	II and III for *Max.* + *M* and + *V*.	II and IV for Reversal.	I and IV or V for *Max.* − *M* and − *V*.	I and V for Reversal.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
a-b-c	− 18,000	+ 37,100	+ 125,100	− 22,800		+ 147,700	+ 162,200	+ 14,300	− 18,000	+ 129,700
c-d-e	− 162,000	+ 3,400	+ 79,800	− 68,400		+ 147,700	+ 83,200	− 65,000	− 162,000	− 14,300
B D	+ 72,000	− 38,300	− 146,500	+ 45,600		− 190,200	− 184,800	+ 7,300	+ 72,000	− 118,200
D E	+ 227,700	+ 53,300			+ 127,200		+ 53,300		+ 354,900	
a B	+ 25,500	− 52,500	− 176,700	+ 32,300		− 208,000	− 229,200	− 20,200	+ 25,500	− 183,400
B c	− 76,400	+ 1,600	+ 70,600	− 36,700(1) − 32,300(2)		+ 97,200 − 28,600	+ 72,200	− 35,100	− 105,000	+ 20,800
c D	+ 127,300	+ 49,500	− 18,700	+ 120,500(1) + 32,300(2)	+ 116,200	+ 97,180 − 28,600	+ 30,800		+ 243,500	+ 98,700
D e	− 76,400	− 76,400			− 209,500		− 76,400		− 285,900	

and *D e*, the greatest of the values in Cols. (6) and (7) due to negative shear are to be combined with Case I. The results in Cols. (8) and (10) give full information regarding the greatest and the least stresses in all the members.

If, in addition to the absolute maximum and minimum stresses, it is desired to know the greatest range of stress during the passage of a single train, then the maximum and minimum live-load stress must be combined with each of the dead-load cases, keeping in mind whether the structure is acting as a continuous girder or a simple span. The additional combinations for minimum values occurring during a single-train movement, the condition of the end supports remaining fixed, are given in Cols. (9) and (11). In Col. (9) are combined the stresses under Case II, with the live-load stresses from maximum *negative* moments and shears. As the structure of Case II acts always as a continuous girder, the live-load stresses to be here considered are therefore to be taken from either Col. (5) or (6), whichever is the greater. In Col. (11) the dead load is Case I, and the structure acts as a continuous girder only when loaded on both arms. The live-load stresses are therefore to be taken from Col. (6) or (7). The stresses shown in Cols. (9) and (11) will be needed in the design only when of opposite sign from those in Cols. (8) and (10) respectively. When this is the case then the member will be designed for either combination of stresses, the combination given in (8) and (9) or that given in (10) and (11). Thus member *B D*, for example, will be designed for a maximum compression of 184,800 lbs., with a reversal to 7,300 lbs. tension; and also for a compression of 118,200 lbs., with a reversal to 72,000 lbs. tension, using the greater of the two areas thus found. Similarly *B c* will be designed for a tension of 72,200 lbs., and compression of 35,100 lbs., or a compression of 105,000 lbs., and tension of 20,800 lbs., using the greater area.

Impact allowances made to live-load stresses are made directly to the stresses given in Cols. (4), (5), (6), and (7). The resulting values are then to be combined with the dead-load stresses in the same manner as here described.

61. Use of Influence Lines for Determining Position of Loads.—If desired, the influence lines may be used to determine only the position of loads for maximum and then the value of the maximum itself may be exactly determined from panel concentrations, calculated as explained in Part I, Art. 148. For example, the influence line for moment at *c*, Fig. 7, is very similar to that for moment at *c* in the simple span *a–e*. The *position* of loads will therefore be determined very closely if we use the position for moment at *c* in a simple span *a–e*. This is found by the usual criterion to be with wheel No. 11 at *c*. With this position of loads, the panel concentrations may then be calculated and the moment found by multiplying these loads by the ordinates to the influence line given on p. 59 and adding the results. Or, the position having been determined by applying the criterion for a simple beam, the moment may still be found by using the influence line directly, summing up the products of ordinates times wheel loads.

For moment at *d* notice that the influence line approximates to that for moment at *d* in a simple span *m e*.

The influence line for shear may be used in the same way, but in this case there is less likely to be any error in the selection of the position for maximum values.

62. Use of Equivalent Uniform Loads.—The influence lines may also be used to select a suitable equivalent uniform load, in the same manner as explained in Art. 173 of Part I. Thus for moment at *d* the dotted straight line *d′ m* may be substituted for the broken line, giving the approximate influence line *m–d′–e*. The proper equivalent load is then the uniform load for moment at *d* in a beam *m e* which is at the 0.3 point in a 100-ft. beam. This is found by Fig. 4 to be 3,340 lbs. per foot. This load, then, multiplied by the area of the true influence line, *a–d′–e*, will give the desired moment. The ordinates to this influence line are 2.19, 6.54, and 15.12, respectively, and the area = 715.5. The moment at *d* = 715.5 × 3,340 = 2,390,000 ft. lbs., and the stress in *c–d–e* = 79,700 lbs. The value obtained from the concentrated loads was 79,800 lbs.

Again, the influence line for positive shear in panel *b c* is $n_1\ c'\ e$. This is of nearly the same form as that for moment at *c* in a beam $n_1 e$, or at the 0.2 point in a beam 77 ft. long. From Fig. 4 this is found to be 3,500 lbs. per ft. The area of the influence diagram is 14.58 and the stress in *B c* is therefore 14.58 × 3,500 × 1.414 = 72,200 lbs. The value given in Table A is 70,600 lbs.

A convenient way to carry out this method is to calculate all stresses for a uniform load of unity per foot and then to multiply the stresses so found by the proper equivalent load determined for each member.

63. True Reactions Calculated from Deflections.—It was noted in Art. 34 that in the application of the usual beam formulas to trusses there are two sources of error: (1) that due to the assumption of a constant moment of inertia and (2) that due to the neglect of the web distortions. The first can be avoided by the use of formulas for variable moment of inertia, but the latter only by the application of the theory of deflections or of redundant members as explained in Chap. VII, Part I. Generally it is found that the two sources of error mentioned tend to compensate, so that if correction is made for variable moment of inertia and not for web distortion, the result is apt to be more in error than when both are neglected. If, therefore, it is desired to employ a more exact method than the usual one, the method of redundant members should be used. While the usual methods give results accurate enough for most purposes it is desirable to know in general what the range of error may be, and in some cases to make a detailed analysis by the exact method.

As the theory of redundant members requires a knowledge of the cross-sections of all the members, it is necessary, before this method can be applied, to determine these sections from calculations based on the usual beam theory, or some other approximate method. The sections being known the exact stresses may then be found. If the preliminary sections are too greatly in error they can be corrected accordingly. Such a change in section will cause some change in the reactions and the stresses, but not generally enough to require a second calculation.

In the application of the method of redundant members it will be convenient to consider the centre reaction R_2 as redundant. Then, as in Art. 32, the value of R_2 is given by the equation

$$R_2 = -\frac{\Sigma \frac{S' u l}{E A}}{\Sigma \frac{u^2 l}{E A}} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

in which

S' = stress in any member due to the given loads, with the structure supported at the two ends;

u = stress in any member due to one pound upward load applied at the centre;

l and A = length and cross-section, respectively, of any member.

By the application of eq. (1) for a unit load at each of the loaded joints, the influence line for R_2 can be drawn, which can be utilized very readily in the construction of the true influence lines for moments and shears. The labor of the calculations will be reduced if two loads of unity be assumed, symmetrically spaced. Thus in Fig. 7, to determine R_2 for a load at c, assume a load unity at both c and g and determine the value of R_2 for these two loads. For a symmetrical load such as assumed the stresses S' need be calculated for one-half only. Also the stresses u need be calculated for one-half only. Then substituting in (1) the result will be the reaction for the two loads. But the reaction for the two loads is equal to twice that for one of them, hence the result obtained by using summations for one span only is twice the correct value for R_2 for a single load.

The graphical method is very convenient in such problems as these. A single displacement diagram for the bridge supported at the ends, and loaded with unit load at the centre is all that is required. From this diagram the deflection curve of the lower chord joints is drawn. Let Fig. 8 represent such deflection curve. Then by the principles explained in Arts. 220 and 223, Part I, the deflection at e caused by a unit load at any other point d is equal to δ_d. But a unit load (or reaction) at e causes a movement of δ_e, hence the reaction at e due to a unit load at d is equal to δ_d/δ_e, or, *in general*

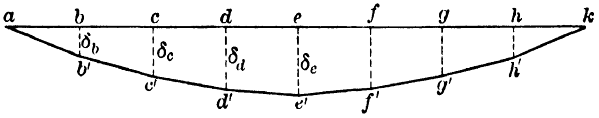


FIG. 8.

$$R_2 = \frac{\delta}{\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which δ is the deflection of any given joint due to one pound at e, bridge considered supported at the ends only, δ_e is the deflection at joint e, and R_2 is the center reaction due to one pound placed at the given joint. The influence line for R_2 is therefore obtained by plotting the deflection curve to such a scale that δ_e is unity, or by replotting the values δ/δ_e.

64. Application to the Truss of Fig. 7.—The method of redundant members will be applied to the truss of Fig. 7 and the true influence lines constructed. The graphical method will be used. The lengths, cross-sections, stresses, and values of ul/A for unit load at e are given in the table below. As the direction of R_2 is not in question the unit load at e will be taken as acting downward.

Member.	Length l.	Cross-section A.	u.	$\frac{ul}{A}$.
a-b-c	720	17.3	+ .50	+ 20.8
c-d-e	720	19.8	+ 1.50	+ 54.6
B-C-D	720	28.0	− 1.00	− 25.7
a B	506.0	36.1	− .707	− 9.9
B c	506.0	19.8	+ .707	+ 18.1
c D	506.0	21.4	− .707	− 16.7
D e	506.0	44.8	0	
D E	379.4	27.9	− 1.58	− 21.5
E e	480	33.4	+ 1.00	+ 14.4

The quantities in the last column, divided by E, are the deformations to be used in constructing the displacement diagram. The quantity E may be omitted as only relative results are needed. The diagram may be started at the centre vertical, the point e being assumed stationary. The diagram is shown in Fig. 9. One-half only is drawn. From this diagram the vertical movement of all joints with respect to a horizontal line a–k may be measured. Dividing each deflection by the deflection of joint e and plotting the results, gives the influence line for R_2 shown in Fig. 10 by the full line $a\,e'\,k$. For convenience the ordinates are plotted above the axis. The numerical values of the ordinates, or values of R_2, are as follows:

Point Loaded.	R_2
b	0.354
c	0.685
d	0.940
e	1.000

By the approximate method previously used the corresponding values of R_2 are: 0.368; 0.688; and 0.914, respectively.

Having the influence line for R_2, the other reactions, and the moment and shear at any section for a load at any point can be determined and any desired influence line drawn. Or, we may proceed more quickly as follows: Consider the moment at *b*, Fig. 10. A load *P* placed on the structure at any point, as *c*, will cause a moment at *b* which may be calculated in two parts, (1) the moment which would result if the structure were supported at the ends only, and (2) the moment due to the centre reaction R_2. The moment due to (1) is equal to $\frac{P(2l - kl)}{2l} \times a = \frac{P(2 - k)}{2} \cdot a$; and that due to R_2 is equal to $\frac{R_2}{2} \times a$. If y_c = ordinate $c c'$, then $R_2 = P y_c$, and the total moment $= P[(2 - k) - y_c]\frac{a}{2}$. But ordinate $c c'' = 2 \times \frac{2l - kl}{2l} = 2 - k$, hence $c' c'' = (2 - k) - y_c$. Therefore if $c' c''$ be multiplied by $a/2$ the result will be the moment at *b* for a unit load at *c*. Likewise for a load at any other joint, the ordinate at the joint between the straight lines $a b'' - b'' k$, and the curve $a e' k$, if multiplied by the constant $a/2$, will represent the moment at *b*. The shaded area therefore serves as the influence area for moment at *b*.

The influence diagram may be reconstructed to a horizontal base and true scale by plotting the ordinates $b' b''$, $c' c''$, etc., multiplied by $a/2$. The replotted diagram will be similar to the diagrams of Fig. 7 (*b*).

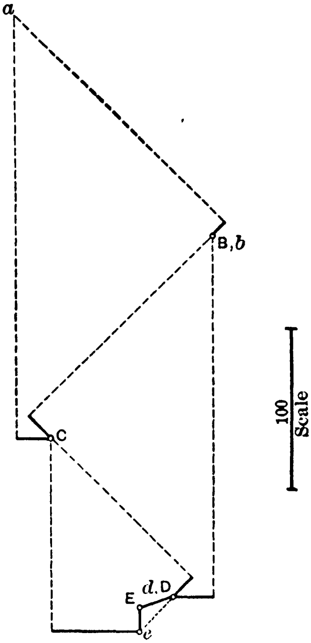


FIG. 9.

For the other points, *c* and *d*, the influence lines are obtained by drawing the straight lines $a c''$ and $a d''$. The true ordinates are then equal to the ordinates between these lines and the curve for R_2, multiplied in each case by $a/2$, where *a* is the distance of the moment centre from the left end of the span.

The true influence lines being thus drawn the exact stresses can be determined in the same manner as the approximate values; but a comparison of the two methods can readily be made by comparing the curves for the approximate and exact values of R_2. In Fig. 10 the dotted curve $a e' k$ gives the values by the approximate method, corresponding to the analysis already made. The error in stresses is clearly indicated by the area enclosed between these two curves as compared to the area of the influence diagram for any particular moment (or shear). In this case the errors are seen to be very small. For example, the influence areas for positive moments (multiplied by $a/2$), are as follows:

	Exact.	Approximate.	Per Cent Error.
M_b	113.5	113.8	+ 0.3
M_c	137.0	137.7	+ 0.5
M_d	70.5	71.6	+ 1.5

For shears a similar method of constructing influence lines may be used. Consider the shear in panel *b c*, Fig. 10. For unit load at *c*, truss supported at

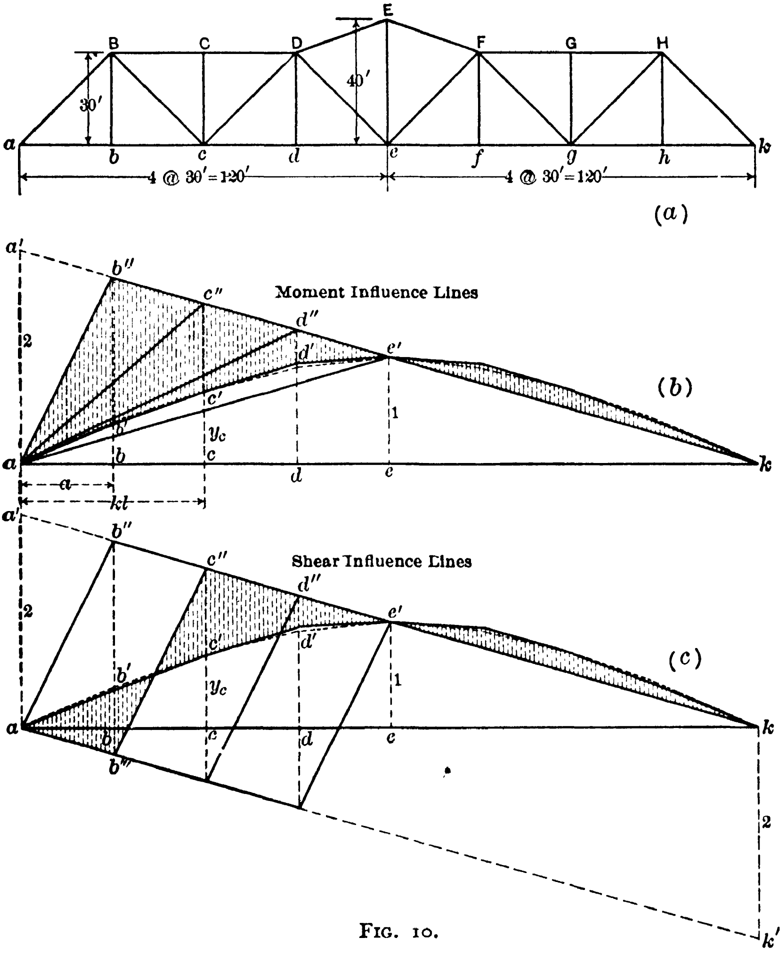


Fig. 10.

ends only, the shear in panel *b c* is equal to $\frac{2l - kl}{2l} = \frac{2-k}{2}$; and the shear due to $R_2 = R_2/2 = \frac{y_c}{2}$. The total shear $= [(2 - k) - y_c] \times \frac{1}{2}$ The ordinate $c'c'' = (2 - k) - y_c$, hence the shear $= c'c'' \times \frac{1}{2}$. The ordinates of the shaded area, in general, multiplied by ½, will therefore give the true influence line for shear in panel $b\,c$. The line $a\,b'''\,k'$ is parallel to $c''\,k$; and the line $a\,b'''\,c''\,k$ has double the ordinates of the influence line for shear in a simple truss. The shear influence lines for the other panels are drawn as indicated.

The method of constructing influence lines here described may of course be used in the usual approximate analysis. The curve for R_2 is first constructed, after which all the influence diagrams are drawn without further calculation. They may then be redrawn, if desired, on a straight base line and to a correct scale. The method here used will be found of general utility also in the analysis of arches and suspension bridges.

65. Deflection under Dead Load and Amount of Uplift Required.— In order to design the end-lifting mechanism it is necessary to know the amount of uplift required and the force necessary to produce it. The full dead-load deflection, bridge swinging, is given by the general formula $\Delta = \Sigma \dfrac{S\,u\,l}{E\,A}$, where S = dead-load stress, bridge swinging, and u = stress for 1 pound applied at the end. The amount of upward deflection produced by a 1-pound force is $\delta = \Sigma \dfrac{u^2\,l}{E\,A}$. The total force required to remove all the dead-load deflection (the true dead-load reaction for a continuous girder) is therefore $\Sigma \dfrac{S\,u\,l}{E\,A} \Big/ \Sigma \dfrac{u^2\,l}{E\,A}$, exactly as found in Art. 63 for the centre reaction R_2. If the ends are to be lifted only sufficiently to prevent the live load from raising the end from its support, then the movement required must be sufficient to develop a reaction equal to the maximum negative live-load reaction. If this be R' then the required movement = $\delta\,R'$. If any other desired amount of reaction is to be obtained, the distance to be moved is readily determined in the same manner.

In the truss of Art. 64 the value of $\Sigma \dfrac{S\,u\,l}{E\,A}$ for the dead load, Case I, is found to be 1.47 in., which is the full deflection from normal position, and is the uplift required if the ends are to be fully lifted. The value of $\Sigma \dfrac{u^2\,l}{E\,A} = 0.0000268$ in. The resulting reaction (true dead-load reaction) = 1.47 ÷ 0.0000268 = 54,700 lbs. The maximum negative live-load reaction = 22,800 lbs. (stress in *a b c* in Col. (5), Table A). To prevent hammering, therefore, the uplift must be at least 22,800 × 0.0000268 = 0.612 in. In practice, the uplift would be made somewhat greater than this in order to provide for variations in temperature (Art. 66), and imperfect adjustment.

66. Effect of Temperature Variations.—A very important question, and one of especial significance when discussing the accuracy of working formulas, is that of the effect of a variation of temperature between different members of a swing bridge. The effect on reactions of any given difference of temperature can very quickly be found in the same manner as the true reactions for loads.

Let t = change of temperature of any member; ω = coefficient of expansion, = .0000065 per 1° F.; l = length of member. Then $\omega t l$ = total change of length of any member. The deflection of the end of the truss due to the change of length of this member will be $u \omega t l$, where u has the same significance as in the previous calculations. For a change of temperature in any number of members the end deflection will be

$$\Delta_t = \Sigma u \omega t l. \qquad \ldots \ldots \quad (3)$$

The reaction necessary to produce a like deflection, or the reaction caused by the assumed temperature changes, will be

$$R_t = \Delta_t / \delta = \Sigma u \omega t l / \Sigma \frac{u^2 l}{EA}. \qquad \ldots \ldots \quad (4)$$

For example, suppose the lower chord of the bridge of Fig. 7 drop in temperature 10° below that of the remaining members. In this case $\omega t l$ for each chord member = 0.0234 in. and $\Sigma u = 8$ (one pound at a), whence the deflection = 0.187 in. and R_t = 0.187/0.0000268 = 6,960 lbs.

Observation of the variation in deflection of swing bridges indicates that the differences in temperature may readily amount to 30°, thus giving rise to variations in reaction of as much as 50 per cent of the normal dead-load reaction.

The great effect of temperature variations evidently renders useless any great refinement in calculations and indicates that the ordinary formulas are accurate enough for all ordinary cases.

67. Counter-balanced Swing Bridges.—Frequently in the case of narrow channels it is necessary to avoid the construction of a pier in the channel. In such a case the pivot pier must be built at the margin and a single arm of the swing span made of the length required for the opening. The other span is needed only for balancing the channel span and is made as short as convenient, and weighted with pig-iron or other heavy material. It may or may not support a panel of the live load. The stresses are found by the application of the general formula for unequal spans (Art. 23), and involves no new principles.

SECTION III.—THE RIM-BEARING SWING BRIDGE

68. Rim-Bearing Swing Bridge,—Truss Continuous over Four Supports.—A method of construction formerly quite general is shown in Fig. 11. The central panel is fully braced and the truss is continuous over four supports.

With some writers it has been customary to apply to this form of truss the formulas for a continuous beam of four supports,—formulas based on the assumption of uniform moment of inertia and a neglect of deflection due to shearing stresses. These formulas give results closely approximate for beams and long-span trusses, and fairly good results for two-span swing bridges; but their application to trusses of such short spans as here considered leads to very erroneous conclusions. With only one span loaded large negative reactions are obtained at B or C, reactions much greater than can ever really occur, and which greatly exceed the dead-load positive reactions. To furnish these negative reactions some form of anchorage would have to be provided at B and C, a thing quite impracticable in a swing bridge.

Again, the central span BC is so short that the effect of the web distortions (neglected in the beam theory) becomes of great importance and an exact analysis will show that the resultant dead- and live-load reaction at B or C will never be negative (the negative live-load reaction will never be large and always much less than the dead-load positive reaction). An example will be given showing the results obtained by the two methods of calculation.

69. *Calculation by Continuous-girder Formulas.*—Considered as a beam continuous over four supports, and with equal end spans, the

reactions due to a single load on the first span are given by the following formulas derived in Art. 25:

$$\left.\begin{aligned}
R_1 &= P(1-k) - \frac{2+2n}{N}P(k-k^3)\\
R_2 &= Pk + \frac{2+5n+2n^2}{nN}P(k-k^3)\\
R_3 &= -\frac{2+3n+n^2}{nN}P(k-k^3)\\
R_4 &= \frac{n}{N}P(k-k^3)
\end{aligned}\right| \quad . \; . \; . \; . \quad (1)$$

in which n = ratio of centre to end span and $N = 4 + 8n + 3n^2$.

Assume, for example, a truss of the same span lengths as the structure of Art. 76, and suppose this to be rigidly braced in the centre panel and to be treated as a beam continuous over four supports (Fig. 13). Assume a dead joint load of 30,000 lbs., and a live joint load of 75,000 lbs. The value of $n = 0.107$ and $N = 4.89$. Applying eq. (1) we find, for dead load, $R_1 = R_4 = +56{,}100$ lbs., and $R_2 = R_3 = +93{,}900$ lbs. The value of R_3 for full live load on the first span

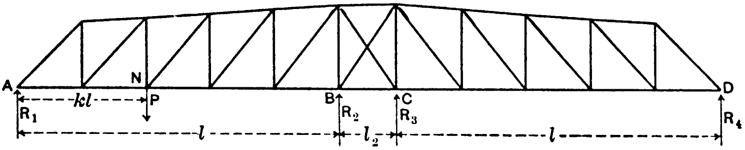


FIG. 11.

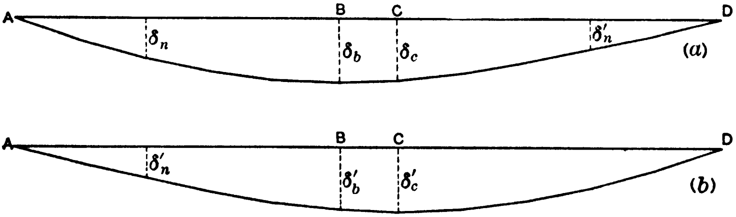


FIG. 12.

is by eq. (1), $R_3 = -\frac{2+3n+n^2}{nN} P \Sigma (k-k^3)$, in which $\Sigma (k-k^3)$ is the summation for the several load points on the first span. From this we find $R_3 = -486{,}000$ lbs. The total reaction at h is therefore $-486{,}000 + 93{,}900 = -392{,}100$ lbs. To provide this amount of negative reaction would require a strong and rigid anchorage, an arrangement of much difficulty. It is further to be noted that if R_3 is negative, R_4 is at the same time positive. One assumption sometimes made is that whenever the net reaction at h becomes negative this point rises slightly from the support and the bridge becomes a two-span structure with supports at a, g, and n. Such action would probably occur, but would also be objectionable as it would cause a very unequal distribution of load on the drum and turntable. It will be shown, however, that the true reaction at h is never likely to be negative.

70. *Calculation by Method of Redundant Members.*—Applying the theory of redundant members to the truss of Fig. 11, it is noted that the structure is doubly indeterminate. The two redundant forces may conveniently be taken as the reactions R_2 and R_3. With these removed the truss is a simple span, the stresses in which are readily calculated for any given loading. As developed in Art. 225, Part I, let

S' = stress in any member due to the given loads, truss supported at A and D only;

u = stress under the same conditions for 1-pound reaction at B;

v = stress under the same conditions for 1-pound reaction at C.

S = total actual stress in any member due to given loads and all reactions.

Then

$$S = S' + u R_2 + v R_3. \qquad (2)$$

Also, since the deflection at B is zero,

$$\Sigma \frac{S u l}{E A} = 0, \qquad (3)$$

and likewise, as the deflection at C is zero,

$$\Sigma \frac{S v l}{E A} = 0. \qquad (4)$$

Substituting the value of S from (2) we have

$$\Sigma \frac{S' u l}{E A} + R_2 \Sigma \frac{u^2 l}{E A} + R_3 \Sigma \frac{u v l}{E A} = 0, \qquad (5)$$

and $$\Sigma\frac{S'vl}{EA} + R_2\,\Sigma\frac{uvl}{EA} + R_3\,\Sigma\frac{vl}{EA} = 0. \quad . \quad . \quad . \quad (6)$$

The graphical method explained in Art. 223, Part I, can be utilized to calculate the variations summations of eqs. (5) and (6), and it will be found convenient to determine the reactions for a load of 1 pound at any joint.

In accordance with the method above referred to, it will first be necessary to construct a displacement diagram of the truss for a one-pound load applied at B, truss supported at A and D. Then from this diagram the deflections of each of the lower joints is determined with respect to the supports A and D, and a deflection curve constructed. Suppose Fig. 12 (*a*) represent such deflection curve. Then suppose a similar curve to be drawn for a one-pound load at C. For a symmetrical truss, like the one considered, this will be similar to the curve for load at B and need not be actually constructed, but for clearness it is shown in Fig. (*b*).

Now consider a unit load applied at any joint N. In accordance with the principles explained in Art. 220, Part I, the ordinate δ_n will be the deflection of joint B due to this unit load; and the ordinate δ'_n, Fig. (*b*), will be the deflection of joint C for this load. Furthermore, the ordinate δ_c is the deflection of B for one pound at C, and the ordinate δ'_b is the deflection of C for one pound at B. Noting the significance of the various summations in eqs. (5) and (6), and assuming S' to be the stress in any member for a one-pound load at N, we have the following identities:

$$\left.\begin{aligned} &\Sigma\frac{S'ul}{A} = -\delta_n;\ \Sigma\frac{S'vl}{A} = -\delta'_n \\ &\Sigma\frac{u^2l}{A} = \delta_b;\ \Sigma\frac{v^2l}{A} = \delta'_c \\ &\Sigma\frac{uvl}{A} = \delta_c = \delta'_b. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

Substituting in (5) and (6), and neglecting the quantity E, we have

$$\left.\begin{aligned} -\delta_n + R_2\delta_b + R_3\delta_c &= 0 \\ -\delta'_n + R_2\delta'_b + R_3\delta'_c &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

Solving these for R_2 and R_3 we derive

$$\left.\begin{aligned} R_2 &= \frac{\delta_n \delta'_c - \delta'_n \delta_c}{\delta_b \delta'_c - \delta'_b \delta_c} \\ R_3 &= \frac{\delta_n \delta'_b - \delta'_n \delta_b}{\delta_c \delta'_b - \delta'_c \delta_b}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For a symmetrical truss, as here assumed, $\delta_b = \delta'_c$, and we also have, in all cases, $\delta_c = \delta'_b$, hence (9) reduces to

$$R_2 = \frac{\delta_n \delta_b - \delta_n' \delta_c}{\delta_b^2 - \delta_c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$R_3 = \frac{\delta'_n \delta_b - \delta_n \delta_c}{\delta_b^2 - \delta_c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The value of δ'_n is obtained from Fig. 12 (*a*) on the right of the centre.

From eqs. (10) and (11) the reactions R_2 and R_3 can be very quickly calculated for a unit load at any joint, and from them the true reactions for any loading. It will be noted that R_3 will be + or − according as the value of $\delta'_n \delta_b$ is greater or less than $\delta_n \delta_c$.

Applying this method to Fig. 13, the displacement diagram is first to be drawn for one pound at *g*, truss supported at *a* and *n* only. The cross-sections and lengths of the various members, excepting the centre diagonals, are given in Art. 76. To avoid the multiple system the centre diagonals will be assumed as shown in Fig. 13 and to be of 24 sq. in. each in section. The stresses *u* are then calculated, the values of $u\,l/A$, and finally the displacement diagram drawn. The values of the several deflections (times *E*) measured from this diagram are as follows, δ denoting the deflection of any joint on the left and δ' that of the corresponding joint on the right:—

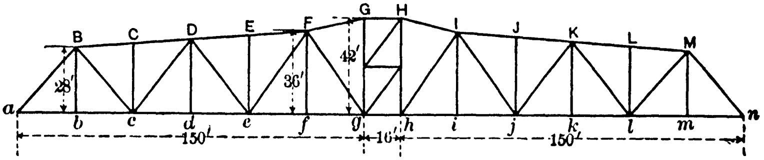


FIG. 13.

Joint	δ	δ'
b	126	119
c	235	225
d	338	322
e	416	398
f	465	447
g	480	
h	466	

Substituting in eqs. (10) and (11) we derive the following values for R_2 and R_3 for unit loads:

Joint Loaded.	R_2	R_3
b	0.378	−0.118
c	0.609	−0.122
d	0.934	−0.236
e	1.089	−0.229
f	1.137	−0.173
First span fully loaded	4.147	−0.879

From this we find that for the first span fully loaded with $P = 75{,}000$ lbs., $R_2 = 311{,}000$ lbs., and $R_3 = -66{,}500$ lbs. The total dead- and live-load reaction at $h = +98{,}000 - 66{,}500 = +31{,}500$ lbs. We also find, from moments about D or A, the true values of R_1 and R_4 for first span loaded, to be + 155,000 lbs. and − 24,500 lbs., respectively.

71. *Comparison of Results.*—The value of R_3 was found in Art. 69, by the continuous-girder theory, to be −486,000 lbs. Comparing this with the correct value of − 66,500 lbs. given above it will be seen that the continuous-girder method is quite inapplicable to such a case. Again, the value of R_4 is found by the two methods to be −24,500 and + 2,400 lbs., respectively.

It may be noted here that the method of analysis of Art. 76, for the truss partially continuous over four supports, may be applied with little error for end reactions to the case here considered. By that method the value of R_1 for first span fully loaded is 163,900 lbs., and $R_4 = -23{,}600$ lbs. $R_2 = 211{,}100$ lbs.

In trusses of this type the formulas for a two-span bridge are often used for calculating the end reactions, with fairly satisfactory results. Thus for the 1st span loaded the coefficients on page 47 give a value of $R_1 = 160{,}000$ lbs., and $R_3 = -27{,}300$ lbs., compared to values of 155,000 and − 24,500 lbs. respectively.

72. *Stresses in Centre Diagonals.*—If the reaction at *h* had the large negative value found by the application of the beam formulas the shear in span *g h* would be 484,000 lbs., giving a stress in each of the diagonals of 304,000 lbs., which would require the use of a large member as assumed. Considering the true reactions, however, the shear is equal to $R_3 + R_4 = -91{,}000$ lbs., giving a maximum stress of only 57,200 lbs. in each diagonal or only 2,400 lbs. per sq. in. for the section assumed. If smaller members had been used their *unit* stress would have been only slightly larger as the deformation is necessarily about the same. This is an important deduction, for it shows that whatever the size of member used in this panel the *unit* stress will be low.

From these considerations it is evidently advisable to employ for the diagonals of this panel, members of only sufficient sectional area to stiffen the structure, when open, against wind and any unbalanced dead load. Then in the analysis these small members may be neglected and the calculations made as explained in the following article. Large members serve no useful purpose. In recalculating old structures, where large members have been used, the foregoing analysis shows that they may likewise be neglected. In modern practice small diagonals are generally employed, making a truss partially continuous over four supports (Art. 74).

73. **Truss Continuous over Three Supports.**—To avoid the objection to the truss continuous over four supports, noted in Art. 68, some trusses have been constructed of the form shown in Fig. 14. In this form the link *E F* carries the load at the centre to the rigid frame *B F C*. The length of the panel *B C* is made equal to the width of the

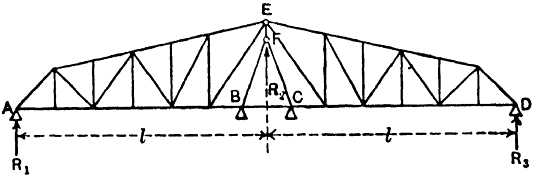


FIG. 14.

truss in order that the weight upon the turntable may be uniformly distributed. The length of the other panels in the lower chord is independent of *B C*. The analysis is precisely the same as for a simple two-span continuous girder, with centre support at *F*.

74. Truss Partially Continuous over Four Supports; Equal Moments at the Centre Support.—Fig. 15 shows a form of construction which also avoids the objection noted in Art. 68. In this form the rigid frame *E F B C* supports the truss by means of the short links *E G* and *F H*. The portion *B C* of the frame is a part of the lower chord of the bridge; in other respects the frame may be considered a part of the pier. There being no diagonals in the panel *E F H G*, there can be no shear transmitted across this panel, and the moments at *E* and *F* must always be equal.

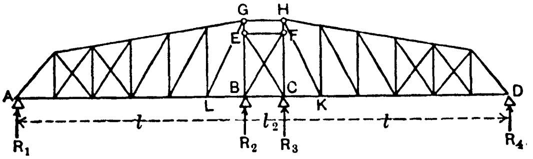


FIG. 15.

The same object may be secured in a simpler way by using full diagonal bracing (Fig. 17) of small section, as mentioned in Art. 72. These members are then neglected in the calculations, giving a condition of equal moments at the centre supports exactly as in Fig. 15.

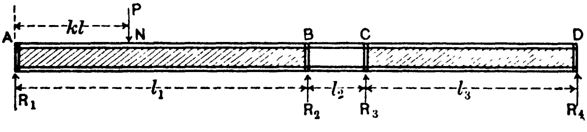


FIG. 16.

Formulas for reactions for such a case may be derived as for beams and will be found to apply with about the same accuracy as in the case of the two-span truss.

75. *Reactions for a Beam Partially Continuous over Four Supports; Equal Moments over Centre Supports.*—Fig. 16 represents a continuous beam on four supports. It is assumed, however, that throughout the middle span the beam has the same moment of inertia as elsewhere but is not capable of resisting shear, so that the bending moment is constant from B to C. The value of R_1 will be derived for a load P on the first span.

It will be convenient to solve this problem by the use of the general formula for deflection as applied to beams, derived in Chapter VII, Part I. It is

$$D = \int \frac{M\,dx}{EI} \cdot m, \quad \ldots \ldots \quad (12)$$

in which

D = deflection of any point;

M = moment at any section due to the given loads;

m = bending moment at any section due to a force of 1 pound applied at the point whose deflection is desired.

The reaction R_1 will be considered as a redundant reaction and its value determined by equating to zero the deflection of the beam at this point. Let M' = moment at any section due to the given loads, with the reaction at A removed, and m = moment due to a reaction of one pound at A. Then, as in Art. 63,

$$M = M' + R_1 m. \quad \ldots \ldots \quad (13)$$

Substituting in (12), placing $D = 0$ and solving, we have (assuming E and I constant)

$$R_1 = -\frac{\int_A^D M' m\,dx}{\int_A^D m^2\,dx}. \quad \ldots \ldots \quad (14)$$

The values of the integrals for the case of a single load P in the first span will now be determined. With the support at A removed the other reactions are determinable by statics from the condition of zero shear in the span $B\,C$. If R'_2, R'_3, R'_4, and M'_2 and M'_3 represent the reactions and moments due to the load P, with R_1 removed, we have

$$R'_2 = P. \quad \ldots \ldots \quad (15)$$

Also $M'_3 = M'_2 = -P\,l_1\,(1 - k)$, and hence

$$R'_4 = \frac{M'_3}{l_3} = -P\,(1 - k)\,\frac{l_1}{l_3}, \quad \ldots \ldots \quad (16)$$

and

$$R'_3 = -R'_4 = P\,(1 - k)\,\frac{l_1}{l_3}. \quad \ldots \ldots \quad (17)$$

The values of the quantities M' and m, and the several integrals, for the various parts of the beam $A N$, $N B$, $B C$, and $C D$ are given in the following table. The origin for $A B$ is conveniently taken at A and for $C D$ at D. Substituting the summations in (14) we derive the formula

Section.	M'	m	$M' m$	m^2	$\int M' m \, d x$	$\int m^2 d x$
$A N$	0	x	0	x^2	0	$\frac{l_1^3}{3}$
$N B$	$-P(x - k l_1)$	x	$-P(x^2 - k l_1 x)$	x^2	$-\frac{P l_1^3}{6}(2 - 3k + k^3)$	
$B C$	$-P l_1 (1 - k)$	l_1	$-P l_1^2 (1 - k)$	l_1^2	$-P l_1^2 (1 - k) \, l_2$	$l_1^2 l_2$
$C D$	$-P \frac{l_1}{l_3} (1 - k) x$	$x \frac{l_1}{l_3}$	$-P \frac{l_1^2}{l_3^2} (1-k) x^2$	$x^2 \frac{l_1^2}{l_3^2}$	$-\frac{P l_1^2}{3} (1 - k) \, l_3$	$\frac{l_1^2 l_3}{3}$

$$R_1 = P \left[(1 - k) - \frac{l_1 (k - k^3)}{2 l_1 + 6 l_2 + 2 l_3} \right]. \quad . \quad . \quad . \qquad (18)$$

If $l_1 = l_3 = l$, then

$$R_1 = P \left[(1 - k) - \frac{l (k - k^3)}{4 l + 6 l_2} \right]. \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

Compared to eq. (4), Art. 42, it is observed that (19) differs from (4) only in the addition of the term $6 \, l_2$ in the denominator.

Having determined R_1, the following formulas are readily derived.

$$R_2 = P - R_1 = P \left(k + \frac{l_1 (k - k^3)}{2 l_1 + 6 l_2 + 2 l_3} \right). \quad . \quad . \qquad (20)$$

$$R_3 = - R_4 = P \frac{l_1 (k - k^3)}{2 l_1 + 6 l_2 + 2 l_3} \cdot \frac{l_1}{l_3}. \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

76. Example of Truss Bridge, Partially Continuous over Four Supports.—A complete analysis will be made of the truss of Fig. 17. The diagonals in the centre panel are of small cross-section and will be omitted in the calculations, thus making the truss partially continuous over four supports. The formulas of Art. 75 will be used, after which the true reactions will also be determined by the method of redundant members.

Assume a dead load of 1,325 lbs. per foot or 33,100 lbs. per joint. A load of $\frac{3}{4} W$ will be assumed for joint *a*. For live load use Cooper's *E*-50 loading with 50% increase for impact on all members excepting vertical hangers, which

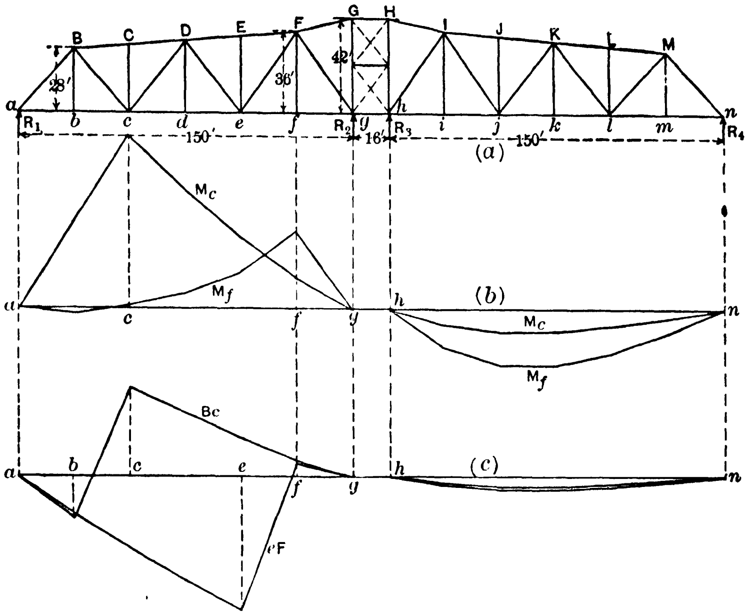

A B C D E F G H I J K L M
a b c d e f g h i j k l m n
28′
36′
42′
R₁ R₂ R₃ R₄
150′
16′
150′
(a)
Mc
Mf
(b)
Bc
(c)
cF

FIG. 17.

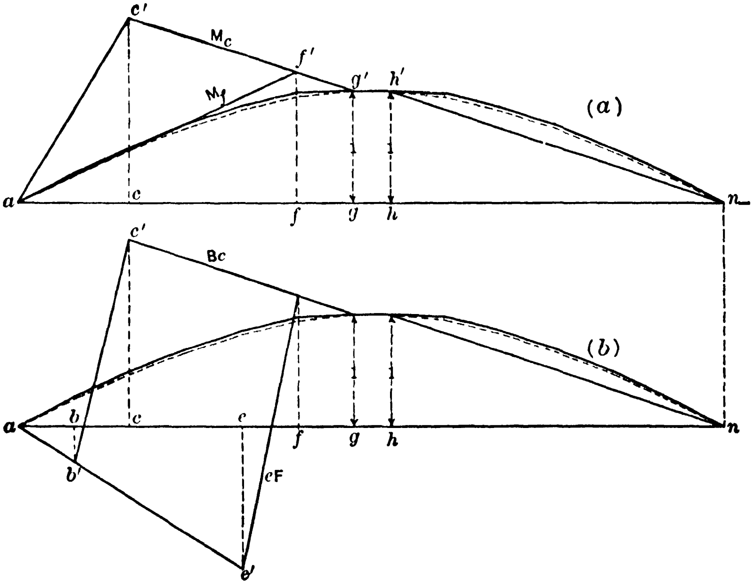

c′
Mc
f′
Mf
g′ h′
(a)
a c f g h n
c′
Bc
(b)
a b c e f g h n
b′
e′
cF

FIG. 18.

need not be considered in this analysis. This is equivalent to the use of loading of *E*-75 without impact.

Reaction Coefficients. Applying eqs. (19) and (21) we have the following values of left reactions for unit loads:

Joint Loaded.	R_1	Joint Loaded.	R_1
b	.798	*i*	− .0549
c	.603	*j*	− .0797
d	.419	*k*	− .0808
e	.254	*l*	− .0638
f	.112	*m*	− .0349
	Σ = 2.186		Σ = − .3141

Dead-Load Stresses.—For Case I, the structure is a double cantilever. The stresses are given in Col. (2) of Table *B*. For Case II, the truss is a continuous girder and $R_1 = (2.186 - .314) \times W = 61{,}960$ lbs. The stresses are given in Col. (2) of the table on p. 85.

Live-Load Stresses.—These are found from influence lines drawn for moments and vertical components of web stress. Fig. 17 shows the influence lines for moments at *c* and *f* and for vertical components of stress in *B c* and *e F*. Having these lines drawn the stresses are found by trial, using unbroken loads. The results are given in Cols. (4), (5), and (6) of the table. Col. (7) gives the stresses when considered as a simple span, also determined from influence lines.

Combination of Stresses.—The combinations of stresses are made as in Art. 60. Cols. (8) and (9) taken together, and Cols. (10) and (11), give the greatest range of stress which can occur during the passage of a single train. It should be remembered that Col. (5) can be combined with Col. (3) only, as the structure becomes a simple span when ends are not raised and one arm only is loaded. In Col. (10) several of the values are a maximum for Case I alone.

Sectional Areas.—From the maximum and minimum stresses here found the cross-sections of the several members have been determined, using the following working stresses:

for tension, 16,000 lbs. per sq. in. on net section;

for compression, $16{,}000 - 70\dfrac{l}{r}$.

For alternating stresses add 50 per cent of the lesser to each stress and take the greater of the two areas so determined. The results are as follows:

Member.	Gross Area sq. in.	Member.	Gross Area sq. in.
a b c	27.0	*a B*	18.8
c d e	36.6	*B c*	19.8
e f g	27.5	*c D*	19.8
g h	41.3		
B C D	33.0	*D e*	30.3
D E F	25.0	*e F*	39.9
F G	39.8	*F g*	45.1
G H	36.0	*G g*	19.8

TABLE B

STRESSES IN A RIM-BEARING SWING BRIDGE

Note: + = tension − = compression.

Member	DEAD LOAD.		LIVE LOAD.				COMBINATIONS.			
	Case I Cantilever	Case II Cont. Gird.	Case III $+ M$ and $+ V$ Cont. Girder.	Case IV $- M$ and $- V$ Cont. Girder.		Case V $+ M$ and $\pm V$ Simple Span.	Dead Load Case II with Live Load Case III or IV.		Dead Load Case I with Live Load Case IV or V.	
				One Arm Loaded.	Both Arms Loaded.		II and III for *Max.* $+ M$ and $+ V$.	II and IV for Reversal.	I and IV or V for *Max.* $- M$ and $- V$.	**I and V for Reversal.**
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
a b c	− 22,200	+ 55,300	+ 237,800	− 32,700		+ 270,900	+ 293,100	+ 22,600	− 22,200	+ 248,700
c d e	− 138,300	+ 67,500	+ 326,900	− 85,700		+ 412,500	+ 394,400	− 18,200	− 138,800	+ 273,700
e f g	− 316,000	− 15,650	+ 81,600	− 126,500		+ 210,600	+ 66,000	− 142,100	− 316,000	− 105,400
B C D	+ 69,100	− 75,400	− 326,200	+ 61,100		− 388,500	− 401,600	− 14,300	+ 69,100	− 319,400
D E F	+ 219,600	− 27,100	− 234,000	+ 107,800		− 343,500	− 261,100	+ 80,700		− 123,900
F G	+ 385,000	+ 76,300			+ 237,300			+ 313,600	+ 622,300	
a B	+ 33,200	− 85,400	− 360,300	+ 48,400		− 406,000	− 445,700	− 37,000	+ 33,200	− 372,800
B c	− 72,000	+ 29,300	+ 173,200	− 42,300		+ 230,700	+ 202,500	− 13,000	− 72,000	+ 158,600
c D	+ 112,700	+ 5,100	− 99,800	+ 105,500		− 126,700	− 94,700	+ 110,600	+ 112,700	− 14,000
D e	− 143,700	− 52,900	+ 37,000	− 193,700	− 196,800	− 169,700	− 15,900	− 249,700	− 340,500	− 313,400
e F	+ 174,200	+ 89,800	− 11,450	+ 311,700	+ 334,600	+ 278,300	+ 78,400	+ 424,400	+ 508,800	+ 452,500
F g	− 119,600	− 95,200			− 400,800	− 369,800		− 496,000	− 520,400	− 489,400
G H	+ 375,000	+ 73,400			+ 204,300			+ 277,700	+ 579,300	
g h	− 375,000	− 73,400			− 204,300			− 277,700	− 579,300	
G g	− 90,000	− 17,600			− 55,400			− 73,000	− 145,400	

77. True Reactions for Trusses Partially Continuous over Four Supports.—The method of redundant members is applied to this type of truss in the same way as for the two-span bridge, Art. 63. Consider R_1 as redundant. Then with this support removed the other reactions are determined by statics. For any load P on the first span these reactions are given by eqs. (15)-(17). For a load P on the third span, $R_2 = 0$ and R_3 and R_4 are the same as for a simple span. The stresses S' of eq. (1), Art. 63, are thus readily determined, and likewise the stresses u due to a one-pound reaction at A. The value of R' is then given by eq. (1).

The graphical process may be employed in the same manner as in Art. 63. Where the spans are equal, as in Fig. 17, the two centre reactions may be taken as redundant and a displacement diagram drawn for a load of one pound placed at each of the points g and h, considering the truss supported at the two ends. From this diagram the deflection curve of the lower joints with respect to the ends a, n, may be constructed as in Art. 64. Taking the deflection at g and h equal to unity this becomes a reaction influence line and may be used directly for constructing influence lines in the same manner as in Fig. 10. Such a construction has been carried out for Fig. 17, and Fig. 18 shows the reaction curve and the influence lines plotted thereon, for moments at c and f, and for V. comp. of stress in $B\,c$ and $e\,F$. The dotted line is the reaction line calculated by the beam formulas of Art. 75. The diagram shows clearly the error involved in the use of the usual formulas. It amounts, for example, to a maximum of 4.6 per cent for moment at c and 5.6 per cent for stress in $B\,c$.

The scale of the influence diagram for moments is found by multiplying the ordinate by $a/2$ as in Art. 64, where a is the distance of the moment centre from the left end. The correctness of the construction is shown as follows: Consider any moment centre E, Fig. 19, and a load P_1 distant $k\,l$ from A. Let M_2 = moment at B due to this load. Then

$$R_1 = P_1(1 - k) - \frac{M_2}{l} \text{ and}$$

$$M_E = R_1 a = \left[P_1(1 - k) - \frac{M_2}{l}\right] . a. \qquad (a)$$

Now if a second load P_2 be applied on the second span, distant $k\,l$

from D, the moments at B and C will be double their value for a single load. By the graphical analysis above given the ordinate $ff' \times P$ is the value of R_2 and R_3 for these two loads. The value of R_1 is $P\ (1 - ff')$, and the moment at B for the two loads $= R_1\,l - P\,(l - k\,l) = P\ (1 - ff')\ l - P\ (l - k\,l)$. For a single load the moment is one-half of this, or $M_2 = \frac{P_1\,(1 - ff')\,l - P_1\,(l - k\,l)}{2}$.

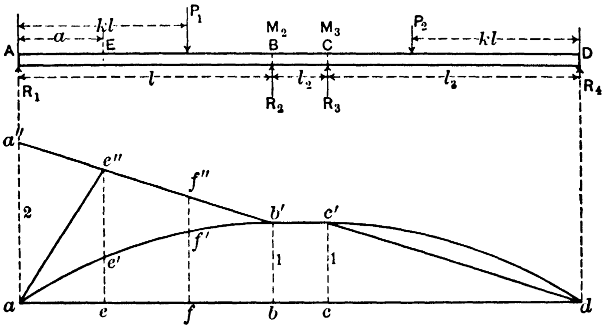


FIG. 19.

Substituting in (a) and reducing we then have

$$M_E = P_1\,[2 - k - ff']\,a/2. \quad . \quad . \quad . \quad . \quad (b)$$

The ordinate $f'f''$ is equal to $(2 - k - ff')$, hence for unit loads the moment $M_E = f'f'' \times \frac{a}{2}$ as already stated. For shear, the ordinates are to be multiplied by ½, and for Vert. comp. web stress in Fig. 17 the factor is $\frac{1}{2} \times \frac{s}{t}$, where s = distance of point of intersection of chords to the left of a and t = distance from this intersection to the foot of the diagonal in question. (See Part I, Art. 162.)

Where the spans are unequal it is simpler to draw the displacement diagram for unit load at one end, as a, and from this construct the deflection diagram for points b and c with respect to a line joining a and d. The resulting diagram may be used in all respects as shown in Fig. 18.

78. Rim-bearing Turntable; Four Supports.—*Equal Moments at* *the Centre Supports.*—Fig. 20.—In this the two triangular frames *B H C* and *C I D* support the truss by means of the short links *F H* and *G I*. The portions *B C* and *C D* of the frames are a part of the lower chord of the bridge. The point *C* is common to both frames. The inclinations of the members *B H*, *H C*, *C I*, and *I D* are such that under maximum loads at *H* and *I* the supports *B*, *C*, and *D* are equally loaded. This arrangement brings the weight on the turntable uniformly distributed over six points. These points can be spaced equal distances apart on the turntable. Under all conditions, the moments at the two centre supports *F* and *G* are always equal. The analysis will then be the same as in Art. 74.

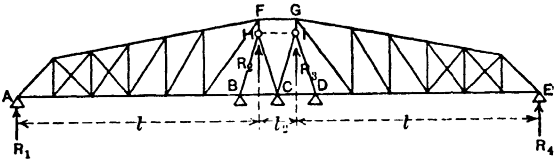


FIG. 20.

79. Lift Swing Bridges.—Various devices have been suggested whereby the bridge may be made continuous when being opened and two simple spans when closed. Fig. 21 shows a form in which, when the bridge is to be swung, the supports at *B* and *C* are lifted far enough to bring the links *E F* and *F G* into action and to raise the ends *A* and *D* from their supports. All the weight is then at the centre and the bridge is swung on a centre-bearing pivot. When closed, and the supports *B* and *C* lowered, the links *E F* and *F G* are under no stress. The analysis then consists in finding the dead-load stresses when open, as in the other forms, and the dead- and live-load stresses when closed, the bridge then consisting of two simple spans. Other forms designed

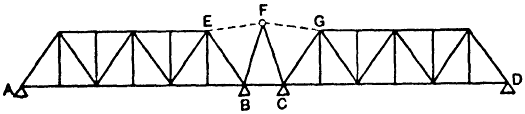


FIG. 21.

to accomplish this object employ toggle joints in the members $E F$ and $F G$, or at other points of the upper chord, whereby it is shortened and the ends of the span lifted free of the supports.

80. Double Swing Bridges.—In a few cases where the clear span required is large two swing spans have been arranged as shown in Fig. 22, giving a clear opening from C to E. When closed the ends at D are locked so that the cantilever ends are forced to deflect equally. Except for this connection the reactions and stresses would be determined from statics, as in the cantilever bridge. The problem will be solved by determining the reaction at D for a single load placed at any point on one of the structures.

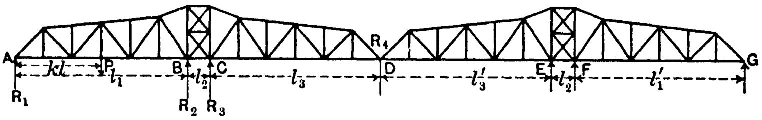


FIG. 22.

Suppose the joint at D not to exist. The structure $A B C D$ will then be statically determinate, and the reactions and deflection at any point may be determined by methods explained in Art. 77. Likewise the structure $D E F G$ will be statically determinate and its deflections may be found. Under these conditions let the following deflections be determined: (1) The deflection at D of the structure $A D$, for a single load P placed at any point of $A D$; (2) the deflection at D of the structure $A D$ for a load of one pound applied at D; and (3), the deflection at D of the structure $D G$ for one pound placed at D. Represent these deflections by δ'_n, δ'_d, and δ''_d, respectively. If the two structures now be connected the common point D will deflect a distance δ_n under the load P, which will be less than δ'_n by reason of the restraining action of the structure $D G$.

Let R_4 be the reaction at D, assumed upward on $A D$. Then we have, with respect to the structure $A D$,

$$\delta_n = \delta'_n - R_4 \delta'_d,$$

and with respect to the structure $D G$

$$\delta_n = R_4 \delta''_d.$$

Eliminating δ_n we have

$$R_4 = \frac{\delta'_n}{\delta'_d + \delta''_d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

When δ'_n is downward then R_4 is positive, or upward on $A\ D$. The values of R_4 being known the other reactions are found by statics.

If the two structures are alike and symmetrical about D, then $\delta'_d = \delta''_d$ and we have $R_4 = \frac{1}{2}\frac{\delta'_n}{\delta'_d}$; but $\frac{\delta'_n}{\delta'_d}$ is the right reaction for a swing bridge, $A\ D$, on rigid supports, hence the reaction at D is one-half that of a structure on rigid supports.

For a symmetrical double bridge of the form considered in Art. 74, the value of R_4, for a single load on span $A\ B$ is, therefore,

$$R_4 = -\frac{1}{2} P \frac{l_1 (k - k^3)}{2 l_1 + 6 l_2 + 2 l_3} \cdot \frac{l_1}{l_3}, \quad . \quad . \quad . \quad . \qquad (23)$$

and for a single load on span $C\ D$, distant $k\ l$ from D

$$R_4 = \frac{1}{2} P \left[(1 - k) - \frac{l_1 (k - k^3)}{2 l_1 + 6 l_2 + 2 l_3} \right]. \quad . \quad . \qquad (24)$$

81. Stresses in Lateral Trusses.—The same general arrangement of lateral bracing is employed as in the ordinary through bridge. The loads upon the upper lateral system are transferred to the lower chord by portals at the ends and centre. Where the arrangement of the main truss provides a diagonal compression member in the panel next the centre, as member $h\ I$, Fig. 17, the portal is placed in the plane of this member and the upper laterals may be considered as terminating at I. The tower is then braced independently. The lower lateral system is complete from end to centre.

Owing to the relative flexibility of a portal frame the upper lateral system is little affected by the continuity of the structure and may be treated in the same manner as in a simple span $h\ n$, whether the bridge is open or closed. The lower lateral system requires consideration of continuity. The maximum shears near the end will occur for bridge closed and live load covering one span, corresponding to Cases II and III of the analysis for the vertical truss. For the shears near the centre the maximum may occur for bridge open, or for bridge closed and loaded with live load. In the first case the shears are determined as in Case I, but to these should be added the shear due to one-half the load on the upper lateral system. For the case of bridge closed the shears are determined as for Case II and Case IV for main trusses.

The wind pressures assumed are generally on the basis of 50 lbs. per sq. ft. on truss alone when closed; and 30 lbs. per sq. ft. on truss when open, or on truss and train when closed and loaded.

Chord stresses resulting from wind loads should be considered where they increase the chord stresses in the main trusses.

CHAPTER III

CANTILEVER BRIDGES

82. General Arrangement of Spans.—A cantilever bridge is one in which one or more of its trusses are extended beyond their supports thus forming cantilever arms. The ends of these cantilever arms then furnish supports for other trusses. In the usual form each complete truss has only two supports and therefore the reactions and stresses are statically determined. Two general arrangements of spans are illustrated in Figs. 1 and 2. Fig. 1 illustrates the arrangement used in the case of numerous spans and Fig. 2 the more usual case of a crossing requiring one long span and two short ones. In Fig. 1 the structure consists of a series of main trusses *A C*, *D G*, etc., supported so as to form two cantilever arms. Then, resting upon these cantilever arms, are simple trusses *C D*, *G H*, etc. In Fig. 2 the main trusses are *A C* and *D F*. They rest upon supports at *A*, *B*, *E*, and *F*, and are cantilevered at one end only. Obviously a combination of the single and double cantilevers may be the best arrangement in some cases. Fig. 3 represents the arrangement of the Niagara bridge, and Fig. 2 several Ohio River bridges. The Mississippi River bridge at Memphis consists of one double cantilever similar to *D G*, Fig. 1, one single cantilever like *A C*, Fig. 2, and two simple spans, one forming a shore span and the other a suspended span between the cantilevers. The Thebes bridge over the Mississippi River consists of two double cantilevers and three simple spans.

In Fig. 1, the spans *A B* and *E F* are called *intermediate* spans, the spans *B C*, *D E*, *F G*, etc., *cantilever* spans, or *cantilever arms*, and *C D* and *G H*, *suspended* spans. In Fig. 2 spans *A B* and *E F* are called *anchor* spans instead of intermediate spans. The points *A* and *F* require anchorage to balance the loads on *B C* and *D E*.

83. Advantages of the Cantilever Bridge.—The chief advantage of the cantilever type for spans of ordinary lengths is in the fact that certain spans can be erected without the use of falsework by building out from the portion already constructed. Thus in Figs. 1 and 2, spans *A B* and *E F* would first be erected on falsework, as usual. This being done the cantilever arms *B C* and *E D* are constructed by building out from the points *B* and *E*. These arms being designed to act as cantilevers when in service their construction in this manner involves no unusual erection stresses. The points *C* and *D* being reached the suspended span is then built by continuing the same mode of construction from both sides until connection can be made at the centre. Temporary members are used at the ends *C* and *D* so that the two ends of the simple span may act temporarily as cantilevers. When connection is made at the centre these temporary members are disconnected so that the span *C D* is free to turn at points *C* and *D* and also has free horizontal motion at one end, thus becoming a simply-supported span. In the case of the suspended span the erection stresses need to be specially considered as the chord stresses are of different character from those which occur after erection.

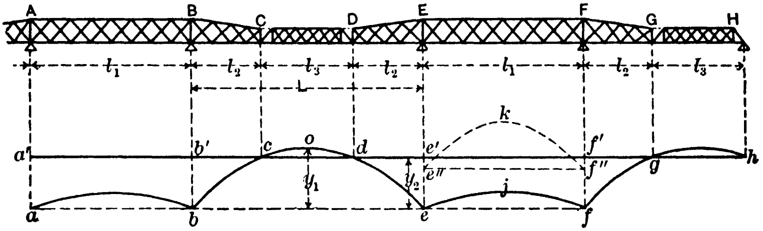


FIG. 1.

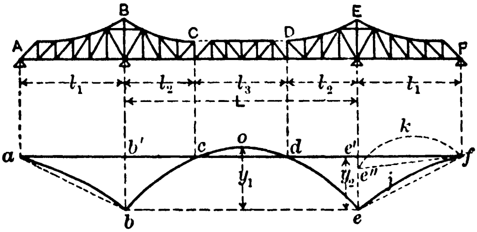


FIG. 2.

The advantages in regard to erection which the cantilever type offers make it especially applicable to crossings where falsework is not practicable, as at the Niagara gorge, or where the depth of channel or navigation interests makes it especially expensive or undesirable.

For spans of ordinary length (up to 500–600 feet), the cantilever type affords no economy over the simple truss, except in the case of expensive falsework, as noted above. For very long spans, however, where the dead load becomes relatively large, the cantilever type becomes the more economical, the advantages in this respect becoming greater the longer the span. A bridge of this type can thus be built of a span-length considerably greater than would be possible for a simple truss.

This gain in economy where the dead load is relatively large is due partly to the fact that the average moments for uniform loads are less than in the simple truss, and partly to the fact that the dead weight in the cantilever type is concentrated largely at points near the support, thus tending to reduce the moments still further. The wind-pressures on the structure will also be concentrated in the same favorable manner. On the other hand, the alternating stresses in the intermediate or anchor spans of the cantilever bridge tend to increase the amount of material required and so reduce the relative economy.

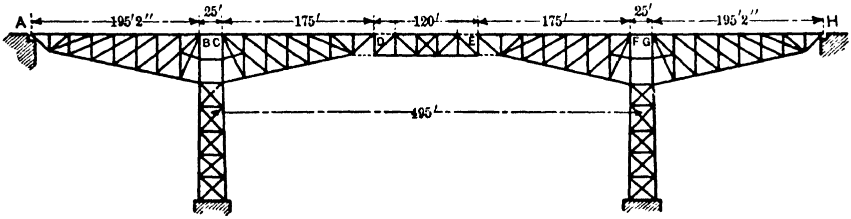


FIG. 3.

An unfavorable element of some importance is the large deflections of the cantilever ends as compared with those of a simple span. Short cantilever arms and long suspended spans are advantageous from this standpoint.

The distribution of moments in a cantilever bridge is much the same as in a continuous girder, but in the former the points of inflection are fixed by the hinges at the ends of the suspended spans. The variation in maximum moments throughout a cantilever structure is shown by the moment diagrams of Figs. 1 and 2. The maximum moments in the suspended and cantilever arms occur for a full load on these spans. For a uniform loading the moment curve for the entire span *B E* will be a parabola, *b c d e*, with centre ordinate $= y_1 = w L^2/8$, the same as for a simple beam of length *L*. The portion *c d* gives the positive moments in the suspended span and the portions *b c* and *d e* the negative moments in the cantilever spans. For the intermediate or anchor arm *E F*, the maximum moments are generally the negative moments which occur when the live load extends over the suspended and cantilever arms. Were there no dead load on *E F* the negative moment curve *e f* would be a straight line; the effect of the dead load is to give to this line a flat curvature, the centre ordinate to which is the dead load positive moment at the centre of the span. The dotted curves $e'' k f''$ (Fig. 1) and $e'' k f$ (Fig. 2) represent the moments for span *E F* fully loaded and show the maximum positive values.

The maximum moments in a series of simple spans, *B E* and *E F*, are shown by the curves *b o e* and $e'' k f''$ or $e'' k f$, measured from the lines *b e* and *e f* as axes. The relative chord sections required in the two types of structures is indicated to a certain extent by the areas of the respective moment diagrams. The sum of the moment areas for the span *B E* is seen to be much less for the cantilever than for the simple span, but for the span *E F* it is somewhat greater.

The moment areas of Figs. 1 and 2 depend to a considerable extent upon the relation between the span lengths *B E*, *E F*, and *C D*. The longer *B E* as compared with *E F* the greater will be the moment ordinate y_1, and the greater the negative moments in span *E F*, but if *E F* is made too long then the positive moments therein become too large. Generally *E F* is made from one-half to two-thirds of *B E*, if not determined by special considerations such as favorable pier locations, width of channel span, etc. Negative reactions in the type shown in Fig. 1 are to be avoided.

The spans *L* and l_1, being assumed as fixed, the length of the suspended span *C D* for maximum economy may be investigated. The position of the ends of the span *C D* fix the points of zero moment and therefore determine the location of the axis $b' e'$ of the moment diagram.

Let y_1 = central ordinate of parabola *b c d e* and y_2 = ordinate at *e*. The sum of the positive and negative moment areas of the span *B E* is equal to (area rectangle *b′ b e e′*) − (area parabola *b c o d e*) + 2 × (area parabola *c o d*) $= y_2 L - \frac{2 y_1 L}{3} + \frac{4 (y_1 - y_2) l_3}{3}$. Considering *e f* a straight line, the moment area for span $e f = y_2 l_1$ (includes both anchor spans of Fig. 2). We have also the relation $\frac{y_1 - y_2}{y_1} = \frac{l_3^2}{L^2}$. Substituting and differentiating with respect to l_3 we find for minimum sum of moment areas

$$l_3 = \tfrac{1}{2} (L + l_1).$$

Considering the span *L* only, it is found that for minimum moment areas $l_3 = \frac{1}{2} L$. The shears in the central span do not vary greatly for considerable variations in ratio of span lengths, but, on the whole are less the shorter the suspended span.

84. Analysis.—The suspended span being a simple truss, its analysis for dead and live load requires no special consideration. Erection stresses need to be considered, these being calculated with each half-span treated as an extension of the cantilever arms. The stresses in the cantilever arm and in the anchor or intermediate span will be discussed in detail. It will serve to illustrate the entire problem to consider a structure of the form shown in Fig. 4.

a. The Cantilever Arm.—Consider the cantilever arm *D E*. The dead-load moments and shears are readily found by first getting the amount of load transferred to *E* from the suspended span and then taking shears, or moments of joint loads, on the right of any section. For live-load stresses the maximum moment and shear at any section in the arm *D E* will result when the span *D F* is fully loaded, although loads to the left of any given section do not affect the stresses of that section.

Influence lines for moment at *I* and shear in panel *I K* are shown in Figs. 4 (*b*) and (*c*). The effect of a load unity on the span *E F* is evidently proportional to the reaction caused at *E*, both as to moment and shear, hence the lines *F′E″* are straight lines, with ordinates at *E′* equal to *a* and unity, respectively. As the load moves from *E* to *I* the resulting bending moment decreases uniformly to zero when the load reaches I. The shear remains equal to unity until the load reaches K and then decreases to zero as the load crosses the panel $K\ I$. For maximum moment at I the criterion is the same as for moment at E in a simple beam $I\ F$. For shear, the same position of loads may be used as for shear in the end panel of a simple truss extending from I to F.

b. The Intermediate Span.—The dead-load moments and shears due to loads on this span are the same as in a simple truss. To these must then be added the moments and shears due to the action of the cantilever arms. Let M_D = moment at D, found from the cantilever arm $D\ E$, and M_C the moment at C, found from the arm $B\ C$. Then at intermediate points in $C\ D$ the moment due to the cantilevers will vary from C to D uniformly. Hence at a distance a from C the moment is $M_G = M_C + (M_D - M_C)\,\frac{a}{l_3}$. The same result will of course be obtained by calculating first the total reaction at C and taking moments of all forces on the left, about the given centre of moments.

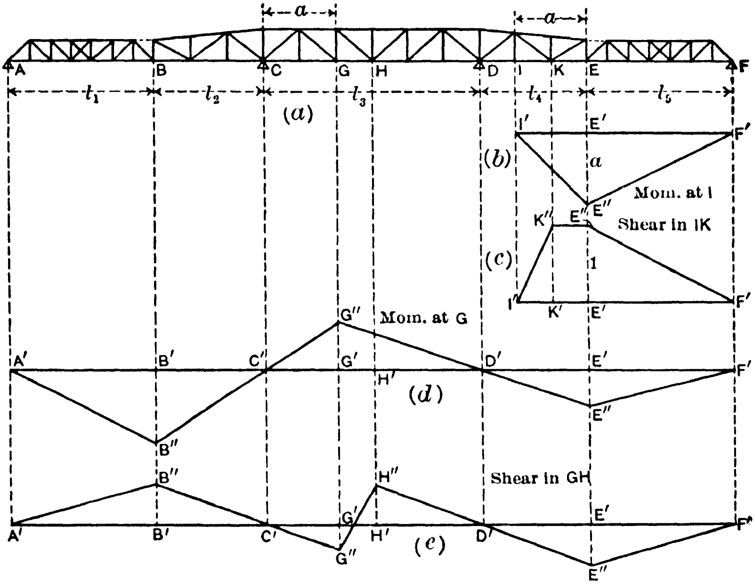


Fig. 4.

The shear due to the cantilever loads will be constant throughout span $C D$ and is to be added to the shears due to loads in $C D$. This constant shear is equal to the reaction at C minus the loads on the cantilever to the left. For symmetrical conditions this shear = 0, and $M_C = M_D$.

For live-load moments and shears, construct the influence lines for moment at G and shear in panel $G H$.

Moment at G.—Fig. (*d*).—A load unity at E causes a negative reaction at C equal to $\frac{l_4}{l_3}$, and hence a negative moment at G equal to $\frac{a l_4}{l_3}$, which is laid off as $E' E''$ in Fig. (*d*). As the load moves to D or F, the moment at G decreases uniformly to zero and the influence line for this portion is $F' E'' D'$. As the load moves from D to G, the moment increases from zero to a value of $\frac{a (l_3 - a)}{l_3}$ at G; beyond G the moment decreases again, becoming zero when the load is at C, then $\frac{l_2}{l_3} (l_3 - a)$ when at B, and finally zero for load at A. Since the ratio of $E'' E'$ to $G'' G'$ is equal to $\frac{l_4}{l_3 - a}$, it follows that $G'' D' E''$ is a straight line. Similarly, $G'' C' B''$ is a straight line. The influence line shows that for a maximum positive moment at G the span $C D$ should alone be loaded, and that for a maximum negative moment the spans $A C$ and $D F$ should be loaded.

If concentrated loads are used the criteria are evident from the diagrams. For positive moment it is the same as for moment at G in a beam $C D$, and for negative moments it is the same as for moment at B in a beam $A C$, and at E in a beam $D F$.

Shear in panel G H.—Fig. (*e*).—The portion $D' H'' G'' C'$ is the same as for a discontinuous span. Between D and F the shear in $G H$ is equal to the reaction at C caused by the load, and is negative, having a value of $\frac{l_4}{l_3}$ for unit load at E. When the load is between A and C the shear is positive and equal to the negative reaction at D. As before, the lines $B'' C' G''$ and $H'' D' E''$ are straight lines. The position of loads for a maximum positive or negative shear is evident from the diagram and also the exact positions of concentrated loads, if desired.

c. The Anchor Span.—If the portion $A C$ be omitted in Fig. 4 the span $C D$ becomes an anchor span. The influence lines are unchanged and the stresses will remain the same excepting those due to the cantilever arm $B C$, which are now zero. The maximum negative reaction at C is required in order to determine the anchorage which will be necessary.

85. *Equivalent Uniform Loads.*—In calculating stresses in large cantilever bridges some equivalent uniform load is usually employed. The influence lines will assist in arriving at a suitable value for such uniform load, as they show clearly the length of load or portion of structure which is significant. The method explained in Art. 173, Part I, will enable the equivalent load for any given system of concentrated loading to be determined for any case. Thus for negative moments throughout span $C D$, due to loads on $D F$, the equivalent load is that for moment at E in a span $D F$, or at a point $l_4/(l_4 + l_5)$ from the end in a span of length $(l_4 + l_5)$. Suppose $l_4 = 200$ feet and $l_5 = 300$ feet. Then $l_4/(l_4 + l_5) = 0.4$ and from Fig. 4, p. 67, the equivalent uniform load for Cooper's E-50 loading for the 0.4 point in a beam 500 feet long = 2,600 lbs. per ft. This loading will give exact result.

86. Divided Supports at the Piers.—For convenience of details the support at a pier is sometimes arranged as a double support, like the centre supports of a certain type of swing bridge (see Fig. 15, Chapter II). Such a form is used in the Niagara bridge, Fig. 3. The diagonals are omitted over the support so that no shear can be transmitted and the moments at each of the two points over the support are equal. The same arrangement may be made at both supports in the type of Fig. 1. In the calculations of stresses the panel over the support may be neglected and the calculations made exactly as for the case already explained. The moments throughout the structure will be the same and therefore also the shears. The load carried by each of the two supports will then be equal to the respective shears in the adjacent panels, plus any joint load applied at the supports. The sum of the two shears will be the total pier reaction as found in the other form.

87. Deflection of Cantilever Bridges.—The deflection of a cantilever truss may be calculated by the usual formula $\Delta = \Sigma \frac{S u l}{E A}$, or by the Williot diagrams. The end of the cantilever arm is subject to a relatively large deflection, as its movement is due not only to the deformations of the members of the cantilever arm itself, but to the deformations also of the members of the anchor or suspended span which act to cause a change of angle in the truss at the pier. Some idea of the relative deflections of a simple span and a cantilever can be obtained by means of the deflection formulas for beams. Consider a load P placed at C, Fig. 5. The deflection of C relative to the tangent at $B = \Delta' = \frac{P l_2^3}{3EI}$, and that of A is $\Delta'' = P\frac{l_2}{l_1} \cdot \frac{l_1^3}{3EI}$. The effect of the distortion in $A\,B$ is to cause the tangent at B to change angle an amount equal to $\frac{\Delta''}{l_1}$, which causes an additional deflection of C equal to $\frac{\Delta''}{l_1} \times l_2$, hence the total movement of C is

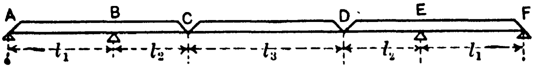


FIG. 5.

$$\Delta' + \Delta''\frac{l_2}{l_1} = \frac{P l_2^2 (l_2 + l_1)}{3EI}.$$

A simple span of length $2\,l_2$ will deflect at the centre a distance $\frac{P l_2^3}{6EI}$. If $l_2 = l_1$, the deflection of the cantilever becomes $\frac{2\,P l_2^3}{3\,E\,I}$, or four times as much as a simple span of length $2\,l_2$. Short cantilever arms and long suspended spans are therefore favorable from the standpoint of deflection. An especially unfavorable condition is such as occurs in the Niagara bridge, Fig. 3, where the deformation of the metal towers gives rise to still greater deflections.

88. Cantilever Bridge Without Suspended Span.—If the suspended span be omitted and the ends of the cantilevers connected, the structure becomes a partially continuous girder and is no longer statically determinate. For two such cantilevers the reactions are found as explained in Art. 80, Chapter II. For additional spans so connected the number of redundant reactions becomes greater, requiring the solution of two or more general equations as illustrated in Art. 70. The Blackwell's Island bridge, New York City, is of this type.

89. Wind Stresses.—The wind pressure is carried to the supports by means of lateral trusses arranged according to the same cantilever system as the vertical trusses. The stresses are therefore determined in the same manner as explained in the preceding articles.

CHAPTER IV

ARCH BRIDGES

SECTION I.—GENERAL CONSIDERATIONS

90. An Arch as distinguished from a simply supported structure, is a beam or truss whose general form is that of a curve, or arch, and which is so supported on its abutments that horizontal as well as vertical motion is resisted. The reactions are, therefore, in general, not vertical but are inclined, the horizontal components of which act in a general direction toward each other and constitute the horizontal thrust of the abutment upon the arch.

In the case of roof arches the horizontal reactions are generally supplied by means of a tie rod uniting the two supports, thus relieving the foundations of this duty.

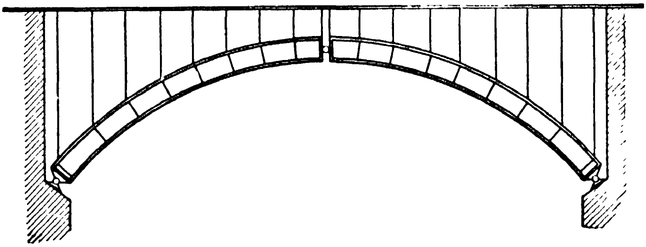


FIG. 1.

91. Kinds of Arches.—Arches of timber or metal may consist of curved beams with solid webs and flanges, or they may be curved trusses with upper and lower chords and web members, either riveted or pin-connected.

With reference to the ordinary modes of support arches may be—

1st. Hinged at the abutments and at the crown.

2d. Hinged at the abutments and continuous throughout.

3d. Fixed rigidly to the abutments and continuous throughout.

Figs. 1, 2, and 3 illustrate three forms of the three-hinged arch;

Fig. 4 illustrates a two-hinged arch, and Fig. 5 an arch with no hinges. In Fig. 1 the arch is made in the form of a curved plate girder. Two-hinged arches are also often built in this manner. Fig. 2 is a roof arch in which the horizontal reactions are furnished by a tie rod. Occasionally such a tie rod is employed in bridge construction. Fig. 3 is a *spandrel braced* arch, the bracing occupying the entire space up to the roadway. This form is also well adapted to the two-hinged arch. Arches of one hinge, placed at the crown, have been constructed, but they are not advantageous and will not be considered in this work.

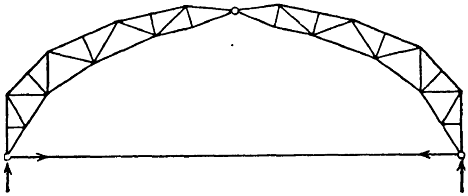


FIG. 2.

In Figs. 1, 4, and 5 the roadway is supported on the arch by means of vertical members, which serve merely to transmit the loads to the arch at certain joints or load points. Arch ribs as well as braced arches are generally loaded at certain load points only, as in the case of the simple truss, or the plate girder with steel floor system. In certain cases it is expedient to suspend the roadway from the arch, as indicated in Fig. 4.

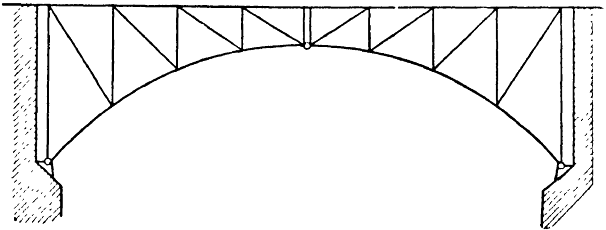


FIG. 3.

92. *Masonry Arches* of concrete or stone masonry are often built with two or three hinges, but generally with no hinge. While arches of brick or stone masonry can hardly be considered as arch ribs, those of solid or reinforced concrete may be classed as such. In any case the most satisfactory method of analysis of such structures is by the

so-called "elastic theory" as developed hereafter for the arch rib in general.

The masonry arch may support the load continuously through the medium of spandrel filling of earth or other material, or the roadway may be supported on piers or spandrel arches, as in the metallic bridge, thus concentrating most of the load at certain load points.

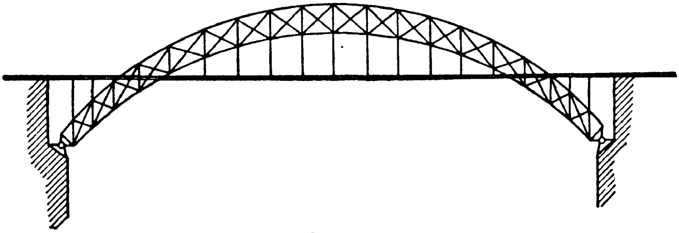


FIG. 4.

93. **Loads and Reactions.**—The methods of supporting the roadway of an arch bridge have been mentioned in Art. 91.

The amounts of the dead, live, and wind loads are fixed upon in the same manner as for other structures. For long-span structures the live load is generally assumed as a uniform load, but as arches are sometimes built of spans of very moderate length, concentrated loads may need to be considered. The use of influence lines enables this to be done with little more labor than is involved in the analysis for uniform loads.

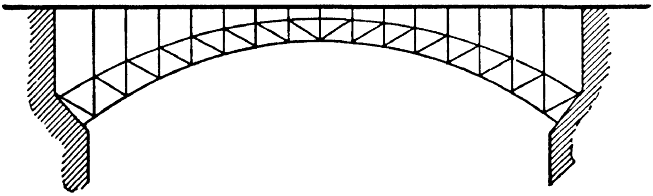


FIG. 5.

The reactions are generally considered as resolved into two components, vertical and horizontal. In the arch with hinged ends these four components fully determine the reactions (Fig. 6). In the arch with fixed ends (Fig. 7), the reactions also include moments M_1 and M_2. In the three-hinged arch the four reactions may be determined by the principles of statics, as explained in Part I, Chapter II, by reason of the fact that the moment is zero at the centre hinge. In the arch of two hinges there are four unknowns and in the arch without hinges, six. In both of these types the reactions cannot be determined by statics alone as they are dependent upon the flexibility of the arch and hence upon its form and dimensions. To solve these cases requires the consideration of the deformations of the structure as in the case of the continuous girder.

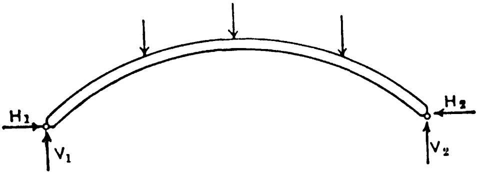


FIG. 6.

In the process of such an analysis, formulas may be developed for arches of solid beams of certain definite forms and proportions. These formulas may then be applied approximately to other forms of beams and even to trusses not too unlike the forms assumed. In this way an approximately correct design may be made. More exact stresses may then be found, if desired, by a calculation of the deformations of the actual arch, as in the case of the swing bridge.

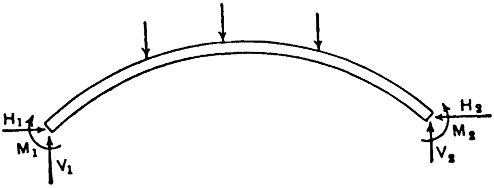


FIG. 7.

For two-hinged spandrel-braced arches, such as shown in Fig. 3 but with two hinges only, a formula developed for a solid beam will not give satisfactory results, and resort must be had to the method of deflections or redundant members. Where, however, the truss is of small depth, as in Fig. 4 or 5, a suitable beam formula may often be applied with satisfactory results.

In the arch of two and of no hinges the reactions are modified by changes of temperature, since the structure is not free to expand. Also, in this form of structure, we can no longer neglect the effect upon the reactions of the changes in dimensions produced by the stresses themselves. It will be seen, therefore, that the analysis of these two types of arches involves not only the given loads but also the possible temperature changes and a close knowledge of the dimensions of the structure itself.

In the analysis of arches it is generally convenient to determine the reactions for a single (vertical or horizontal) load placed at any load point. The reactions being known the stresses in any member are readily found by the principles applicable to simple beams and trusses. Then by summation the reactions or stresses for any given loading may be found and thus the effect of dead and live loads determined.

94. Internal Stresses.—After the loads and reactions have been found the stresses in any part of the arch can be determined by the general methods applicable to simple beams and trusses. However, the presence of horizontal external forces and the special form of the structure, while involving no new principles, modifies somewhat the general formulas, methods, and definitions.

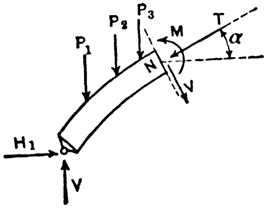


FIG. 8.

95. *The Arch Rib—(a) Algebraic Method.*—Consider an arch rib of constant depth, and let Fig. 8 represent the portion to the left of any section *N* taken at right angles to the axis. The external forces are supposed to be known.

Let ΣV = sum of the vertical components of the external forces on the left of the section;

ΣH = sum of the horizontal components of these forces.

ΣM = sum of the moments of the forces about the gravity axis at N.

α = inclination of axis at N to the horizontal.

The internal stresses at the section *N* are commonly expressed as a moment *M*, constituting a stress couple as in a simple beam, a thrust *T*, consisting of a direct compression uniformly distributed over the section, and a shear *V*, at right angles to the axis, as in a simple beam. Then, equating external and internal forces, we have in general

$$\left.\begin{aligned} M &= \Sigma M \\ T &= \Sigma V \sin\alpha + \Sigma H \cos\alpha \\ V &= \Sigma V \cos\alpha - \Sigma H \sin\alpha \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

(*b*) *Graphical Method.*—The moment, shear, and thrust are readily obtained from the force and equilibrium polygons. Let $A\ b\ c\ d$, Fig. 9 (*a*), represent the equilibrium polygon drawn for the external forces acting. Then R is the line of action of the resultant of all forces on the left of the section. Let β = angle between R and the axis at N and z = lever arm of R about the gravity axis of the section. We then have

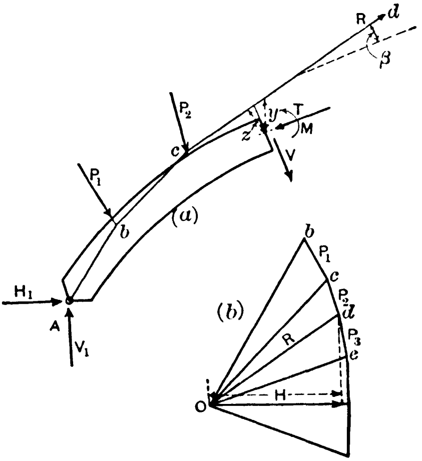


FIG. 9.

$$\left.\begin{aligned} M &= R z \\ V &= R \sin\beta \\ T &= R \cos\beta \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

The moment $R\ z$ is also equal to (Hor. comp. R) × y; or if in Fig. (*b*) H = Hor. comp. R, we have also

$$M = H\ y. \quad \ldots\ldots\ldots \quad (3)$$

When the loads are all vertical then H is the pole distance of the force polygon. Hereafter the loads will generally be assumed as vertical.

The values of M, V, and T having been found at any section the stress intensities are deduced as for the case of combined flexure and compression in straight beams.

If A = area of cross-section;

I = moment of inertia of cross-section;

y_1 and y_2 = the distances from the centroid of the section to extreme upper and lower fibres, respectively;

f_1 and f_2 = the corresponding fibre stresses, then, assuming M as positive when causing compression in the upper fibres, we have

$$\left.\begin{aligned} f_1 &= \frac{T}{A} + \frac{M y_1}{I} \\ f_2 &= \frac{T}{A} - \frac{M y_2}{I} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

The distribution of shearing stress follows the same law as in a straight beam.

The application of these formulas involves a small error due to the curvature of the beam, which brings the centroid of stress slightly inside the centre of gravity of the section. This error is large only in the case of beams of very short curvature, such as rings and hooks. It is negligible in the case of most arches.

Moment Centres for Maximum Fibre Stresses.—In determining maximum total fibre stress, f_1 or f_2, and in studying the effect of moving loads, it is convenient to reduce eq. (4) to a different form. If r = radius of gyration, $= \sqrt{\frac{I}{A}}$, then eq. (4) may be written in the form

$$\left.\begin{aligned} f_1 &= \frac{T r^2 + M y_1}{I} \\ \text{and} \qquad f_2 &= \frac{T r^2 - M y_2}{I} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

Referring to Fig. 10, $M = Te$, hence

$$f_1 = \frac{T\,(r^2 + e\,y_1)}{I} = \frac{T\left(\frac{r^2}{y_1} + e\right) y_1}{I}.$$

Now $\frac{r^2}{y_1} + e$ is a length and $T\left(\frac{r^2}{y_1} + e\right)$ is a moment, which may be called M'_1. Hence we have, $f_1 = \frac{M'_1 y_1}{I}$, in which M'_1 is equal to the thrust T, multiplied by the arm $\left(e + \frac{r^2}{y_1}\right)$. If now we lay off the distance $\frac{r^2}{y_1}$ *below* the gravity axis, fixing a point o_1, this point may be taken as the centre of moments for M'_1, for then $M'_1 = T\left(e + \frac{r^2}{y_1}\right)$. (This point o_1 is known as the *kern* point of the section.) A similar point *above* the neutral axis, distant $\frac{r^2}{y_2}$ therefrom, gives the centre of moments o_2 for the lower fibre stress. These moments may be called M'_2. Having the moments M'_1 and M'_2, the fibre stresses are

$$f_1 = \frac{M'_1 y_1}{I} \text{ and } f_2 = \frac{M'_2 y_2}{I}. \quad . \quad (6)$$

The extreme fibre stresses are proportional to the moments M'_1 and M'_2 and hence will be a maximum when these moments are a maximum. In finding maximum fibre stresses for moving loads it is therefore convenient to determine the maximum moments, using points o_1 and o_2 as moment centres. The values of the moments M' may be found in the same manner as for any other moment centre, using either T, R, or H in the equation, H being the horizontal component of R, as in Fig. 9 (b). Thus

$$M'_1 = T\left(e + \frac{r^2}{y_1}\right) = R\,z' = H\,y'. \quad . \quad . \quad . \quad (7)$$

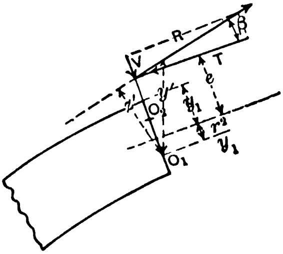


FIG. 10.

If the beam is a plate girder and the moment of resistance of the web is neglected, or is taken account of by adding a certain fraction of its area to the flange, then the *kern* point or moment centre becomes the centre of gravity of the flange.

96. *The Braced Arch.*—(*a*) *Algebraic Method.*—If the arch is a truss then the stresses in the members are found most conveniently by moments; or, if the chords are parallel or nearly so, the web stress may also be found conveniently by shears as in a parallel or curved-

chord truss. Algebraically the stress in $A\,B$ (Fig. 11) $= M_D\,/t_1$; stress in $C\,D = M_A\,/t_2$. The stress in $A\,D$ may be found by taking moments about the intersection of $A\,B$ and $C\,D$, if convenient, or by an equation of horizontal or vertical components. Generally the co-ordinates of each joint, referred to horizontal and vertical axes, will be known, and hence horizontal and vertical projections of any

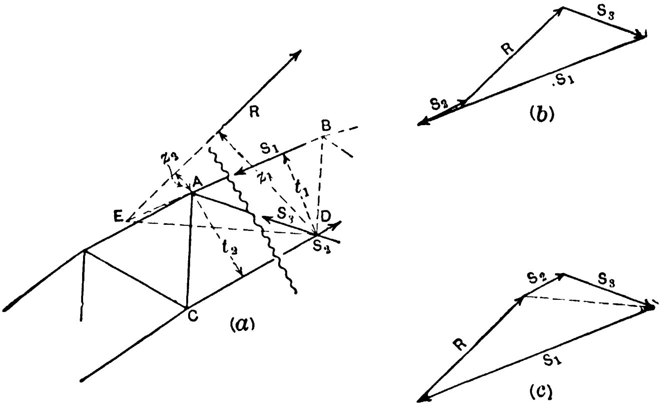


FIG. 11.

member can be readily found. The moments M, of the external forces on the left of the section, are also readily obtained from a summation of the moments of the loads and reactions in detail.

(*b*) *Graphical Method.*—The resultant R being known, the stresses may be found by various graphical methods (Fig. 11). The bending moment at $D = M_D = R\,z_1$, whence $S_1 = \dfrac{R\,z_1}{t_1}$; $M_A = R\,z_2$ and $S_2 = \dfrac{R\,z_2}{t_2}$. For stress in S_3 the force polygon for R, S_1, S_2, and S_3 may be drawn, as in Fig. 11 (*b*). Or, draw R to an intersection E, with one of the forces S_1. Then the resultant of R and S_1 must pass through the intersection D of the other two. This gives the direction of the resultant of R and S_1 and enables the three forces to be determined by the force polygon as shown in Fig. (*c*).

97. Advantages of the Arch Bridge.—If all the loads were fixed, the form of an arch could be so selected that the equilibrium polygon of the forces would follow the axis of the arch throughout, as in Fig. 12. The only stresses then existing in the arch would be the thrust, and the design would be of maximum economy. Compared to a truss, such an arch rib would correspond somewhat to the upper chord, where the chord is curved just enoughto take all the shear. In the arch the lower chord is replaced by the thrust from the abutments. If the arch does not exactly fit the equilibrium or pressure line, as in Fig. 13, it will be subjected to small bending moments, represented by the shaded areas. The truss, however, is subjected as a whole to moments as represented in Fig. 14, and the advantage of the arch is still large. When the live load is large as compared to the dead load, the moments which occur in the arch under partial loading also become large, and the advantages of the arch become still less, but the moments are yet on the average smaller than in the truss. Offsetting the advantage of such reduced

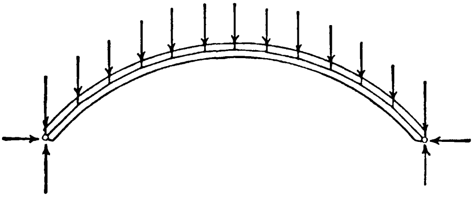


FIG. 12.

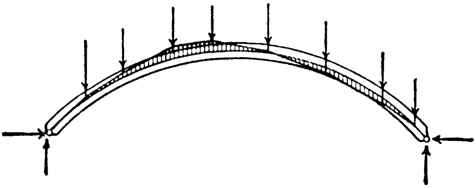


FIG. 13.

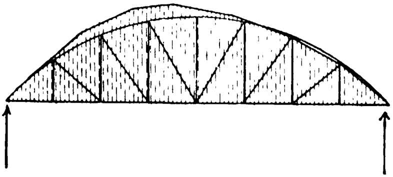


FIG. 14.

bending moments are the increased stresses in the piers or abutments, the serious modification of stresses which are caused by any slight settlement of supports (except in the three-hinged type), the stresses arising from temperature changes, and the large deflections under moving loads, with increased secondary stresses, as compared with deep trusses.

Arches are well adapted to high crossings, especially those of a single span. They can be constructed without falsework, in the same manner as cantilever bridges, and in many locations their form fits the contour of the ground very well, requiring a minimum amount of substructure. The use of arches at the Niagara gorge illustrates these points.

As compared to the suspension type of bridge the arch is more rigid and better suited for heavy railroad traffic for any but the longest spans. It is not so well adapted as the suspension bridge for very long spans, as the long compression rib of the arch cannot compare with a tension cable in certainty of design or economy of material. Deformations of a tension cable develop moments tending to resist further deformations, while deformations of a compression rib develop moments tending to increase still further such deformations.

98. Deflection of Curved Beams.—The equations of Arts. 1, 2, and 3 apply only to beams which are straight before bending. If a beam is curved in its unstrained condition, as in an arch rib, the formulas will be somewhat modified, due to the fact that the length of an element is no longer approximately equal to its horizontal projection.

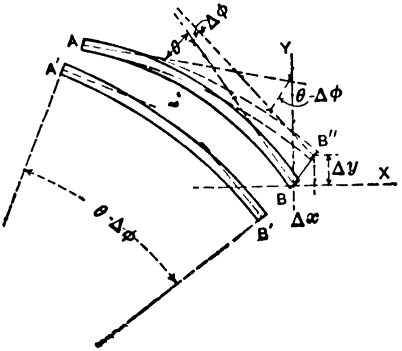


Fig. 15.

Let $A\ B$, Fig. 15, be any portion of a curved beam in its unstrained form. Suppose now that under the action of certain forces the beam is bent so that this portion is brought into the position $A'\ B'$. We wish now to find the movement of B and the tangent at B, with reference to A and the tangent at A, these two points, A and B, being any two points in the beam. This relative motion will be made apparent by making A' coincide with A, and the tangent at A' coincide with the tangent at A. This new position is represented by the dotted outline $A\,B''$. The absolute movement $B\,B''$, now shown, is the relative movement required. The tangent at B has moved through an angle $\Delta\,\varphi$, making now an angle with the tangent at A of $\theta - \Delta\,\varphi$, θ being the original angle. The point B has also moved in space a distance $B\,B''$, the components of which motion, referred to any two rectangular axes with origin at B, will be called $\Delta\,y$ and $\Delta\,x$. The effect of flexure alone will be first considered, after which the effects of direct stress and temperature change will be taken account of.

99. *Angular Change*, $= \Delta\,\varphi$. Let $C\,E\,F\,D$, Fig. 16, be an element of the beam, of length $d\,s$, whose end faces $C\,E$ and $D\,F$ are at right angles to the axis, and whose end tangents make an angle with each other originally equal to $d\,\theta$. Let $\delta\,\varphi$ be the change in angle between end faces or end tangents due to bending. The change in length of a fibre at a distance y from the neutral axis will be equal to $y\,\delta\,\varphi$, and the corresponding stress per unit area will be equal to $f = E\dfrac{y\,\delta\,\varphi}{d\,s}$. If $d\,a$ is an element of area of the cross-section, the total moment of resistance of the beam is equal to $\displaystyle\int_F^D f\,d\,a\,y = \int_F^D E\,y^2\,d\,a\frac{\delta\,\varphi}{d\,s}$.

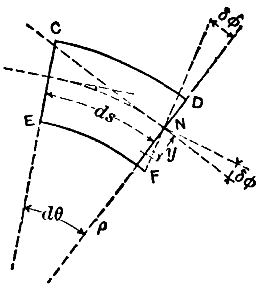


Fig. 16.

But for any particular section, E and $\dfrac{\delta\,\varphi}{d\,s}$ are constant; and if M is the bending moment at the section, taken about the axis N, and I is the moment of inertia of the section, we have

$$M = E\frac{\delta\,\varphi}{d\,s}\int_F^D y^2\,d\,a = E\,I\frac{\delta\,\varphi}{d\,s},$$

from which we have

$$\delta\,\varphi = \frac{M\,d\,s}{E\,I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and in Fig. 15,

$$\Delta\,\varphi = \int_A^B \delta\,\varphi = \int_A^B \frac{M\,d\,s}{E\,I}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It has been assumed that the curvature of the beam is large as compared to the width $D\,F$ so that the length of all fibres of an element $d\,s$ may be assumed equal. For beams of very short curvature, as in the case of hooks, machine frames, etc., this assumption cannot be made.

Note that eq. (9) reduces to eq. (3) of Chap. I, if $d\,s$ is replaced by $d\,x$.

100. *Components of Deflection,* $= \Delta\,y$ and $\Delta\,x$.—Let $A\,C\,D\,E\,B$, Fig. 17, represent the axis of the unstrained form of the beam, and $A\,C\,D'\,E''\,B^n$ the strained form $A\,B''$, of Fig. 15. Now conceive the beam to pass into its strained form by the successive bending of each $d\,s$ in turn. The bending of the element $A\,C$ through the angle $\delta\,\varphi$ causes the portion $B\,C$ to turn through the same angle $\delta\,\varphi$ about C as a centre, with radius u, the point B moving to B' through a distance $\delta\,v$, having the components $\delta\,y$ and $\delta\,x$. Then from the bending of $C\,D'$ the point B' moves to B'', etc. If x and y are the co-ordinates of any point C, origin at B, we have, by similar triangles,

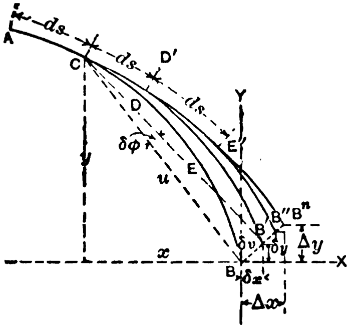


FIG. 17.

$$\frac{\delta\,y}{\delta\,v} = \frac{x}{u} \text{ and } \frac{\delta\,x}{\delta\,v} = \frac{y}{u}.$$

Solving for $\delta\,y$ and $\delta\,x$ and substituting for $\delta\,v$ the value $u\,\delta\,\varphi$, we have

$$\delta\,y = x\,\delta\,\varphi \text{ and } \delta\,x = -\,y\,\delta\,\varphi.$$

Substituting the value of $\delta\,\varphi$ from eq. (8), we have

$$\Delta\,y = \int \delta\,y = \int_A^B x\,\delta\,\varphi = \int_A^B \frac{M\,x\,d\,s}{E\,I}, \quad . \quad . \quad . \quad (10)$$

and

$$\Delta\,x = \int \delta\,x = -\int_A^B y\,\delta\,\varphi = -\int_A^B \frac{M\,y\,d\,s}{E\,I}. \quad . \quad (11)$$

Equations (9), (10), and (11) are the fundamental equations employed in the analysis of arch ribs of steel and in the elastic theory of masonry arches. These formulas are also used for approximate analysis of trussed arches, in the same manner as the formulas for the solid continuous girder are used for continuous trusses. The deformation due to shear is negligible in the case of the solid beam, but is of importance in the case of the truss unless the depth be relatively small.

Equations (10) and (11) can readily be derived directly from the general formula for the deflection of a beam given in Art. 214, Part I. For a curved beam the formula would be $\Delta = \int \frac{M\,d\,s}{E\,I} \,.\, m$, in which m = bending moment for a one-pound load applied at the point whose deflection is desired. For $\Delta\, x$ the unit load would be applied at B and acting horizontally. The value of m would be equal to y, hence $\Delta x = -\int \frac{M\,d\,s}{E\,I} \,.\, y$, $\Delta\, x$ being negative. Likewise $\Delta\; y = \int \frac{M\,d\,s}{E\,I} . x$.

101. *Effect of Direct Stress on the Values of* $\Delta\,\varphi$, $\Delta\,y$ and $\Delta\,x$.— Arch ribs are generally subjected to combined compression and bending. In the foregoing analysis the effect of bending alone has been considered; we will now determine the effect of the direct compression upon the form of a curved rib.

Suppose the element $C\,D$, Fig. 18, represent any element subjected to the thrust T, applied at the gravity centre and producing a uniform compressive stress f per unit area. As a result of this stress all the fibres will be shortened a proportionate amount and $C\,D'\,F'\,E$ may represent the deformed element. The distortion $N\,N' = \frac{f}{E}\,d\,s$. Accompanying the longitudinal shortening $N\,N'$ there will be a slight expansion transversely, but this will be very small and will be neglected. The width $C\,E$ will therefore be unchanged and hence the radius and centre of curvature of the arch at this point will remain unchanged.

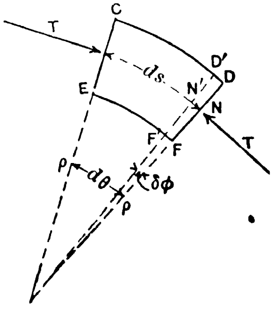


Fig. 18.

If ρ = radius of curvature of the axis then we have

$$\delta\,\varphi = \frac{f}{E} \,.\, \frac{d\,s}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and for the entire rib between any two sections A and B, as in Fig. 15,

$$\Delta \varphi = \int_A^B \frac{f\,d\,s}{E\,\rho} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

The components of the motion, $\Delta\, x$ and $\Delta\, y$, are determined as follows: Suppose in Fig. 19 each element $d\,s$ is shortened by the amount $\frac{f}{E}\,d\,s$. The shortening of the element $A\,C$ affects the movement of B in two ways, (1) by the direct shortening of the rib and (2) by the change of angle $\delta\,\varphi$ produced. The first causes a shortening in the value of $d\,x$ equal to $\frac{f}{E}\,d\,s \cos \alpha = \frac{f}{E}\,d\,x$, and in the value of $d\,y = \frac{f}{E}\,d\,s \sin \alpha = \frac{f}{E}\,d\,y$. The change of angle $\delta\,\varphi$ causes a movement of B determined as in Art. 99. The x-component $= \delta\,x = -\,y\,\delta\,\varphi$ and the y-component $= \delta\,y = x\,\delta\,\varphi$. Substituting the value of $\delta\,\varphi$ from eq. (12), we have for the total movement due to the shortening of the element $A\,C$,

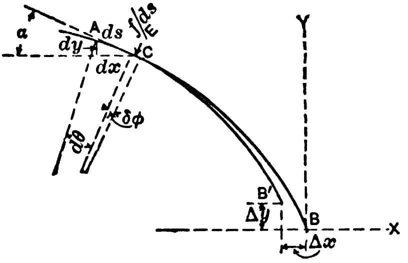


Fig. 19.

and

$$\left.\begin{aligned} \delta\,y &= \frac{f}{E}\,d\,y + \frac{f\,x\,d\,s}{E\,\rho} \\ \delta\,x &= \frac{f}{E}\,d\,x - \frac{f\,y\,d\,s}{E\,\rho} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Integrating between A and B we have finally

$$\left.\begin{aligned} \Delta\,y &= \int_A^B \frac{f\,d\,y}{E} + \int_A^B \frac{f\,x\,d\,s}{E\,\rho} \\ \Delta\,x &= \int_A^B \frac{f\,d\,x}{E} - \int_A^B \frac{f\,y\,d\,s}{E\,\rho} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

The origin is at B and x is measured positively toward the left.

102. *Effect of Temperature Changes.*—Let ω = coefficient of expansion and t = change of temperature, an increase of temperature

being called plus. An increase of temperature throughout the arch of t degrees will cause a proportionate increase of dimension, each $d\,s$ being increased by $\omega\,t\,d\,s$. The x- and y-components will be increased by the amounts $\omega\,t\,d\,x$ and $\omega\,t\,d\,y$ respectively, and hence for any portion $A\,B$ of a beam (Fig. 15),

$$\Delta\,x = -\,\omega\,t\int_A^B d\,x \text{ and } \Delta\,y = -\,\omega\,t\int_A^B d\,y. \quad . \quad . \qquad (16)$$

The angle θ will not be affected.

103. *Total Values of $\Delta\,\varphi$, $\Delta\,y$ and $\Delta\,x$.*—Combining the results Arts. 99–102, we have for the total movement of B, due to loads and temperature change

$$\Delta\,\varphi = \int_A^B \left[\frac{M\,d\,s}{E\,I} + \frac{f\,d\,s}{E\,\rho}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

$$\Delta\,y = \int_A^B \left[\frac{M\,x\,d\,s}{E\,I} + \frac{f\,d\,y}{E} + \frac{f\,x\,d\,s}{E\,\rho} - \omega\,t\,d\,y\right] \quad . \quad . \qquad (18)$$

$$\Delta\,x = \int_A^B \left[-\,\frac{M\,y\,d\,s}{E\,I} + \frac{f\,d\,x}{E} - \frac{f\,y\,d\,s}{E\,\rho} - \omega\,t\,d\,x\right] \quad . \qquad (19)$$

104. Application of Deflection Formulas to Arches.—In an arch of two hinges each part is, in general, subjected to bending moments and direct stresses, and therefore the arch is distorted at all sections. While the end points remain in a fixed position the end tangents do not, each one turning more or less about the hinge. In Fig. 20 let the line $A\,C\,B$ represent the initial form of the arch axis and the line $A\,C'\,B$ the bent form under load. The tangent at A has turned through an angle β and at B through some other angle β'. To represent the motion of B

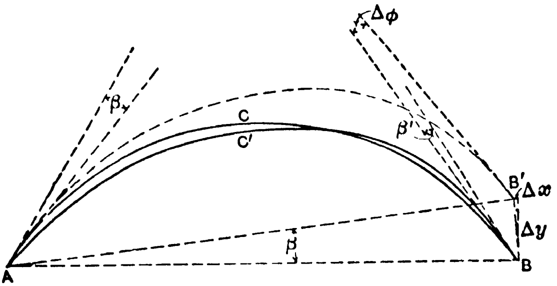


FIG. 20.

relative to the tangent at A the arch may be rotated about A through the angle β until the bent form coincides at that point with the original position. The movement at B is now correctly shown and is from B to B'. The values of Δy, Δx, and $\Delta \varphi$ at B' are given by the formulas of the preceding articles, but we are as yet unable to calculate these as the reactions and stresses remain undetermined.

Now Δx is a small quantity as compared to Δy, which is itself small as compared to the dimensions of the truss. Hence we may place $\Delta x = 0$ and therefore from (19) we have

$$\Delta x = \int_A^B \left[-\frac{M y\, d s}{E I} + \frac{f\, d x}{E} - \frac{f y\, d s}{E \rho} - \omega t\, d x \right] = 0. \quad . \quad (20)$$

This equation of condition, together with the three equations of statics, supplies the four equations necessary for the determination of the reactions for a two-hinged arch.

If the supports are not fixed, as in the case where a tie rod is used, then Δx is not zero but is equal to the elongation of the tie rod under the stress H. Δx is to be assumed as positive when corresponding to a *shortening* of the arch.

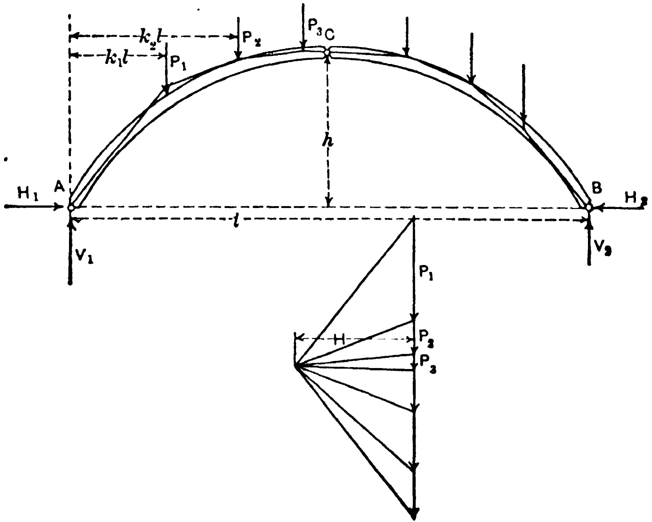


FIG. 21.

For arches with fixed ends the values of both Δx and Δy are zero, and also the total change in angle, $\Delta \varphi$, from A to B, is zero. Hence in eqs. (17), (18), and (19) the right-hand members reduce to zero, giving three condition equations. These, with the three equations of statics, enable the six unknown elements of the reactions in this case to be determined.

105. *Application to Trussed Arches.*—The beam formulas above developed may be applied approximately to trusses, as was done in the case of continuous girders and trusses. In many cases, however, the results thus obtained are not sufficiently accurate and recourse must be had to the method of redundant members whereby due account is taken of the deformation of the truss as actually built.

SECTION II.—ARCHES OF THREE HINGES

106. Reactions and Stresses for Dead Load.—The reactions for the three-hinged arch for any given loading are fully determined by statics, as explained in Chap. II, of Part I. A summary of the equations as applied to vertical loads will be here given. Fig. 21 shows an arch of three hinges supporting vertical loads. *A* and *B* are at the same level. Taking moments about *B* we have at once

$$V_1 = \Sigma P\,(1 - k). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Likewise, with *A* as moment centre

$$V_2 = \Sigma P\,k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Then with *C* as moment centre, considering only the structure *A C* (the moment at *C* being zero), we have

$$V_1 \frac{l}{2} - \Sigma_A^C P\left(\frac{l}{2} - k\,l\right) - H_1\,h = 0,$$

whence

$$H_1 = \frac{l}{2\,h}\Big[\,V_1 - \Sigma_A^C P\,(1 - 2\,k)\,\Big], \quad . \quad . \quad . \quad . \qquad (3)$$

also

$$H_1 = H_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

whence all reactions become known.

Graphically, the reactions are determined by constructing an equilibrium polygon which shall pass through the hinges, *A*, *C*, and *B*, as shown in Fig. 21. See also Art. 45, Part I. In the manner there described the dead-load reactions may be found at once, and thence the dead-load stresses either algebraically or graphically. (See Art. 60, Part I, for example of the graphical method applied to a roof-arch.) In the case of a trussed arch a stress diagram is the most expeditious way of determining dead-load stresses.

107. Reactions for a Single Load.—Let $A\,B\,C$, Fig. 22, represent a three-hinged arch of span l and rise h. P is any load, distant $k\,l$ from the left end.

(*a*) *Graphical Method.*—The equilibrium polygon is constructed by drawing $B\,C$ to intersect the load vertical at i and drawing $A\,i$.

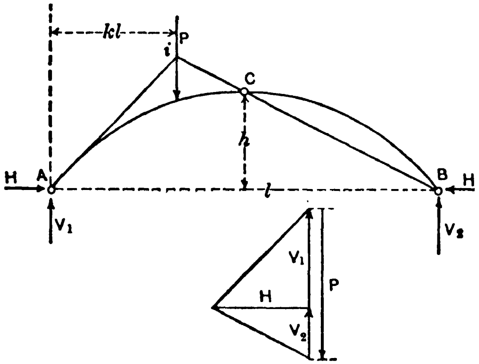


FIG. 22.

The force polygon determines V_1, V_2, and H. For a load on the right half, the left reaction line would pass through A and C.

(*b*) *Algebraic Method.*—As in Art. 106, we have

$$V_1 = P\,(1 - k) \qquad (5)$$

$$V_2 = P\,k \qquad (6)$$

For a load on $A\,C$,

$$H = \frac{V_2\,l}{2\,h} = \frac{P\,l}{2\,h}\,k \qquad (7)$$

For a load on $C\,B$

$$H = \frac{V_1\,l}{2\,h} = \frac{P\,l}{2\,h}\,(1 - k) \qquad (8)$$

It will be seen that the values of V_1 and V_2 are the same as for a simple beam, and that H is the same for loads symmetrically placed on the two halves, or for k in (7) equal to $(1 - k)$ in (8).

108. Use of Reaction Lines to Determine Position of Live Loads for Maximum Stresses.—The nature of the stress produced in any member, by a load placed at any given point, can readily be determined from a study of reaction lines, constructed as explained in the preceding article. This information is sufficient to determine the position of live load for a maximum stress in any member. The method of influence lines is also well adapted for this purpose, and for the calculation of the stresses themselves. Both methods will be explained.

Let q, Fig. 23, be any section cutting three members of a braced arch of three hinges, A, B, and C. The centre of moments for $D\,E$ is G, and a load at i, the intersection of $B\,C$ and $A\,G$, will cause no stress in $D\,E$, since the line $A\,G$ is the line of action of the abutment reaction at A, the only external force on the left of the section. For loads between i and B the left reaction line will pass below G, passing through C for loads on $C\,B$. The moment of the reaction at A, about G, will then be negative and cause tension in $D\,E$. For loads between i and E, inclusive, the reaction line from A lies above G, and therefore the stress in $D\,E$ will be compressive. For loads between D and A the only force on the right of the section is the reaction at B, acting in the line $B\,C$, which also causes compression in $D\,E$. Therefore for a maximum tension in $D\,E$ all joints from i to B should be loaded and for maximum compression all joints from A to i.

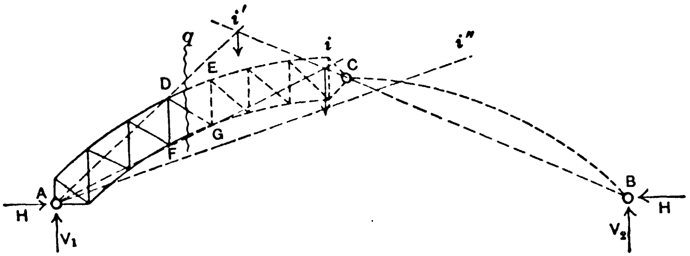


FIG. 23.

The centre of moments for $F\,G$ is at D, and from considerations similar to the preceding, it is found that the maximum tension and the maximum compression in this member occur when the load extends to the left and right, respectively, of point i'.

For the web member $D\,G$, draw $A\,i''$ parallel to $D\,E$ and $F\,G$, if these members are parallel, or toward their intersection if not parallel. For all loads between G and B the reaction line from A lies above $A\ i''$; since this line never passes below C. Hence the component of the reaction perpendicular to $A\ i''$ (in case of parallel chords), or the moment of the reaction about the intersection of $D\,E$, $F\,G$, and $A\ i''$ (in case of non-parallel chords), produces tension in the member $D\,G$. For loads from D to A the right reaction, acting in the line $B\,C$, causes compression in $D\,G$. Hence for maximum tension in $D\,G$, $G\,B$ should be loaded, and for maximum compression $D\,A$ should be loaded. If $A\ i''$ should pass to the left of C, then all loads between its intersection with $B\,C$, and B, would cause compression in $D\,G$.

In the case of an arch rib the centres of moments are taken as explained in Art. 95, and in finding the load for maximum shear, the limiting line, corresponding to $A\ i''$, is drawn parallel to the flanges at the section considered.

109. Influence Lines for the Three-Hinged Arch.—For some purposes the use of influence lines is preferable to the method just explained. In the case at hand their construction leads at once to methods of calculating maximum stresses for concentrated loads which are as readily applied as in the case of simple trusses.

110. *Influence Line for H.*—Consider the arch of Fig. 24, with a single load P at any point D or D'. The value of H for loads on the left half is $\frac{P\,l}{2\,h}\,k$, and for loads on the right half is $\frac{P\,l}{2\,h}\,k'$. The influence line for H then consists of the two straight lines $A'\,C''$ and $C''\,B'$ of Fig. (*b*), the ordinate $C'\,C''$ being equal to $l/4\,h$.

111. *Influence Lines for Moment.*—Consider the moment at G for stress in $D\,E$. The value of this moment for a load on the right of the section is $V_1\,a - H\,y$, and for a load on the left it is $V_1\,a - P\,(a - k\,l) - H\,y$. Noting that the value of V_1 is the same as in a simple truss, it is seen that in all cases the moment at G, due to the vertical forces V_1 and P, is the same as in a simply supported truss, and that the total moment is equal to this moment, less the moment $H\,y$. Hence to construct the influence line for M_G we may first construct the influence line for moment at G' in a simple beam $A'\,B'$ (Fig. *c*), making $G'\,G'' = \frac{a\,(l - a)}{l}$ as in Art. 122, Part I. Then on this same base construct a diagram $A'C''B'$, in which $C'C'' = \frac{l}{4h} \cdot y$. The ordinate to this diagram at any point will then be $H\,y$, and the total bending moment at G, due to a load unity at any point, will be given by the ordinates of the shaded area between the lines $A'G''B'$ and $A'C''B'$. This shaded diagram is therefore the in-

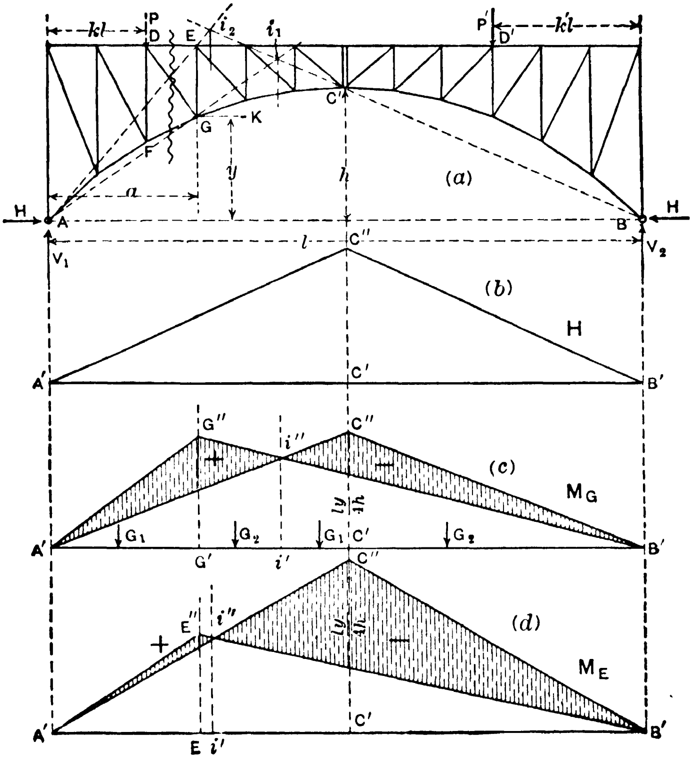


Fig. 24.

fluence diagram for moment at G. It is unnecessary to replot this diagram to a straight base as the shaded area gives all needed information.

Fig. (*d*) shows the influence diagram for moment at E for stress in GK. The diagram is the same as Fig. (*c*) excepting the ordinate $C'C''$, the value of y being now the ordinate to point E, which is equal to h plus length of center strut.

The points i'' in Figs. (*c*) and (*d*) are the critical points, corresponding to the points i_1 and i_2 of Fig. (*a*). Loads placed in those verticals cause no moments at the respective moment centres. The influence areas indicate clearly the relative effect of loads on different parts of the arch. They can readily be used for calculating maximum stresses due to either uniform or concentrated loads, as in the case of simple structures.

112. *Criterion for Maximum Moment.*—The form of the influence lines of Figs. (*c*) and (*d*) gives at once the criterion for maximum moments, as in Arts. 123 and 134 of Part I. The influence areas are all triangular in form and hence the same criterion for maximum will apply as for moment in a simple beam of a length equal to the length of the triangular area in question. Thus for maximum positive moment at G the criterion is the same as for moment at G' in a beam $A'i'$ (Fig. *c*). Hence if the loads on the portions $A'G'$ and $G'i'$ be represented by G_1 and G_2 the criterion for maximum is

$$\frac{G_1}{A'G'} = \frac{G_2}{G'i'} = \frac{G_1 + G_2}{A'i'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The train should probably head toward the right. For the maximum negative moment, load $i'B'$, heading train toward the left, the criterion is

$$\frac{G_1}{i'C'} = \frac{G_2}{C'B'} = \frac{G_1 + G_2}{i'B'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

It will be seen that the determination of the distances needed in eqs. (9) and (10) requires the fixing of point i' only, which can be done by the reaction lines shown in Fig. (*a*). The above method is equally applicable to arch ribs, using the proper *kern* point or flange centre for the moment centre.

113. *Influence Lines for Web Stress or Shear.*—Consider the stress in DG, Fig. 25. It will be convenient to construct the influence line for the vertical component of this stress. Produce FG to the intersection I. The stress in DG is found from the moments about I of the forces to the left (or right) of the section. For loads from G to B this moment $= V_1 s - Hy$, and Vert. comp. $DG = V_1 \frac{s}{t} - \frac{Hy}{t}$, where y = ordinate to moment centre I. And for loads from A to F, Vert. comp. $DG = V_2 \frac{l - s}{t} - \frac{Hy}{t}$. For unit load the influence lines for $V_1\, s/t$ and $V_2\, (l - s)/t$ are given in Fig. (*b*) by the lines $A''\, B'$ and $A'\, B''$. (These are the same as for web stress in a simple truss of the form shown.) Then, as in Art. 111, we may plot to the same base a diagram for $H\, y/t$, making the ordinate $C'\, C''$ equal to $l\, y/4\, h\, t$. The shaded areas are then the influence areas for the vertical component in $D\, G$. The point i' is readily found also by the indicated construction in Fig. (*a*).

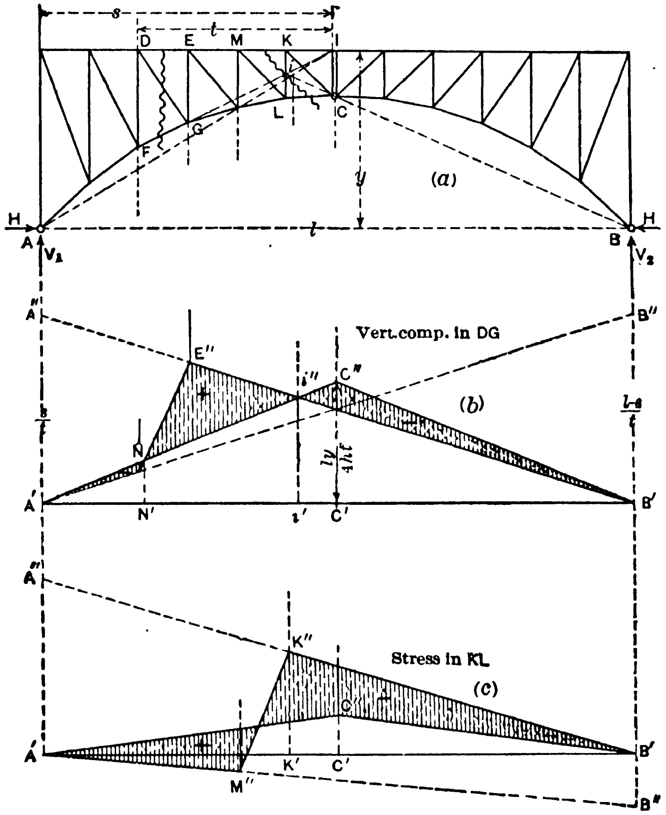


FIG. 25.

For maximum compressive stress in $D\, G$, the portions $A'\, N'$ and $i'\, B'$ should be loaded, and for maximum tension the portion $N'\, i'$. The criteria may be stated as for moment, but the question of separate loadings, or broken loads, for sections $A'\, N'$ and $i'\, B'$ requires special consideration.

Fig. (*c*) shows the influence diagram for stress in $K L$. Here it will not be convenient to get the intersection of the chords, but instead the method of shears may be used. The points K'' and M'' are determined by placing a unit load at K and then at M and calculating the stress in $K L$ as for a simple span. Then for a unit load at C calculate the stress in $K L$ due to the *horizontal force* H equal to $l/4\,h$. Plot this as $C''\,C'$. This enables the complete diagram to be drawn.

In the rib or solid beam, Fig. 26, the shear for loads on the right of the section is $V = V_1 \cos \alpha - H \sin \alpha$, and for loads on the left it is $- V_2 \cos \alpha - H \sin \alpha$. The resulting influence diagram is shown in Fig. (*b*). The lines $A''\,B'$ and $A'\,B''$ are the same as for shear in a simple beam, multiplied by $\cos \alpha$, and the term $H \sin \alpha$ is given by the diagram $A'\,C''\,B'$.

114. *Influence Line for Thrust.*—The thrust, T, Fig. 26, is equal to $V_1 \sin \alpha + H \cos \alpha$, for loads on the right of the section, and $-V_2 \sin \alpha + H \cos \alpha$, for loads on the left. The influence diagram for the thrust is given in Fig. (*c*).

115. Equivalent Uniform Loads for Three-Hinged Arches.—The influences lines of Figs. 24 and 25 aid greatly in selecting a suitable equivalent uniform load. It will be noted that only a portion of the span is covered for maximum moments and that therefore the equivalent uniform load should be selected with reference to a span length considerably shorter than the arch span.

The special method of selection explained in Art. 174, of Part I, is directly applicable to this case. Thus for the maximum moment at G, Fig. 24 (*c*), the equivalent uniform load is that for moment at C' in the beam $i'\,B'$. If, for example, l = 180 feet, then $i'\,B'$ = about 110 feet, $i'\,c'$ = 20 ft., and $i'\,C'/i'\,B'$ = 0.18; hence the equivalent uniform load is that for moment at the 0.18 point in a beam 110 feet long. Referring to Fig. 4, p. 53, we find this to be 3,400 lbs. per foot for Cooper's E-50 loading. The maximum live-load moment at G is therefore equal to area $i''\,C''\,B' \times 3{,}400$.

116. Deflection of the Three-Hinged Arch.—The methods of calculating deflections are the same as given in Chap. VII, Part I. Under full loads arches deflect less than simple trusses, but under partial loads the deflection at the quarter point may be much greater.

In applying the graphical method to the three-hinged arch, a separate diagram must be drawn for each half in case of unsymmetrical loading. Then since all members in general change their direction the diagram will require correction, which must be made on the basis of fixed abutments and a hinge at the centre. As the construction of displacement diagrams is of special importance in the case of arches, such

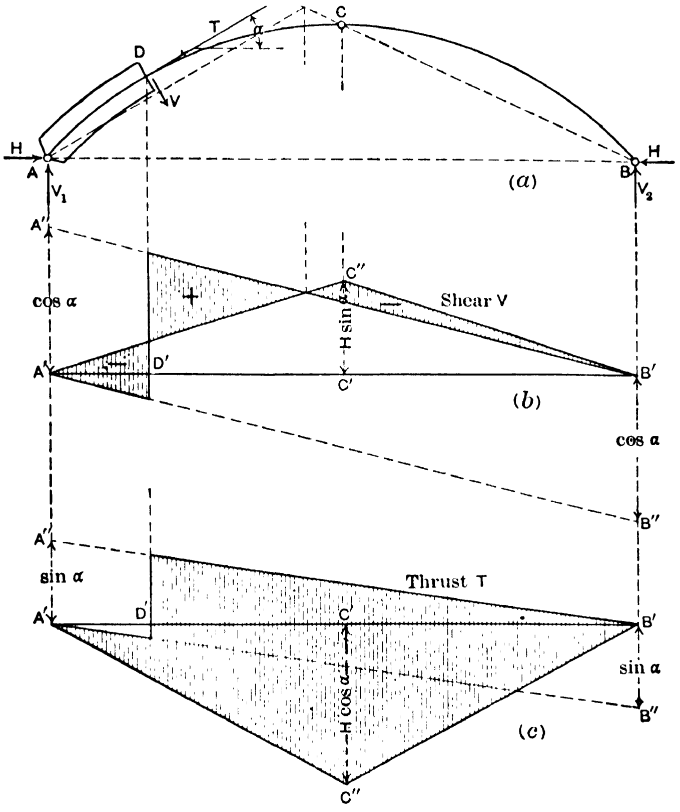


FIG. 26.

a diagram will be drawn for the arch shown in Fig. 27 (*a*), which is an outline of the arch bridge of the Chicago, Milwaukee and St. Paul Railway Company across the Menominee River, near Iron Mountain, Michigan.* The span length is 207 feet.

A load of 1,000 lbs. per joint will be assumed as applied on the left half only, giving unsymmetrical conditions. The resulting stresses, determined graphically, are tabulated below, together with lengths and cross-sections of members and values of deformation times E, $= \frac{Sl}{A}$.

---

* See *Engineering News*, Nov. 20, 1902.

Member	Length l	Cross-Section A	Stress S	$\frac{Sl}{A}$
a b	276	27.2	− 910	− 9,200
b c	276	27.2	− 1,800	− 18,300
c d	276	24.6	− 2,560	− 28,700
d e	276	24.6	− 2,380	− 26,700
e F	153	31.2	− 2,650	− 13,000
A B	313	46.2	− 2,800	− 19,000
B C	310	38.0	− 1,760	− 22,800
C D	302	38.0	− 730	− 5,900
D E	240	38.0	+ 100	+ 800
E F	153	38.0	− 90	− 400
a A	624	40.7	− 2,560	− 39,300
a B	550	20.6	+ 1,810	+ 40,300
b B	476	32.4	− 2,080	− 31,300
b C	435	14.7	+ 1,400	+ 41,600
c C	336	23.5	− 1,600	− 22,800
c D	348	14.7	+ 970	+ 23,000
d D	213	19.8	− 910	− 9,800
d E	306	14.7	− 200	− 4,200
e E	132	19.8	+ 160	+ 1,100
F g	153	31.2	− 80	− 400
g h	276	24.6	− 70	− 800
h i	276	24.6	+ 480	+ 5,400
i j	276	27.2	+ 330	+ 3,300
j k	276	27.2	+ 150	+ 1,500
F G	153	38.0	− 2,660	− 10,700
G H	240	38.0	− 3,070	− 19,400
H I	302	38.0	− 3,050	− 24,200
I J	310	38.0	− 2,930	− 23,800
J K	313	46.2	− 2,620	− 17,700
g G	132	19.8	+ 20	+ 200
G h	306	14.7	+ 600	+ 12,600
h H	213	19.8	− 260	− 2,800
H i	348	14.7	− 180	− 4,400
i I	336	23.5	+ 100	+ 1,500
I j	435	14.7	− 260	− 7,700
j J	476	32.4	+ 200	+ 3,000
J k	550	20.6	− 250	− 6,700
k K	624	40.7	+ 210	+ 3,200

Fig. (*b*) is the displacement diagram for the left half, assuming *A a* to remain vertical; and Fig. (*c*) is the diagram for the right half, *K k* remaining vertical. It remains now to correct these diagrams on

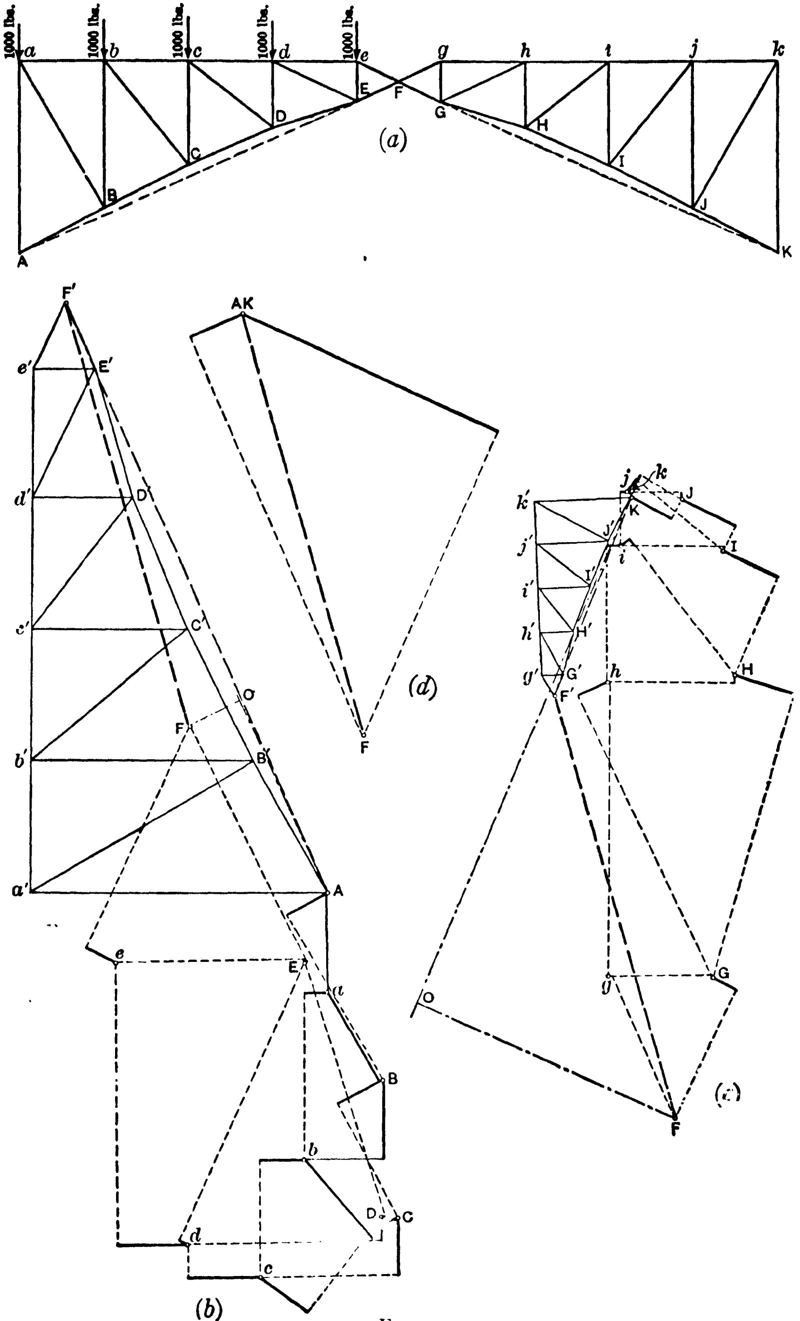

1000 lbs.
1000 lbs.
1000 lbs.
1000 lbs.
1000 lbs.
a
b
c
d
e
g
h
i
j
k
A
B
C
D
E
F
G
H
I
J
K
(a)
F′
e′
E′
d′
D′
c′
C′
O
F
b′
B′
a′
A
e
E
a
B
b
D
C
d
c
(b)
AK
(d)
F
j
k
k′
J
K
J′
j′
i
I
I′
i′
h′
H′
h
H
g′
G′
F′
O
g
G
F
(c)

FIG. 27.

the basis that the movement of the point F is determined by the deformation along the lines $A F$ and $F K$. The actual deformation of $A F$ is determined in Fig. (*b*) by drawing $A O$ perpendicular to line $A F$ of Fig. (*a*). The distance $F O$ is the deformation required; it is compression. Then likewise, in Fig. (*c*), the distance $F O$, from F to the line $K O$, is the true deformation of line $F K$ of Fig. (*a*). These deformations being known, a diagram, Fig. (*d*), assuming A and K to stand fast, determines the true movement of F, which is the distance $A F$. Then, in Figs. (*b*) and (*c*), the correction diagrams, shown by light full lines, are drawn, making $F F'$ in each one equal to $F K$ of Fig. (*d*). The true movement of any joint is then found as usual from Figs. (*b*) and (*c*), measuring from the correction diagram.

117. *Deflection Due to Temperature Changes.*—For a uniform change of temperature throughout the arch, all the members are affected proportionately and each half of the arch will be exactly similar to the original form, but with all dimensions reduced or increased in the same ratio. From this condition the movement of the crown, or any joint, can readily be stated algebraically.

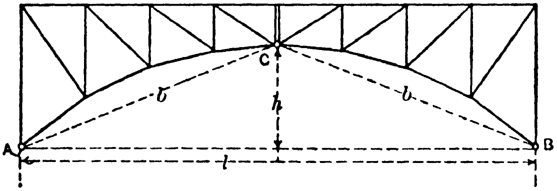


FIG. 28.

Consider the arch of Fig. 28. Let t = change of temperature. ω = coefficient of expansion, and Δ = rise of crown hinge C due to the change of temperature. The length b is increased by the amount $\omega t b$ which, since the span length is fixed, acts to raise point C. From the relation $h^2 = b^2 - \left(\frac{l}{2}\right)^2$ we have, by differentiation, $2\,h\,d\,h = 2\,b\,d\,b$, and hence $d\,h = d\,b\,\frac{b}{h}$. But for small movements $\Delta = d\,h$, and $\omega\,t\,b = d\,b$, hence we have

$$\Delta = \omega\,t\,\frac{b^2}{h} = \omega\,t\,h\left(1 + \frac{l^2}{4h^2}\right). \quad . \quad . \quad . \quad . \quad (11)$$

For other points the movement is readily deduced from that at C.

118. Stresses in Lateral Systems.—A complete lateral system for a three-hinged arch requires an arrangement of bracing as shown in Fig. 29 or in Fig. 30. The dotted lines indicate the planes in which diagonal bracing is required in order to give lateral rigidity. In Fig. 29 a complete upper lateral system is provided, transferring its load to points *a* and c_1 on the left and c_2 and *b* on the right. From these points the loads are transferred by means of vertical transverse bracing to the point *C* and to the abutments at *A* and *B*. The upper laterals cannot be made continuous at the centre, as motion must be free at this point. The lower laterals extend continuously from *A* to *B*. In Fig. 30 the loads acting at the upper panel points are transferred at each panel by means of vertical transverse bracing to the lower lateral system, and thence to the abutments.

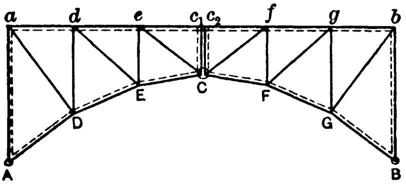


FIG. 29.

Generally, for the sake of added stiffness, transverse bracing is used at every panel and an upper lateral system also provided. In this case the loads will travel from the upper joints to the abutments along the two paths offered, in proportion to their rigidity. By reason of the large section of the lower chord, and the fact that the shortest path to the abutment is down the vertical and thence along the lower chord, it is economical and convenient to provide bracing along this line of travel sufficient to take most or all of the loads on the upper

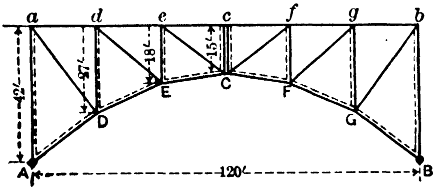


FIG. 30.

joints. The upper laterals along the roadway should then be added for sake of increased rigidity and should be designed to carry one-third to one-half of the loads at the upper joints. The analysis of the several systems will be considered on this basis.

(*a*) *The Upper Lateral Truss.*—The stresses in the upper lateral truss are calculated as for a simple truss with supports at a and c_1, c_2 and b. Lateral forces applied in a plane above the lateral truss will cause an overturning moment, giving rise to vertical loads on the main trusses, as in the case of an ordinary bridge. These loads may be combined with other vertical loads developed by the transverse bracing and the lower laterals, as discussed later.

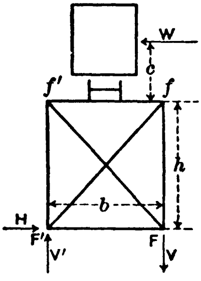


FIG. 31.

(*b*) *The Transverse Bracing.*—Fig. 31 represents the transverse bracing at any panel, and W the amount of the lateral panel load assumed as carried by the transverse bracing (the total upper panel load in Fig. 30), applied at a distance c above the top laterals. Then Hor. comp. $f'F = W$; $V = V' = W\frac{c+h}{b}$; $H = W$. Stress in $f'F' = W\frac{h+c}{b}$. The reactions V and V', reversed in direction, act as vertical loads upon the main trusses at F and F'. The reaction H is supplied by the lower lateral system.

If upper laterals are used then the transverse frames at c and b receive the loads from these laterals and are stressed the same as the end bracing in a simple-span deck bridge.

(*c*) *The Lower Lateral Bracing.*—The lateral forces acting at the several joints are equal to the lower joint loads, plus the lateral forces, if any, brought to the joints by the transverse bracing in the several panels. Having these lateral forces, the shears in the lateral system are found as for a simple truss of the same span. These shears give the lateral components of the web stresses. Since the diagonal members of the lower lateral system do not lie in a horizontal plane, the resultant of the diagonal stress and that of the lateral strut meeting at the same joint is inclined, this inclination being different at each joint. These resultants being determined they can in turn be resolved into horizontal and vertical components in the plane of the main truss. The stresses in the main truss due to these forces, together with those due to the loads brought by the transverse bracing, are then found.

In the case of an arch rib, lateral bracing is usually inserted along both the top and bottom flanges. The lateral load brought down from the roadway together with that applied along the girder itself, may be assumed to be equally divided between the two systems of bracing.

EXAMPLE.—For illustration, the lower lateral stresses of Fig. 30 will be determined. Width between trusses = b = 16 ft. Suppose the lateral force acting along the top chord = 500 lbs. per lineal foot, applied 10 ft. above the plane of the upper chord, and the lateral force acting along the lower chord = 300 lbs. per lineal foot. The total lateral force acting at each joint, C, F, and G, including the load transferred by the cross-bracing = $(500 + 300) \times 20$ = 16,000 lbs. The overturning effect at each panel causes vertical loads at the several panels as follows: At panel C, $(500 \times 20) \times \frac{25}{16}$ = 15,600 lbs.; at panel F, $10{,}000 \times \frac{28}{16}$ = 17,500 lbs.; and at G, $10{,}000 \times \frac{37}{16}$ = 23,200 lbs. These forces act as downward loads on the leeward truss and upward loads on the windward truss.

Fig. 32 represents the lower lateral system, with lateral loads W, each equal

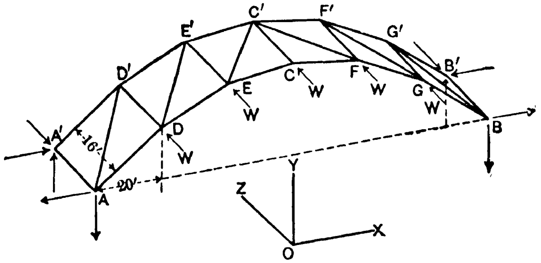


FIG. 32.

to 16,000 lbs. Tension diagonals are assumed. For convenience in dealing with the diagonal members their stresses will be resolved into three components parallel to the axes X, Y, and Z, as indicated; the Z-axis being perpendicular to the plane of the truss. The Z-component in each diagonal will be equal to the shear in the panel, and will have the following values: in panel $C F$ = 8,000 lbs., panel $F G$ = 24,000 lbs., and panel $G B$ = 40,000 lbs. The stress in any diagonal = Z-component $\times \frac{\text{length}}{16}$. The stresses in the several lateral struts are equal to the respective shears.

The effect of the lateral stresses upon the main truss remains to be determined. At each joint resolve the stress in the diagonal into X, Y and Z components. The Z-component is balanced by the stress in the strut; the X and Y components act in the plane of the main truss as loads upon that truss. At joint C' the diagonals $C'F$ and $C'E$ act. The Z-component in each = 8,000 lbs. The X-component $= 8{,}000 \times \frac{20}{16} = 10{,}000$ lbs.; and the Y-component $= 8{,}000 \times \frac{3}{16} = 1{,}500$ lbs. On the leeward truss the X-components act toward the abutments from the centre and the Y-components act downward. At F and E the same forces act, but in an opposite direction. Then in panel FG the X-component of $F'G = 24{,}000 \times \frac{20}{16} = 30{,}000$ lbs.; and the Y-component $= 24{,}000 \times \frac{9}{16} = 13{,}500$ lbs. In $G'B$ the corresponding values are 50,000 lbs. and 37,500 lbs., respectively. Combining the vertical or Y-components with the vertical loads from the transverse bracing, we have the following downward loads acting on the truss $A'B'$. At C', $1{,}500 \times 2 + 15{,}600 = 18{,}600$ lbs.; at F', $13{,}500 + 17{,}500 = 31{,}000$ lbs.; and at G', $37{,}500 + 23{,}200 = 60{,}700$ lbs. The upward loads on truss ACB will be the same numerically, but applied on joints nearer the abutment in each case. The X-components of the diagonal stresses act as horizontal forces at the joints. The total vertical and horizontal forces acting on the truss $A'C'B'$ are shown in Fig. 33. The stresses resulting therefrom are readily determined in the usual manner.

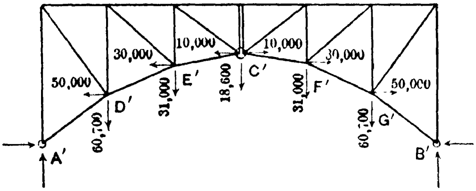


FIG. 33.

SECTION III.—ARCHES OF TWO HINGES

119. General Formulas for Reactions for an Arch Rib of Two Hinges.—Some of the general notation employed will be here given.

ρ = radius of curvature;

α = inclination of arch axis at any point;

α_1 = inclination of arch axis at springing line;

L = length of arch axis;

M = bending moment at any section due to given loads and the true reactions;

M' = bending moment at any section due to given loads and vertical reactions only.

The reactions V_1 and V_2, Fig. 34, are obtained at once from moments at B and A. They are

$$V_1 = \Sigma P (1 - k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$V_2 = \Sigma P k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These are the same as in a simple beam or truss.

The value of H is obtained from the condition explained in Art. 104,

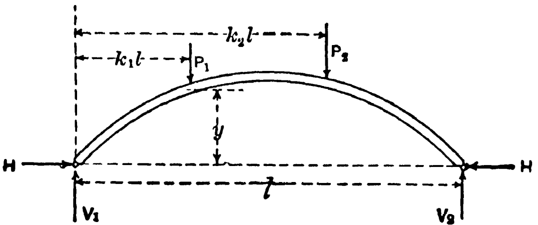


FIG. 34.

that the horizontal deflection of B referred to A and the tangent at A is zero. From eq. (19), Art. 103, noting that $\int_A^B dx = l$, we have, therefore, the condition that

$$\int_A^B \frac{M y\, ds}{E I} - \int_A^B \frac{f\, dx}{E} + \int_A^B \frac{f y\, ds}{E \rho} + \omega t l = 0, \quad . \quad . \quad (3)$$

in which M = bending moment and f = average compressive stress at any section, ω = coefficient of expansion, t = change of temperature, and ρ = radius of curvature. The first term takes account of the effect of bending moment and the second and third terms the effect of direct compression.

To bring eq. (3) into a form to solve for the unknown reaction H, it is necessary to express the quantities M and f in terms of H and known quantities. The bending moment M may be separated into two parts: a moment M' due to vertical forces only (loads and vertical reactions), and a moment M'' due to the forces H. The moment M' is the same as in a simply supported structure and is readily calculated. The moment $M'' = H\,y$, hence

$$M = M' - H\,y. \quad . \quad . \quad (4)$$

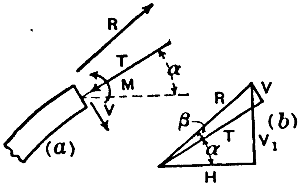


FIG. 35.

To determine f, consider the forces acting at any section, Fig. 35. Fig. (*b*) shows the relation between the resultant R, the thrust T, and the horizontal component H, of the reaction. The angle β is generally small, as the direction of the resultant follows approximately the line of the arch. Hence we have, approximately, $T = H \sec \alpha$, and therefore,

$$f = \frac{T}{A} = \frac{H \sec \alpha}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\int_A^B \frac{f\,y\,d\,s}{E\,\rho} - \int_A^B \frac{f\,d\,x}{E} = \frac{H}{E}\int_A^B \frac{\sec \alpha}{A}\left(\frac{y\,d\,s}{\rho} - d\,x\right) \quad . \quad (6)$$

Again, if the curvature is uniform or approximately so, we have, from Fig. 36, $y = \rho \cos \alpha - \rho \cos \alpha_1$, and $d\,x = d\,s \cos \alpha$, hence $\frac{y\,d\,s}{\rho} - d\,x = (\cos \alpha - \cos \alpha_1)\,d\,s - d\,x = -\,d\,s \cos \alpha_1$, in which α_1 is the inclination at the springing line. The second member of eq. (6) therefore becomes

$$-\frac{H}{E} \cos \alpha_1 \int_A^B \frac{d\,s}{A \cos \alpha}.$$

For any given arch the quantity $A \cos \alpha$ can be calculated for various sections along the arch and an average value determined. Generally the cross-section A will vary approximately with $\sec \alpha$ and for the purposes of the present calculations it is sufficiently accurate to assume this to be the case. If then A_0 represents the value of the cross-section at the crown, or the average value of $A \cos \alpha$, we have $\int_A^B \frac{d\,s}{A \cos \alpha} = \frac{L}{A_0}$, in which L = length of the arch axis. We have then, finally, the approximate expression for the deformation due to thrust, eq. (6),

$$\int_A^B \frac{f\,y\,d\,s}{E\,\rho} - \int_A^B \frac{f\,d\,x}{E} = -\,\frac{H \cos \alpha_1\,L}{E\,A_0}. \quad . \quad . \quad . \quad (7)$$

As this term representing the effect of thrust is relatively small in any case, the errors involved in the approximations used are generally of no practical consequence.

Substituting from (4) and (7) in eq. (3), we have, in terms of H and known quantities,

$$\int_A^B \frac{M' y\,d s}{E I} - H \int_A^B \frac{y^2\,d s}{E I} - \frac{H L \cos \alpha_1}{E A_0} + \omega t l = 0. \quad . \quad (8)$$

Solving for H we get, finally, the general expression,

$$H = \frac{\displaystyle\int_A^B \frac{M' y\,d s}{I} + E \omega t l}{\displaystyle\int_A^B \frac{y^2\,d s}{I} + \frac{L \cos \alpha_1}{A_0}}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eq. (9) is the complete expression for H, and includes the effect of the deformation due to axial thrust and also that due to temperature

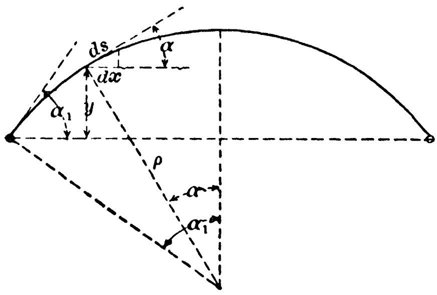


FIG. 36.

Generally the effect of temperature is separately considered, in which case the term $E \omega t l$ in (9) disappears. The effect of temperature alone is given by the equation

$$H_t = \frac{E \omega t l}{\displaystyle\int_A^B \frac{y^2\,d s}{I} + \frac{L \cos \alpha_1}{A_0}}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

120. *Arch on Yielding Supports.*—In case the supports move horizontally a certain distance Δx, due either to settlement or to the use of a tie rod instead of fixed abutments, this term Δx must be added to the second number of eq. (3). The general expression for H in eq. (9) becomes then (neglecting temperature changes)

$$H = \frac{\int_A^B \frac{M' y\,d s}{I} - E \Delta x}{\int_A^B \frac{y^2\,d s}{I} + \frac{L \cos \alpha_1}{A_0}}. \quad . \quad . \quad . \quad . \quad (11)$$

If the tie rod has a cross-section of A_1 and length l, then $\Delta x = \frac{H l}{A_1 E}$ and we have from (11)

$$H = \frac{\int_A^B \frac{M' y\,d s}{I}}{\int_A^B \frac{y^2\,d s}{I} + \frac{L \cos \alpha_1}{A_0} + \frac{l}{A_1}}. \quad . \quad . \quad . \quad . \quad (12)$$

Where a tie rod is used the temperature effect is measured by the difference in temperature between the tie rod and the other truss members.

121. *Relative Effect of Deformation Due to Thrust.*—For arches of considerable rise the effect of distortion due to thrust is small and may be neglected. For $\alpha_1 = 90°$ it becomes zero, and the value of H is then

$$H = \frac{\int_A^B \frac{M' y\,d s}{I}}{\int_A^B \frac{y^2\,d s}{I}}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The relative effect of thrust may be estimated by comparing the two terms in the denominator of eq. (9). Placing approximately $I \cos \alpha =$ constant, $= I_0$, we may write $\frac{d s}{I} = \frac{d s \cos \alpha}{I_0} = \frac{d x}{I_0} = \frac{d x}{A_0 r^2}$, hence the denominator becomes $\frac{1}{A_0}\left[\int_A^B \frac{y^2\,d x}{r^2} + L \cos \alpha_1\right]$. For a parabolic arch with a rise of one-fifth the span length and a depth of rib of one-fifth the rise, the value of the second term is about 1.5 per cent of the first term and hence the error arising from neglecting this term is about 1.5 per cent. For a greater rise or a less depth of rib the error becomes less in proportion to the square of these quantities. Excepting for flat arches the second term may therefore be neglected

However, as the term $\frac{L \cos \alpha_1}{A_0}$ is constant and need be calculated but once, it is very little additional work to include it.

Note that eq. (13) may be written out at once from the general equation for deflection of beams of Art. 214, Part I. The deflection due to vertical loads $= \int_A^B \frac{M' dx}{EI} . m$ and that due to the force $H = H \int_A^B \frac{y\,dx}{EI} m$. Placing these equal and solving for H, we have

$$H = \frac{\int_A^B \frac{M'\,dx}{I} . m}{\int_A^B \frac{y\,dx}{I} . m}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

in which m is the moment due to one-pound loads acting horizontally at B and A, which in this case is equal to y.

122. General Method of Application.—To apply any of the foregoing equations for H it is necessary, first, to assume an arch. Then if it is of such form and section as to permit of integration the solution is directly effected. If the form does not admit of integration the arch must be divided into short lengths and the several values under the integrals determined for the various sections by the process of summation.

123. The Parabolic Arch with Variable Moment of Inertia.—A form of arch which admits of the ready solution of H is one in which

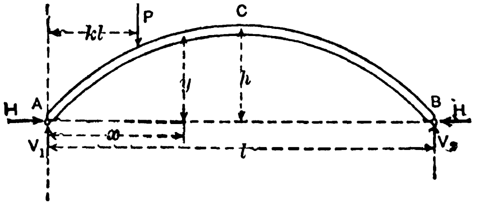


FIG. 37.

the axis is a parabola and the moment of inertia increases in proportion to the inclination of the axis from the horizontal. Even though such form and variation be not strictly followed in the design, the resulting formulas may be applied with little error to flat arches, and may be used for approximate or preliminary values in forms varying more widely from that assumed.

Let $A\,B$, Fig. 37, be a parabolic arch in which I varies with sec α. Let I_0 = moment of inertia at the crown; then in general $I = I_0 \sec \alpha$. With origin at A the equation of the parabolic axis is $\frac{h-y}{h} = \frac{(\frac{1}{2}\,l-x)^2}{(\frac{1}{2}\,l)^2}$,

or
$$y = 4\,h\left(\frac{x}{l} - \frac{x^2}{l^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Also, for any point, $d\,s = d\,x \sec \alpha$. Hence $\frac{d\,s}{I} = \frac{\sec \alpha\, d\,x}{I_0 \sec \alpha} = \frac{d\,x}{I_0}$. Eq. (9) of Art. 119 then becomes

$$H = \frac{\int_A^B M'\,y\,d\,x + E\,I_0 \omega\,t\,l}{\int_A^B y^2\,d\,x + \frac{L\,I_0 \cos \alpha_1}{A_0}}. \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

124. *Value of H for a Single Load P, Temperature Constant.*—Assuming the arch loaded with a single vertical load P, Fig. 37, the value of M' is given by the following expressions:

$$\left.\begin{array}{l} \text{for } x < k\,l,\ M' = V_1\,x = P\,(1 - k)\,x \\ \text{for } x > k\,l,\ M' = V_2\,(l - x) = P\,k\,(l - x) \end{array}\right\} \quad . \quad . \quad . \qquad (17)$$

Substituting these values of M', and the value of y from (15), we have

$$\int_A^B M'\,y\,d\,x = 4\,P\,h \int_o^{k\,l} (1 - k)\,x\left(\frac{x}{l} - \frac{x^2}{l^2}\right) d\,x +$$
$$4\,P\,h \int_{k\,l}^{l} k\,(l-x)\left(\frac{x}{l} - \frac{x^2}{l^2}\right) d\,x, \text{ and } \int_A^B y^2\,d\,x = 16h^2 \int_o^l \left(\frac{x}{l} - \frac{x^2}{l^2}\right)^2 d\,x.$$

Performing these integrations and reducing we have:

For a single load P,

$$H = \frac{\frac{5}{8}\,P\,\frac{l}{h}\,(k - 2\,k^3 + k^4)}{1 + \frac{15}{8}\,\frac{L\,I_0 \cos \alpha_1}{l\,h^2\,A_0}}. \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Except for very flat arches, the second term in the denominator of (18) may be neglected, giving

$$H = \frac{5}{8}\,P\,\frac{l}{h}\,(k - 2\,k^3 + k^4). \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

If a tie rod is used of cross-section A_1, the value of H is found from eq. (12) to be

$$H = \frac{\frac{5}{8} P \frac{l}{h} (k - 2 k^3 + k^4)}{1 + \frac{15}{8} \frac{I_0}{l h^2} \left(\frac{L \cos \alpha_1}{A_0} + \frac{l}{A_1} \right)}. \quad . \quad . \quad . \quad (20)$$

125. *Value of H for Temperature Change.*—For a rise of temperature of t degrees we have, by substituting in (10)

$$H_t = \frac{E I_0 \omega t l}{\frac{8}{15} h^2 l + \frac{L I_0 \cos \alpha_1}{A_0}}. \quad . \quad . \quad . \quad . \quad (21)$$

And for any but flat arches

$$H_t = \frac{15}{8} \frac{E I_0 \omega t}{h^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

126. *Value of H for End Displacement.*—For a horizontal movement of the end of Δx, due to yielding of the support, the value of H is found from (21) by substituting $- \Delta x$ for $\omega t l$, giving

$$H_d = - \frac{E I_0 \Delta x}{\frac{8}{15} h^2 l + \frac{L I_0 \cos \alpha_1}{A_0}}. \quad . \quad . \quad . \quad . \quad (23)$$

In the case where temperature or displacement stresses alone are to be determined the approximate term $\frac{L I_0 \cos \alpha_1}{A_0}$ is more accurately written $\frac{l I_0 \cos \alpha_1}{A_0}$, for in this case the thrust at any point due to the single force H is $H \cos \alpha$, and the value of eq. (7) becomes $- \frac{H \cos \alpha_1 l}{E A_0}$, where A_0 is the average cross-section. The error indeed is, however, of no practical consequence.

127. *The Reaction Locus.*—Since the reaction lines (equilibrium polygon) for load P, Fig. 38, pass through A and B, these lines become known by determining the distance y_0 of the intersection i above the axis. Referring to the force polygon, Fig. 38 (a), we have, by similar figures, $\frac{y_0}{k l} = \frac{V_1}{H}$. Or since $V_1 = P (1 - k)$, we have, in general, for any form of arch,

$$y_0 = k l (1 - k) \frac{P}{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For the parabolic arch, neglecting the effect of rib shortening, we have from (19)

$$y_0 = \frac{1.6\, h}{1 + k - k^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

The locus of y_0, shown in Fig. 38 by the line $m\, n$, is called the "reaction locus." In graphical analysis it is convenient to first construct this locus from the expression for y_0. Having this curve the reaction lines for a load at any point can be drawn at once, and used in the same manner as explained in Art. 108 for the three-hinged arch. The use of this locus is also further expained in Art. 140.

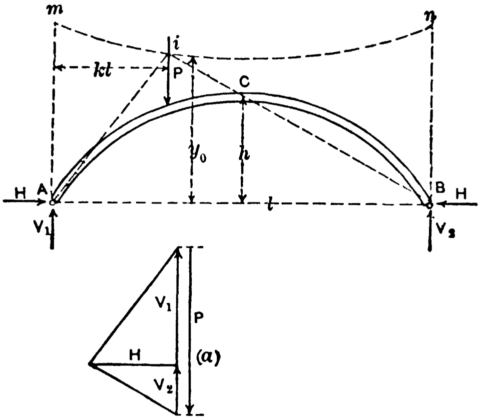


FIG. 38.

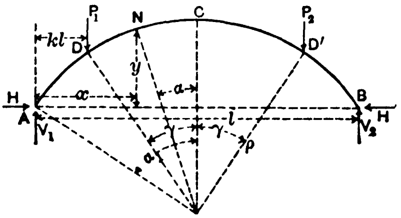


FIG. 39.

128. The Circular Arch of Constant Section.—*Value of H for a Single Load.—Temperature Constant.*—Let $A\,C\,B$, Fig. 39, represent a circular arch of constant section. Let ρ = radius of curvature,

α_1 = inclination of arch at the hinge, γ = inclination at the load point, and α = inclination at any section N. In deriving the value of H for a single load it will simplify the integrations to first determine H for two equal loads, P_1 and P_2, symmetrically placed. The value of H for a single load is then one-half of the value for the two loads.

Referring to the general expression for H, eq. (9), the values of M' and y will first be expressed in polar coordinates. For any section N,

$$x = \rho\,(\sin\alpha_1 - \sin\alpha);\ k\,l = \rho\,(\sin\alpha_1 - \sin\gamma);$$

$$y = \rho\,(\cos\alpha - \cos\alpha_1);\ d\,s = \rho\,d\,\alpha.$$

For two equal loads, P, we have $V_1 = V_2 = P$. Then the values of M' are

$$\text{for } \alpha < \gamma,\ M' = V_1\,k\,l = P\,\rho\,(\sin\alpha_1 - \sin\gamma);$$

$$\text{for } \alpha > \gamma,\ M' = V_1\,x = P\,\rho\,(\sin\alpha_1 - \sin\alpha).$$

Eq. (9) then becomes ($\omega = 0$).

$$H = \frac{P\left[2\rho^3\int_0^{\gamma}(\sin\alpha_1 - \sin\gamma)(\cos\alpha - \cos\alpha_1)\,d\alpha + 2\rho^3\int_{\gamma}^{\alpha_1}(\sin\alpha_1 - \sin\alpha)(\cos\alpha - \cos\alpha_1)\,d\alpha\right]}{2\rho^3\int_0^{\alpha_1}(\cos\alpha - \cos\alpha_1)^2\,d\alpha + \frac{L\,I\cos\alpha_1}{A}}. \quad (26)$$

Performing the integrations, substituting $2\,\rho\,\alpha_1$ for L, and dividing by two, we have, for a single load P

$$H = \frac{P\,\rho^2\left[\sin^2\alpha_1 - \sin^2\gamma - 2\cos\alpha_1\,(\alpha_1\sin\alpha_1 - \gamma\sin\gamma - \cos\gamma + \cos\alpha_1)\right]}{2\,\rho^2\,(\alpha_1 - 3\cos\alpha_1\sin\alpha_1 + 2\,\alpha_1\cos^2\alpha_1) + \frac{2\,\alpha_1\,I\cos\alpha_1}{A}}. \quad (27)$$

For a semicircular arch, $\alpha_1 = 90°$, and the value of H becomes

$$H = \frac{P\cos^2\gamma}{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

129. *Value of H for Temperature Change.*—For a rise of temperature of t degrees the value of H will be obtained from (9), omitting the term containing M'. There results

$$H_t = \frac{E\,I\,\omega\,t\,l}{2\,\rho^3\,(\alpha_1 - 3\cos\alpha_1\sin\alpha + 2\,\alpha_1\cos^2\alpha_1) + \frac{L\,I\cos\alpha_1}{A}}. \quad . \quad . \quad . \quad . \quad (29)$$

For a semicircular arch, $l = 2\rho$ and

$$H_t = \frac{8 E I \omega t}{\pi l^2}. \quad \ldots \ldots \quad (30)$$

130. *Value of H for End Displacement.*—For an end movement of Δx the value of H is given by substituting $-\Delta x$ for $\omega t l$ in eq. (29) or (30).

131. *The Reaction Locus.*—The value of y_0 is given by the general expression of eq. (24). Here we have $k l = \rho (\sin \alpha_1 - \sin \gamma)$; $l = 2 \rho \sin \alpha_1$; and $1 - k = \dfrac{\sin \alpha_1 + \sin \gamma}{2 \sin \alpha_1}$.

Substituting in (24) we have

$$y_0 = \frac{P}{H} \cdot \frac{\rho (\sin^2 \alpha_1 - \sin^2 \gamma)}{2 \sin \alpha_1}. \quad \ldots \ldots \quad (31)$$

For a semicircular arch

$$y_0 = \frac{\pi \rho}{2}, \quad \ldots \ldots \quad (32)$$

that is, the reaction locus for a semicircular arch is a straight line drawn a distance 1.57 ρ above the springing line.

132. Calculation of H for Arch Ribs, for a Single Load P.—Let *A B*, Fig. 40, be a two-hinged arch rib of any form. Neglecting temperature effect the general expression for H is

$$H = \frac{\Sigma \dfrac{M' y\, d s}{I}}{\Sigma \dfrac{y^2 d s}{I} + \dfrac{L \cos \alpha_1}{A_0}},$$

in which M' is the bending moment at any point due to vertical forces only (the same as in a straight beam).

Divide the arch into convenient subdivisions of equal or unequal length. Generally it will be convenient to assume either $d s$ as constant or to make $d s / I$ constant. The constant term $L \cos \alpha_1 / A_0$ is readily calculated. Call this term B. Calculate now the value of $y\, d s / I$ for each subdivision and let Q represent in general this value. Then

$$H = \frac{\Sigma M' Q}{\Sigma y Q + B}. \quad \ldots \ldots \quad (33)$$

The quantity M' for a single load is given by the equations

$$\left.\begin{aligned} &\text{for } x < k\,l,\ M' = P\,(1 - k)\,x \\ &\text{for } x > k\,l,\ M' = P\,k\,(l - x) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (34)$$

Hence we may write

$$\Sigma\, M'\,Q = P\,\Sigma_o^{k\,l}\,(1 - k)\,x\,Q + P\,\Sigma_{k\,l}^{l}\,k\,(l - x)\,Q. \quad . \quad (35)$$

Now suppose a load equal to Q be applied to a beam $A\,B$ at a distance x from A, Fig. 40 (*b*). The bending moment at D due to this load will be equal to $Q\,x\,(1 - k)$ for $x < k\,l$, and $Q\,(l - x)\,k$ for $x > k\,l$. If a series of loads equal to the several values of Q be applied to the beam at the corresponding distances x, the total moment at D due to all these loads will be equal to $\Sigma_o^{k\,l}\,Q\,x\,(1 - k) + \Sigma_{k\,l}^{l}\,Q\,(l - x)\,k$. This quantity is identical with the value of $\Sigma\,M'\,Q$ of eq. (35) for $P = 1$. The value of $\Sigma\,M'\,Q$ for a unit load at other points will likewise be equal to the bending moment in a straight beam at the corresponding point, due to the same loads Q as used before. Hence to determine all the values of $\Sigma\,M'\,Q$, or $\Sigma\,M'\dfrac{y\,d\,s}{I}$, for all sections of the arch, calculate the bending moments in a straight beam at corresponding

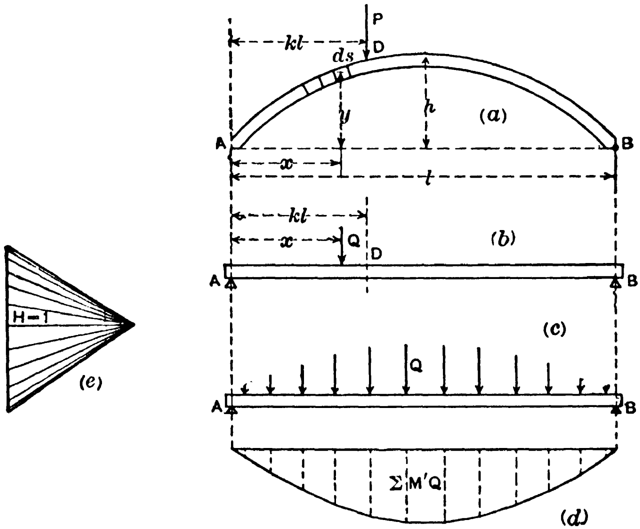


FIG. 40.

points, due to the loads Q. Graphically, all of these values will be given by a single equilibrium polygon, using the several values of Q for loads. Such construction is shown in Fig. (*c*), (*d*), and (*e*). Taking a pole distance of unity, the ordinates to the equilibrium polygon give at once the desired values.

The quantity $\Sigma\, y\, Q$ of eq. (33) may be calculated graphically in a somewhat similar manner, but as this term is constant no great advantage is gained. For graphical construction lay off the values of Q as horizontal loads acting on a vertical cantilever beam of length h, at the proper distances y from the fixed end. Then an equilibrium polygon gives $\Sigma\, y\, Q$.

The values of H for unit loads are now obtained by dividing the values of $\Sigma\, M'\, Q$, or the ordinates of equilibrium polygon of Fig. 40 (*d*), by the quantity $\Sigma\, y\, Q + B$, thus giving the influence line for H. The equilibrium polygon of Fig. (*d*) is itself such an influence line, using as the scale unit the quantity $\Sigma\, y\, Q + B$. Or, this quantity may be used for the pole distance in Fig. 40 (*e*), in which case the polygon becomes the influence line for H. For an example of the use of this method see Art. 146.

133. The Braced Arch.—*General Formula for H.*—For the braced or trussed arch with parallel chords and small depth, the methods and formulas for solid beams can be applied with little error. In such forms the influence of the web members is relatively so small that the deformations are practically the same as for a solid beam of the same moment of inertia (determined from chord sections only). Hence the reactions will be the same as for the beam and may be determined in the same way. The reactions being known the chord and web stresses are found from moments and shears as explained in Art. 96.

For braced arches of variable depth the results obtained from beam formulas are not sufficiently accurate, and for such forms the general method of deflections must be applied to the actual form selected. The application of this method requires a preliminary design to be made in some manner, as the deflections are dependent upon the cross-sections of the members. This preliminary design may be made by the use of an approximate formula for H, such as for the parabolic arch rib, or it may be assumed for the first analysis that the cross-sections of all members are the same, or they may be varied in proportion in accordance with some similar design previously worked out. It is to be noted that the actual cross-sections are not needed, but only the relative sections of the various members.

The preliminary design being made in whatever manner, the true value of H for any loading is given by the general formula for redundant members of Art. 222 of Part I. Here the reaction H will be taken as redundant. The general value of H is (E being assumed constant):

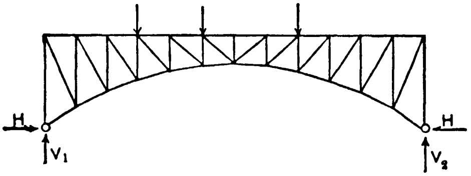


FIG. 41.

$$H = \frac{\Sigma \frac{S' u l}{A}}{\Sigma \frac{u^2 l}{A}}, \quad . \; . \; . \; . \; . \; . \; . \quad (36)$$

in which S' = stress in any member due to the vertical forces (loads and vertical reactions) only;

u = stress in any member due to a force of one pound applied horizontally at the hinges and acting *outward;* and

l and A = length and cross-section of any member.

The numerator of (36) is the outward deflection of one end, with arch supported on rollers and acted upon by the vertical forces, and the denominator is the outward deflection of the same point caused by a one-pound load.

In this case the formula takes full account of all effects of deformation in each member, whether due to moment or thrust, so-called, and hence is not subject to the correction for thrust as in the case of a beam formula.

134. *Use of a Tie Rod.*—If a tie rod be employed to restrain the ends then the problem is exactly the same as that of a single redundant member. The formula is the same as (36), but the summation in the denominator must now include the tie rod for which $u = 1$.

135. *Value of H for a Single Load.*—For the analysis of live-load stresses it is desirable, as in other similar problems, to determine the reactions for a single load at each load point. If the algebraic method is used it requires the application of eq. (36) several times, which is a somewhat tedious operation. The stresses u may conveniently be determined by a single stress diagram for a unit horizontal force at each end. The stresses S' are different for each different position of load, but, noting that for members to the left of the load the stresses are proportional to V_1 and for those on the right they are proportional to V_2, the calculations become relatively simple.

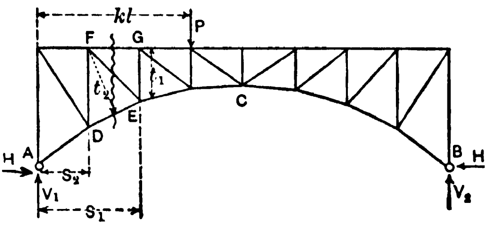


FIG. 42.

Thus in Fig. 42 the stress in FG for load P at G, or at any point on the right of the section, $= V_1 s_1/t_1 = P(1-k)\frac{s_1}{t_1}$; and likewise, stress in $DE = P(1-k)\frac{s_2}{t_2}$; and, in general, the stress in any member $= V_1 s/t$, where s = lever arm of the reaction V_1, and t = lever arm of the member. Hence, for all positions of P on the right of the section cutting the given member, the stress is found by multiplying the quantity $P\,s/t$ by the proper value of $(1-k)$. When the load is on the left of the section the stress is found from the right reaction, and is equal to $P\,k\frac{s_1}{t}$, where s_1 is the arm of the right reaction.

A further simplification may be made in the calculations by the use of symmetrical loads. Assume the truss loaded with two unit loads P, placed symmetrically (Fig. 43). The value of H determined for these two loads will be twice that due to a single load. The stresses due to the symmetrical loads are very readily calculated. The vertical reactions are each equal to P. Then for members to the left of P_1, the stress is the same for all positions of the loads, and is equal to $P_1\, s/t$, in which s is the lever arm of the reaction and t is the lever arm of the member. For all members between the loads, the moment of the external forces is constant and equal to $P\, k\, l$. Hence all these stresses are equal to $P\, k\, l/t$. A single graphical diagram will give all

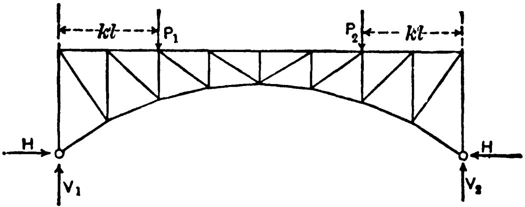


FIG. 43.

the stresses $P\, s/t$. The stresses $P\, k\, l/t$ are obtained by calculating the values of l/t and then multiplying by the various values of k for the different positions of the loads. This method is illustrated in the example of Art. 146.

136. *Graphical Method of Calculating H for a Single Load.*—In calculating H for trussed arches, use may be made of the displacement diagram according to the method employed in Art. 223, of Part I, as follows: Construct a displacement diagram for the arch (Fig. 44),

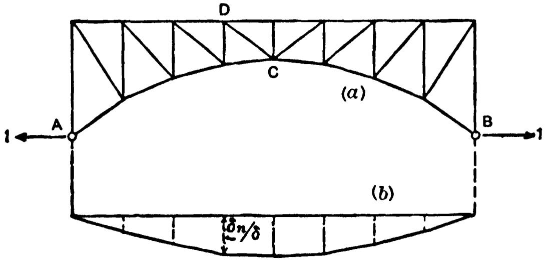


FIG. 44.

loaded with unit loads at A and B and acting outward, no other loads on the structure. The centre vertical may be assumed as standing fast. The stresses for use in this process can be most readily obtained from a stress diagram. Then from this displacement diagram measure the vertical deflections of the several load points relative to the fixed ends A and B, and also the horizontal deflection of B. Let δ_n represent the vertical displacement of any point, and δ the horizontal movement of B with respect to A (twice the displacement for one-half the arch). Then if the values δ_n/δ be plotted as in Fig. (*b*), the resulting curve becomes the influence line for H. For if $\delta_n =$ vertical movement of any load point D, due to one pound at B, then, conversely, the horizontal movement of B caused by a unit load at D will also be δ_n. The deflection due to one pound at $B = \delta$, hence the reaction developed by one pound at D will be δ_n/δ. The value δ_n is, in fact, the value of $\Sigma \frac{S' u l}{A}$ of eq. (36) and $\delta = \Sigma \frac{u^2 l}{A}$ (E being omitted throughout). A single displacement diagram will thus give the necessary data for the determination of H for all load points. See Art. 146, for example.

137. *The Reaction Locus* for the braced arch is determined in the same way as explained in Art. 127 for the arch rib, and the value of y_0 is given by the general formula of eq. (24). The locus is used in the manner described in Art. 140.

138. *Value of H for Temperature Change.*—A rise of temperature of t degrees causes an increase of span length, if unrestrained, of $\omega t l$. The thrust necessary to resist this is found by replacing the numerator of eq. (36) by $\omega t l$ (retaining the value E), giving

$$H_t = \frac{\omega t l}{\Sigma \frac{u^2 l}{E A}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (37)$$

If a tie rod is employed, subjected to the same temperature variation as the arch, there will be no change in H. If the tie rod is subjected to an increase of temperature t', different from t, then

$$H = \frac{\omega (t - t') l}{\Sigma \frac{u^2 l}{E A}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

139. Stress Calculation.—*A. Dead-Load Stresses.*—The reactions having been determined, the dead-load stresses are determined in the same manner as explained for the three-hinged arch. For the parabolic or circular arch, for which convenient formulas exist for H for a single load, the total value of H for dead load is conveniently found by summation for the several joint loads. For the more general case of Arts. 119–122, and for the braced arch, the summations may be made at once for the entire dead load; but as the calculation of live-load stresses requires the determination of H for a single load at each joint, the value of H for dead load is in this case also conveniently found by summation of results for separate joint loads.

140. *B. Live-Load Stresses.—Use of the Reaction Locus.*—In finding the position of loads for maximum stress the reaction lines may be used as in Art. 108, for the three-hinged arch. The reaction locus is first constructed, after which the method is the same for all forms of arches.

Let Fig. 45 represent any two-hinged arch and $m\,n$ the reaction locus. Then for maximum stress in member $C\,D$, with centre of moments at F, a load at i produces no stress, while all loads to the right produce tension and all loads to the left compression. For $F\,E$, all loads to the right of i' cause compression and all loads to the left tension. For the web member $D\,F$, draw $A\,i''$ toward the intersection of $C\,D$ and $F\,E$. Then loads between D and i'' cause compression in $D\,F$, while loads to the right of i'' and to the left of F cause tension in $D\,F$. For such a piece as $E\,I$, with centre of moments at G, loads between i''' and i^{IV} produce positive moment, or tension in $E\,I$, while loads on the remaining portions produce compression.

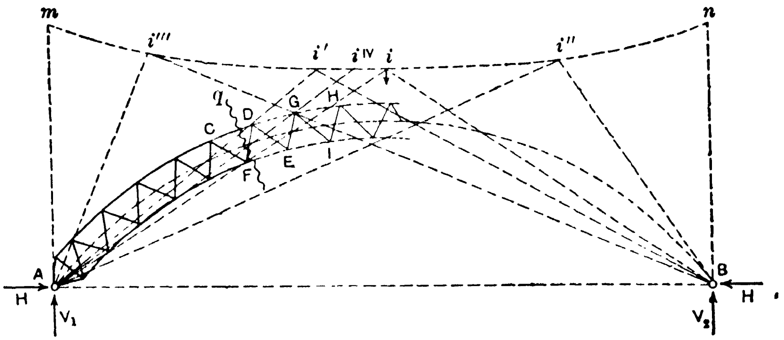


FIG. 45.

141. *Use of Influence Lines.—Influence Lines for Moments.*—The value of the bending moment at any point F, Fig. 46, is

$$M = M' - H\,y_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

in which M' is the moment due to the vertical forces only. As in Art. 111, the influence line for M may be constructed by combining that for M' with the curve for $H\ y_1$. Instead, however, of drawing the curve for $H\ y_1$, thus requiring a different curve for each centre of moments,

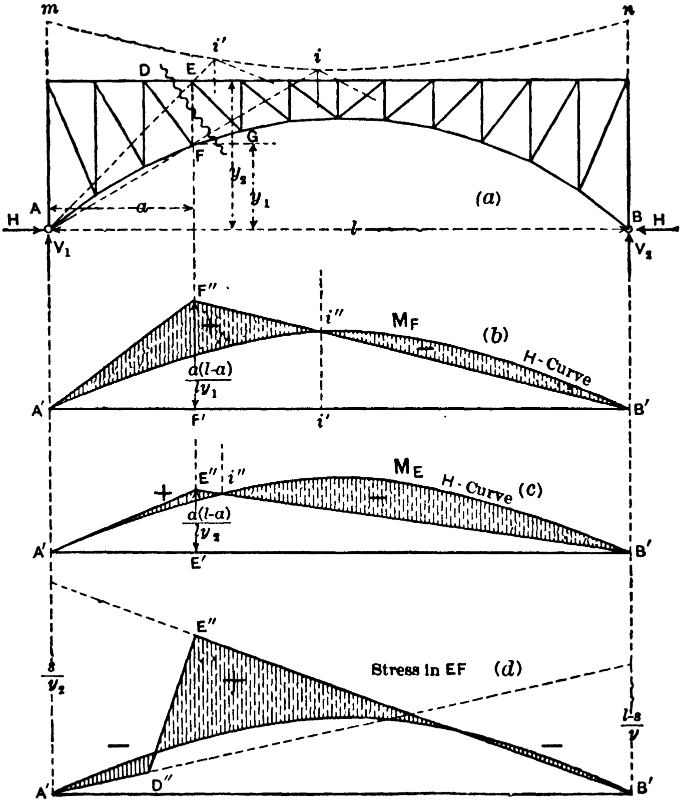


FIG. 46.

the curve for H may be used and the influence line drawn for M'/y_1. Such a diagram for point F is shown in Fig. 46 (*b*). The ordinate $F'F'' = M'/y_1 = \frac{a\,(l - a)}{l\,y_1}$. The shaded area will then serve as the influence diagram for M_F, but each ordinate will need to be multiplied by y_1 to give the true value of M, for $M = \left(\frac{M'}{y_1} - H\right) y_1$.

As in Art. 111 the influence ordinates become zero at point i'', corresponding to point i on the reaction locus. This locus is, however, not needed in the construction of influence lines.

Fig. (*c*) shows the influence line for moment at E.

142. *Influence Lines for Web Stress or Shear.*—For stress in any web member, as $E F$, the intersection of $D E$ and $F G$ is found, as usual. This will be in the line of the top chord, distant y_2 from H. If s is the lever arm of V_1 about such intersection and t is the arm of member $E F$, then for loads on the right the stress in $EF = \left(V_1 s - H y_2\right) \div t = \left(V_1 \frac{s}{y_2} - H\right) \frac{y_2}{t}$, and for loads on the left, stress in $EF = \left(V_1 \frac{l-s}{y_2} - H\right) \frac{y_2}{t}$. The influence line for the quantity in parenthesis is drawn in Fig. 46 (*d*). For actual stress the shaded ordinates are to be multiplied by $\frac{y_2}{t}$.

In the case of a beam, or truss with parallel chords, the influence lines for shear will be needed. These are drawn in a manner similar to that explained in Art. 113 for the three-hinged arch, using the influence line for H as a basis. The shear at D (Fig. 47), for a unit load on the right, $= V_1 \cos\alpha - H \sin\alpha = (1 - k) \cos\alpha - H \sin\alpha$; and for loads on the left it is $- V_2 \cos\alpha - H \sin\alpha = k \cos\alpha - H \sin\alpha$. These values may be written in the form $[(1 - k) \cot\alpha - H] \sin\alpha$, and $(- k \cot\alpha - H) \sin\alpha$. In Fig. 47 (*b*) the line $A' C'' B'$ represents H, and the lines $A'' B'$ and $A' B''$ represent $(1 - k) \cot\alpha$ and $k \cot\alpha$. The shaded area, therefore, is the influence diagram for shear at D. The true values are found by multiplying these values by $\sin\alpha$. For points near the centre $\cot\alpha$ becomes very large and $\sin\alpha$ small, as the effect of H is small. A reduced scale for H may be used for such points. For the centre point the influence of H becomes zero and the influence line is the same as for a straight beam.

143. *Influence Lines for Thrust.*—In analyzing beams it may be convenient to calculate moments with reference to the neutral axis and then to determine the thrust or direct stress separately. For a unit load on the right of point D, Fig. 47, the thrust is $H \cos\alpha + V_1 \sin\alpha = H \cos\alpha + (1 - k) \sin\alpha$; and for a load on the left, the thrust is $H \cos\alpha - V_2 \sin\alpha = H \cos\alpha - k \sin\alpha$. These values

may be written in the form $[H + (1 - k) \tan \alpha] \cos \alpha$ and $(H - k \tan \alpha) \cos \alpha$, respectively. In Fig. 47 (*c*) the curve $A' C'' B'$ is the curve for H. The values $(1 - k) \tan \alpha$ and $k \tan \alpha$ are given by the ordinates to the lines $A'' B'$ and $A' B''$, respectively. The ordinates of the shaded area therefore represent the quantities $H + (1 - k) \tan \alpha$

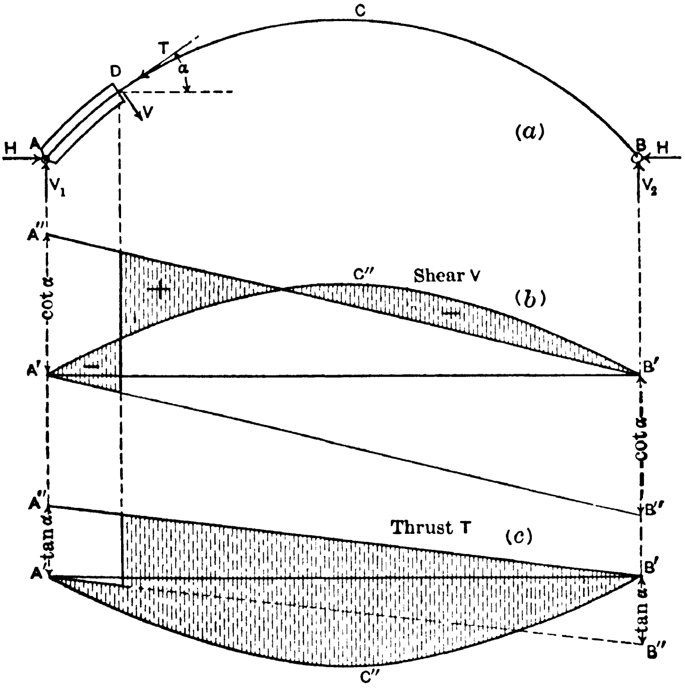


FIG. 47.

and $H - k \tan \alpha$, and hence will serve as the influence diagram for thrust. The true values are found by multiplying all ordinates by $\cos \alpha$.

In general, the maximum thrust at all sections occurs for a fully loaded structure, but the maximum flange or fibre stress occurs for partial load such as will cause large bending moments. The maximum fibre stress can best be found from influence lines for moment, with centre of moment taken at the centre of gravity of the flange or the "kern" point, as explained in Art. 95.

144. *Position of Loads for Maximum Stress.*—The form of the influence lines being curved, no convenient criterion for position of loads can be stated. For all practical purposes, however, each segment of the influence diagram may be considered a triangle and the position of loads, or the suitable equivalent uniform load, determined on this basis. In getting actual stresses, either for a uniform load or for concentrations, the true influence diagram should be used.

145. Examples.—(1) *Plate Girder Highway Arch.* (Fig. 48.)—An analysis will be made of an arch rib in the form of a plate girder. Assume the following data: Span length = 180 ft.; rise = 36 ft.; depth of girder between flange centres = 4 ft. The arch axis will be taken parabolic in form and the load verticals placed 15 ft. apart horizontally. The live load will be taken at 1,500 lbs. per ft. per girder and the dead load at 1,000 lbs. per ft. Stresses will also be calculated for a temperature change of 60 degrees.

146. *Approximate Analysis.*—The first analysis of stresses will be made by the use of the formulas of Arts. 123–125, these being based on the assumption of a parabolic axis and that the moment of inertia varies with sec α. The effect of thrust will be neglected at first. The formula for H is (eq. (19), Art. 124), $H = \frac{5}{8} P \frac{l}{h} (k - 2k^3 + k^4)$, and for y_0, the ordinate to the reaction locus (eq. (25)), $y_0 = \frac{1.6\,h}{1 + k - k^2}$. Values of H and of y_0 for unit loads, calculated from these formulas, are given in Table A. Fig. 48 (*b*) shows the influence line for H plotted from these results. The reaction locus is also given in Fig. 48 (*a*).

TABLE A.

Values of y_0 and H for P = 1.

Load Point.	k	y_0	H
b	1/12	53.5	0.256
c	2/12	50.6	0.495
d	3/12	48.5	0.695
e	4/12	47.2	0.848
f	5/12	46.3	0.943
g	6/12	45.7	0.985

The influence line for H having been constructed, the influence lines for moments are drawn on the same diagram, as explained in Art. 141. Such lines (section d) are shown in Fig. (*b*) for upper and lower flange centres, for the quarter-point and the centre (section g). The values of the various ordinates are given in Table B.

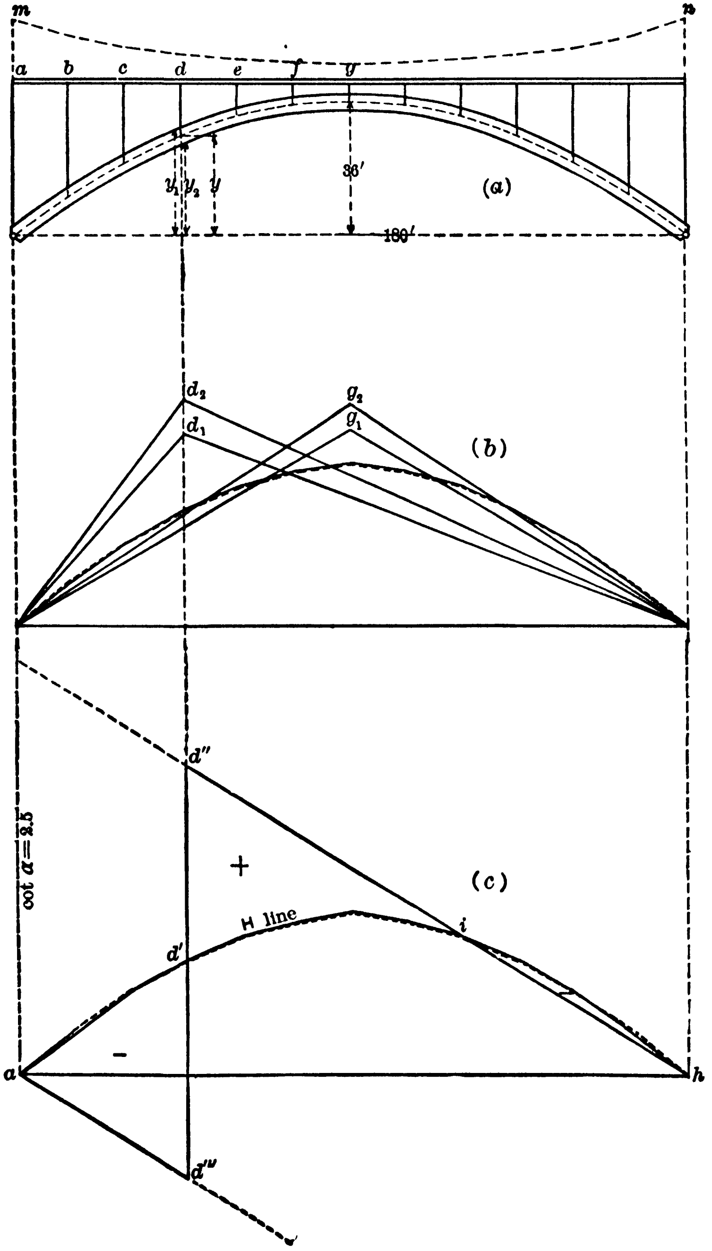


FIG. 48.

TABLE B.

Values of Ordinates y and $\frac{a(l-a)}{l}$.

Load Point.	y	y_1	y_2	$\frac{a(l-a)}{l y_1}$	$\frac{a(l-a)}{l y_2}$
b	11.0	12.72	9.85	1.082	1.396
c	20.0	21.75	18.30	1.150	1.366
d	27.0	28.95	25.25	1.165	1.335
e	32.0	33.95	30.05	1.176	1.330
f	35.0	37.00	33.00	1.182	1.325
g	36.0	38.00	34.10	1.183	1.323

From these influence lines the dead- and live-load moments are obtained. To get the moment in any case, the proper influence area is to be multiplied by the corresponding value of y_1 or y_2 and by the load per foot. Thus for the dead-load moment at section *d*, top flange, the total area of the influence diagram is $-$ 6.29. Multiplying this by y_1 of Table *B* and by the dead load per foot, gives $M = -6.29 \times 28.95 \times 1{,}000 = -182{,}000$ ft.-lbs. The results for sections at *d* and *g* are given in the table below.

Section.	Moment Centre	+ Area.	− Area.	Σ Areas.	D. L. Moment.	Live-Load Moment.	
						+	−
d	Top	15.84	22.33	− 6.29	− 182,000	687,000	967,000
	Bottom	25.92	15.63	+ 10.29	+ 260,000	982,000	395,000
g	Top	3.88	9.04	− 5.16	− 196,000	221,000	515,000
	Bottom	10.54	3.32	+ 7.22	+ 245,000	538,000	169,000

From the values of the maximum moments found in this way a preliminary design was made on the basis of a working stress of 15,000 lbs. per sq. in. on gross area, with resulting flange areas as follows:

Section.	Gross Flange Area.
b,	15 sq. in.
c, d, e,	20 "
f, g	18 "

Any portion of the web to be considered as flange is included in these calculated areas.

147. *Exact Calculation of Stresses.*—Having now an actual design at hand we may proceed to determine the true stresses therein by the methods explained in Art. 132. The first step is to calculate a new set of values for H. The formula is, neglecting temperature effect,

$$H = \frac{\Sigma \frac{M' y\, d s}{I}}{\Sigma \frac{y^2\, d s}{I} + \frac{L \cos \alpha_1}{A_0}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and for temperature changes

$$H_t = \frac{E\, \omega\, t\, l}{\Sigma \frac{y^2\, d s}{I} + \frac{L \cos \alpha_1}{A_0}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Table C gives the calculations required by eq. (a). The value of $d s$ at each section is taken as one-half the sum of the lengths of the two panels adjacent to the section in question. This leaves a half-length of 9.1 ft. at the end, whose middle ordinate, y, is 3.0 ft. Foot-units are used throughout.

TABLE C.

Calculations for H for Unit Loads.

Section.	y	$d s$	I	$\frac{y\, d s}{I}$	$\Sigma \frac{M' y\, d s}{I}$	$\frac{y^2\, d s}{I}$	H
a	3.0	9.1	.833	22	80	70	0.000
b	11.0	17.9	.833	236	32,800	2,600	0.262
c	20.0	17.0	1.111	306	62,000	6,120	0.495
d	27.0	16.1	1.111	391	86,500	10,580	0.690
e	32.0	15.4	1.111	443	105,400	14,200	0.840
f	35.0	15.3	1.000	535	117,700	18,750	0.937
g	36.0	15.1	1.000	544	121,900	19,600	0.973
		105.9		2,477		71,920	

As explained in Art. 132, the values of $\Sigma \frac{M' y\, d s}{I}$ may be obtained by considering the several quantities $\frac{y\, d s}{I}$ as applied as loads to a simple beam, in the same verticals as the several load points, b, c, d, etc., and calculating the bending moments at these several sections. This calculation may be made graphically or algebraically. The reaction for such a beam will be $2{,}477 - \frac{1}{2}(544) = 2{,}205$ (the section g being at the centre). The moment at sec. b, for example, will then be equal to $2{,}205 \times 15 - 22 \times 11.25 = 32{,}800$; that at c $= 2{,}205 \times 30 - 22 \times 26.25 - 236 \times 15 = 62{,}000$, etc. The several values are given in the table. The quantity $\Sigma \frac{y^2\, d s}{I}$ for the entire arch $= 71{,}920 \times$

2 − 19,600 = 124,240. The value of $\frac{L \cos \alpha_1}{A_0}$ of eq. (a) $= \frac{197 \times 0.78}{0.25} = 610.$

The value of H for any load point is then equal to $\frac{\Sigma \frac{M' y\, d s}{I}}{124,240 + 610}$. The several values are given in Table C. The small influence of arch shortening, represented by the term 610, is to be noted; also the fact that the small section $d\,s$ at a, in Table C, may be neglected without affecting the value of H. Compare these values with the values in Table A.

From the new values of H a new influence line is drawn, shown by the dotted lines in Fig. 48. The new influence areas may then be determined and the corrected values of bending moments. For the quarter-point and crown the results are as follows:

Section.	Moment Centre.	D. L. Moment.	Live-Load Moment.	
			+	−
d	Top	− 136,300	694,000	898,000
	Bottom	+ 297,000	1,000,000	552,000
g	Top	− 161,800	262,000	505,000
	Bottom	+ 277,000	583,200	167,000

Compare these values with those given on p. 171.

148. *Temperature Stresses.*—Assuming a change of temperature of 60° from the normal, eq. (b) gives $H_t = + 2,400$ lbs. The moments due to this are equal to $\pm H_t y_1$ and $\pm H_t y_2$. The values for sections d and g are as follows:

Section.	Moment Centre.	Moment Due to Temperature.
d	Top	± 69,500
	Bottom	± 60,700
g	Top	± 91,400
	Bottom	± 81,700

A rise of temperature causes negative moments in the arch.

The value of H_t calculated by the approximate formula of Art. 125 is $H_t = \frac{15\, E\, \omega\, t\, I_0}{8\, h^2} = 2,400$, the same as obtained by the more exact method.

149. *Shearing Stresses.*—In arch ribs the shearing stresses, as compared to the bending stresses, are small. The minimum permissible web thickness will usually be ample, but in designing the riveting of flanges and web it is necessary to determine the maximum shear at a few sections.

Fig. 48 (*c*) shows the influence line for shear at section *d*, plotted as explained in Art. 142. For this section cot $\alpha = 2.5$ and sin $\alpha = 0.37$. Measuring the influence areas we have: area $a\,d'''\,d' = -30.2$; $d'\,d''\,i = +35.7$; $i\,h = -3.3$; total $= +2.2$. Hence, as in Art. 142, dead-load shear $= 1{,}000 \times 2.2 \times 0.37 = +830$ lbs.; maximum live-load shear $= 35.7 \times 1{,}500 \times 0.37 = +19.800$ lbs.

150. *Values of H for a Circular Arch.*—Calculations have been made for a circular arch of the same rise and same cross-sections as the parabolic arch of Fig. 48, resulting in the following values of *H* for unit loads:

Section.	Value of *H*.
b	0.260
c	0.488
d	0.680
e	0.823
f	0.913
g	0.947

The resulting maximum moments are considerably greater at the crown than those of the parabolic arch, but the difference is small between the quarter-point and the end.

151. (2) *Analysis of a Braced Arch.*—Assume an arch of the dimensions shown in Fig. 49 (*a*). The dead load will be taken at 1,200 lbs. per foot per truss and the live load at 3,000 lbs. The joint loads, therefore, will be 30,000 lbs. and 75,000 lbs., respectively.

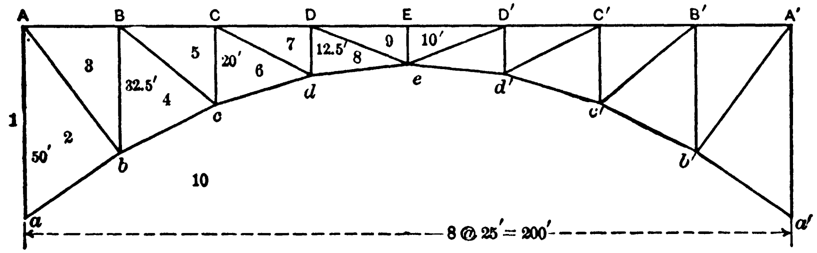


FIG. 49 (*a*).

a First Approximation.—The first analysis will be made on the assumption that the sectional areas of the members are each equal to unity. On this basis the values of *H* for a load unity at each joint are determined. The graphical process will be employed.

The first step is to calculate the stresses in the members for a value of $H = 1$. This is conveniently done by means of a stress diagram, given in Fig. (*b*). From this diagram are obtained the stresses u given in the following table. The values of $u\,l$ are then calculated and used in the construction of the displacement diagram shown in Fig. (*c*). In this construction Ee is assumed to stand fast.

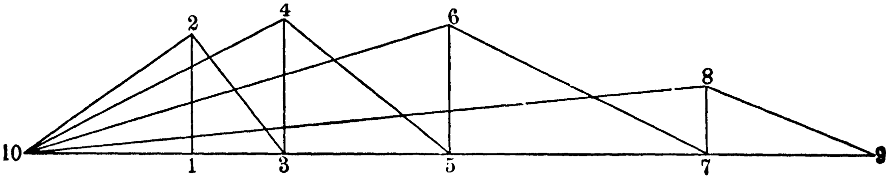


FIG. 49 (*b*).

Member.	Length l.	u (Stress for $H = 1$).	$u\,l$.
AB	300	+ .538	+ 161
BC	300	+1.50	+ 450
CD	300	+3.00	+ 900
DE	300	+4.00	+ 1,200
ab	366	−1.220	− 446
bc	335	−1.720	− 577
cd	313	−2.610	− 817
de	301	−4.019	− 1,212
Aa	600	+ .70	+ 420
Bb	390	+ .769	+ 300
Cc	240	+ .750	+ 180
Dd	150	+ .400	+ 60
Ee	120	......	.....
Ab	492	− .883	− 434
Bc	384	−1.231	− 473
Cd	335	−1.677	− 562
De	323	−1.077	− 348

From the displacement diagram, the horizontal deflection of a is found to be equal to 20,000 (E being omitted in the calculations), or with respect to a' the deflection is 40,000. The vertical deflections are given in the table below. Dividing these values by 40,000 we get the following values of H for load unity at each joint:

Joint.	Deflection.	Value of H.
B	11,700	0.292
C	22,300	0.558
D	31,150	0.779
E	34,950	0.874

From these values the curve (or influence line) for H is plotted in Figs. (*d*), (*e*), and (*f*).

The stresses in the various members are then calculated by means of influence lines plotted as explained in Arts. 141 and 142. Fig. (*d*) shows the influence lines for upper-chord members, and Fig. (*e*) for the lower-chord members. For any chord member, then, the stress is equal to the respective influence line area, multiplied by y, the lever arm of H about the moment center of the member in question, and by p, the load per foot.

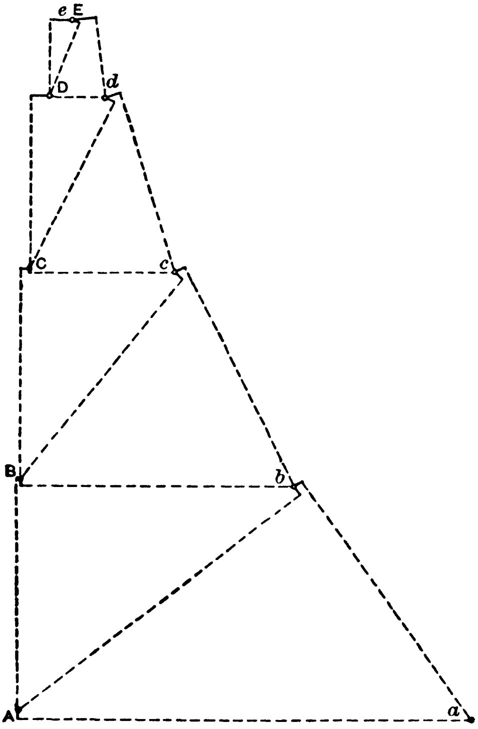


FIG. 49 (*c*).

Fig. (*f*) is the influence diagram for vertical component of stress in the diagonal members. The vert. comp. of stress in any diagonal is, then, load per foot multiplied by the respective influence line area, times the term y/t, which is the vertical distance from H to the moment center divided by the lever arm of the member in question.

The results of this analysis are given in Fig. (*g*) by the figures not enclosed in parentheses. The values given are the combined dead- and live-load stresses. From these stresses an approximate design has been made with sectional areas as shown in Fig. (*g*) by the numbers not in parentheses.

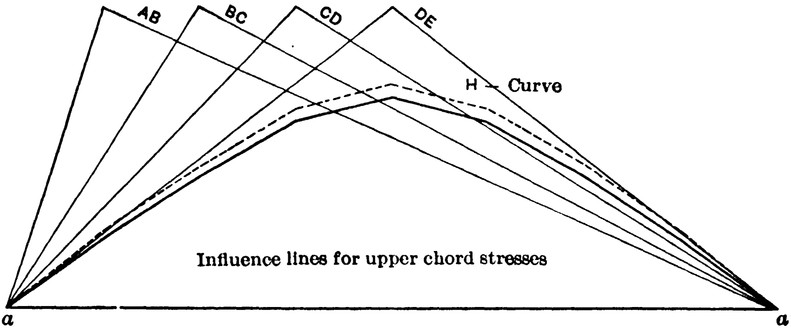


FIG. 49 (*d*).

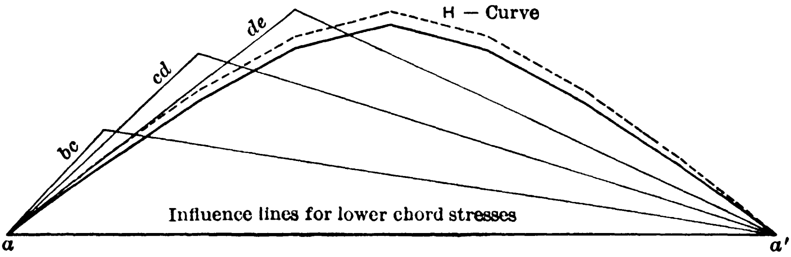


FIG. 49 (*e*).

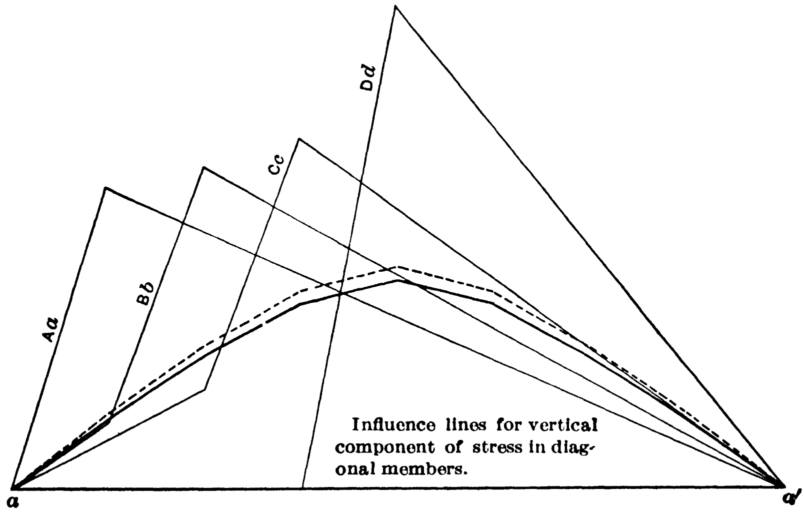


FIG. 49 (*f*).

In the calculations of the sectional areas, 50 per cent. was added to the live-load stresses for impact and the following working stresses used:

For tension members, 16,000 lbs. per sq. in.

For compression members, 16,000 − $70 \times l/r$ lbs. per sq. in.

(*b*) *Second Analysis.*—Having the sectional areas approximately determined, a more exact analysis may now be made by the same method used in the first analysis. The details of this will not be given.

New values of ul/A will need to be calculated and a new displacement diagram for $H = 1$ will be required. The resulting values of H were found to be larger than before, requiring a new curve for H to be plotted in Figs. (*d*), (*e*), and (*f*). The other lines of these diagrams remain the same. The stresses were then calculated using the new influence lines, and the sectional areas revised. In making up the revised areas, the stress used was the *average* of those obtained in the first and second analysis. It was found that in this way it was possible to get a very close agreement between the required area and area provided in the final design.

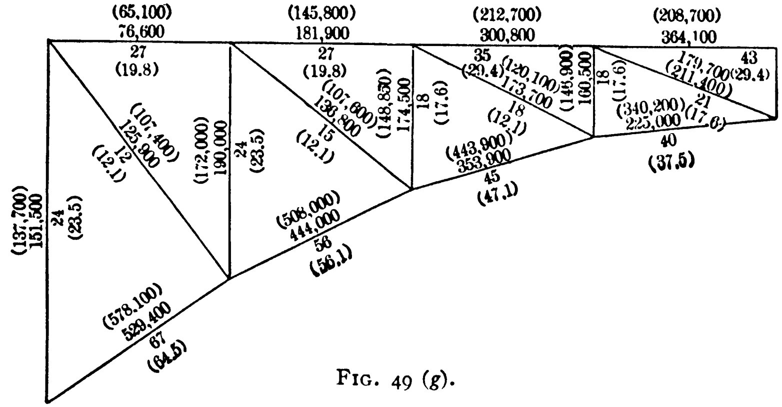


Fig. 49 (*g*).

(*c*) *Third Analysis.*—Using the sectional areas as determined in the second analysis, new values of H were determined, and a new curve for H plotted in Figs. (*d*), (*e*), and (*f*). The new H curve is shown by the dotted lines in the figures. The stresses were again calculated, and it was found that the area provided agreed very closely with the area required in nearly all members. The final sections and stresses are as shown in parentheses in Fig. (*g*).

The final value of the deflection of point a with respect to a' was $1426.0/E$. The values of H are as given in the following table:

Joint	Deflection	Value of H.
B	456	0.320
C	865	0.607
D	1,205	0.845
E	1,365	0.957

d. Temperature Stresses.—The temperature stresses are very quickly found from the calculations already made. Using the final sections determined, the displacement diagram gives a total deflection for $H = 1$, of $1426.0/E = 0.0000492$ in. A rise of temperature of 1° would cause an expansion, if unrestrained, of $.0000065 \times 200 \times 12 = .0156$ in. Hence such a change of temperature would develop a reaction of $H_t = .0156/0.0000492 = 317$ lbs. If a change of 60° is provided for the resulting H is $60 \times 317 = 19{,}000$ lbs. The stresses in the various members are obtained from the table, p. 161, by multiplying the several values of u there given by 19,000. It will be seen that in a few cases the temperature stresses exceed 25 per cent. of the total dead- and live-load stresses.

152.—Deflection of Two-Hinged Arches.—The formulas already developed may be applied to calculate the vertical (or horizontal) deflection at any point of an arch. The general formula for beams is $\varDelta = \int \frac{M\,d\,s}{E\,I}\,.\,m$, in which m is the bending moment due to a load unity applied at the given point and in the direction in which the deflection is desired.

It is generally convenient to calculate the deflection due to the vertical and the horizontal forces separately. Assuming the arch free to move horizontally at one end, the deflection from vertical forces will be

$$\varDelta' = \int \frac{M'\,d\,s}{E\,I}\,.\,m', \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

and from horizontal forces

$$\varDelta'' = \int \frac{H\,y\,d\,s}{E\,I}\,.\,m' = H \int \frac{y\,d\,s}{E\,I}\,.\,m', \quad . \quad . \quad . \qquad (37)$$

in which m' is the bending moment due to the one-pound load, arch considered as a simple beam. Now since $\int \frac{y\,d\,s}{E\,I}\,.\,m'$ is the vertical deflection of the given point for one pound applied horizontally at the hinge, it is also equal to the horizontal deflection of the hinge for a one-pound vertical load applied at the given point, $= \delta_n$ of Art. 136. By that article the value of H for a one-pound load was shown to be equal to δ_n/δ, hence, eq. (37) reduces to $\varDelta'' = H \times H_u\,\delta$, in which H is the total thrust for the given loading, H_u = thrust for one pound applied at the point whose deflection is desired, and δ = horizontal deflection of hinge for $H = 1$. This form of expression is advantageous, as the several values of H_u have already been calculated, and $\delta = \int \frac{y^2\,d\,s}{E\,I}$ of eq. (9). Note, further, that M' and m' of eq. (36) are equal to the moments in a simple straight beam.

Hence in case $\frac{ds}{I}$ = constant = $\frac{dx}{I_0}$, we have $\Delta' = \frac{1}{EI_0}\int_0^l M'\,d\,x\,.\,m'$, which is the deflection of a straight beam of length l and moment of inertia I_0.

The deflections of an arch rib are much less than those of a straight beam of the same depth. They are greatest at about the quarter-point where the moments are a maximum. The value of Δ'', representing the effect of H, is of opposite sign from the term Δ'.

The deflections of braced arches are calculated by the same formula as used for simple trusses, $\Delta = \Sigma\frac{S\,u\,l}{E\,A}$. Here again the vertical and horizontal forces may conveniently be considered separately. The deflection curve determined in Art. 136 serves as the deflection curve for the horizontal forces. That for vertical forces is determined as in a simple span, algebraically or graphically.

The deflection of a two-hinged arch is somewhat less in general than a three-hinged arch, and much less than that of a simple truss of the same span.

153. Wind Stresses.—The lateral bracing of a two-hinged arch is arranged in the same manner as in the three-hinged arch (Art. 118), except that in this case the upper lateral system may be made continuous across the centre. Generally transverse bracing is used at each panel, as well as an upper and a lower lateral system, thus giving a redundant system. [Fig. 50 (*a*).]

The stresses in the upper lateral system and in the transverse bracing are determined by the same general method as explained in Art. 118 for the three-hinged arch. Without the transverse bracing the stresses in the upper laterals and end bracing are found as for a simple span. Also, if the upper laterals are omitted and the transverse bracing used, the stresses in the latter are readily found, as explained in Art. 118. Where both lateral systems and transverse bracing are used, the loads will be carried partly by one system and partly by the other. The depth of the transverse bracing at the centre being small, this bracing is relatively rigid at that point. The lower laterals are also generally more rigid than the upper laterals, especially where the arch is a plate-girder rib. Hence loads applied at joints near the centre will mainly be carried down the transverse bracing and along the lower laterals. Loads applied at joints near the end, as at *d* and *g*, will be carried to a larger extent by the upper laterals, although the path *g G B* is likely to be more rigid than the path *g b B*. Again, the relative rigidity of the two paths depends much upon the size of the laterals themselves, and hence if a certain assumption is made as to the distribution of the loads and the laterals are designed

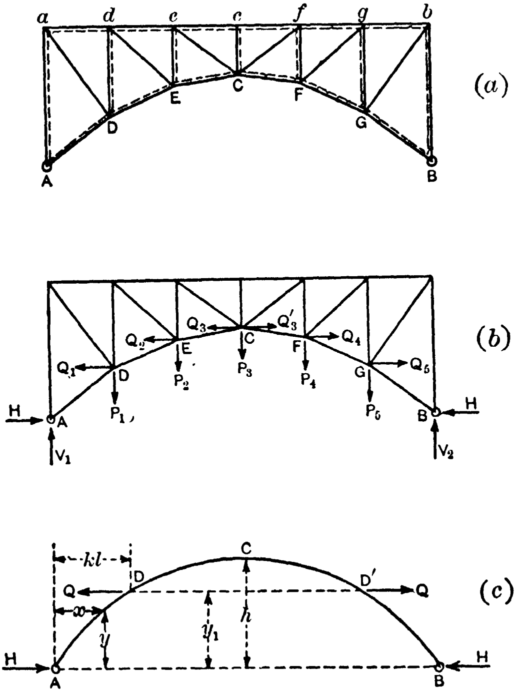


FIG. 50.

accordingly, their relative rigidities tend to follow the same proportions as those assumed in the distribution of the load. It will be safe and sufficiently accurate to design the more rigid path (the transverse bracing and lower laterals in this case) to carry, say, two-thirds or three-fourths of the total lateral pressure and the other system (the upper laterals) to carry the remainder. An exact analysis would require the application of the method of redundant members to the structure as a whole, and as there is one redundant lateral member in each panel and one redundant member (or reaction) in each vertical truss, such a solution would be very laborious.

The loads transferred to the lower laterals having been estimated, these are combined with the loads acting directly thereon, and the shears and stresses in the lower lateral diagonals determined as in Art. 118. The resulting forces acting in the plane of the main truss are also to be determined as in that article (Fig. 33). The effect of these forces upon the main truss remains to be considered.

Fig. 50 (*b*) represents the leeward main truss, and the horizontal and vertical forces acting at the several joints due to the action of the transverse bracing and laterals. The problem is to determine the value of H for these forces in the same manner as for any other system of loads acting in the plane of the truss.

The value of H for the vertical loads P, for either the arch rib or the braced arch, is determined by methods already explained. The value of H for the horizontal forces Q is found by similar methods. In the case of the braced arch, the simplest method is to construct a displacement diagram of the truss for unit horizontal forces at A and B, as in Art. 136. Then from this diagram determine the horizontal displacement of the several joints D, E, etc., with respect to A. Then the horizontal displacement of any joint, divided by that of B, will equal the reaction H due to unit horizontal load at the joint in question. In this manner the values of H for ΣQ can be determined and the stresses then found.

If an arch rib is in question the value of H due to the horizontal loads Q requires the use of the general methods of Art. 119. Considering two equal symmetrical loads Q [Fig. (*c*)], acting at D and D', the resulting values of H will be equal to each other, as for vertical loads, and the vertical reactions will be zero. The bending moment at any section is then equal to $M' - H y$, as in eq. (4) of Art. 119, in which M' is the bending moment due to the forces Q with the reactions H removed. Finally, omitting the temperature term and also the term taking account of the effect of thrust, we have in general, as in eq. (9),

$$H = \frac{\int \frac{M' y \, d s}{I}}{\int \frac{y^2 \, d s}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Let $k\,l$ and y_1 be the coordinates of point D, referred to A as origin. In this case the value of M' for the portion $A\,D$ is zero, and for $D\,C$, $M' = Q\,(y - y_1)$, hence we have

$$\int \frac{M'\,y\,d\,s}{I} = 2\,Q \int_D^C \left(\frac{y^2\,d\,s}{I} - \frac{y_1\,y\,d\,s}{I}\right). \quad . \quad . \quad (39)$$

For the calculation of lateral stresses the arch may be assumed of parabolic form, and also that $\frac{d\,s}{I} = \frac{d\,x}{I_0}$, as in Art. 123. We also have $y = 4\,h\left(\frac{x}{l} - \frac{x^2}{l^2}\right)$. Substituting this value in (39) and integrating, we have

$$\int_D^C \frac{y^2\,d\,s}{I} = \frac{1}{I_0}\left[\frac{4}{15}\,h^2\,l - 16\,h^2\,l\left(\frac{k^3}{3} - \frac{k^4}{2} + \frac{k^5}{5}\right)\right]$$

and

$$\int_D^C \frac{y_1\,y\,d\,s}{I} = \frac{y_1}{I_0}\left[\frac{h\,l}{3} - 4\,h\,l\left(\frac{k^2}{2} - \frac{k^3}{3}\right)\right].$$

Also, as in Art. 124, $\int_A^B \frac{y^2\,d\,s}{I} = \frac{8\,h^2\,l}{15\,I_0}$. Substituting in (38) and reducing we get

$$H = Q\left[(1 - 20\,k^3 + 30\,k^4 - 12\,k^5) - \frac{y_1}{4\,h}(5 - 30\,k^2 + 20\,k^3)\right]. \quad . \quad (40)$$

By the application of eq. (40) for each of the pairs of loads Q, the final value of H is obtained and thence the stresses in the arch truss or rib.

SECTION IV.—ARCHES WITHOUT HINGES

154. General Formulas for Arch Ribs.—Let $A\,B$, Fig. 51, be a symmetrical arch, fixed at the ends and loaded in any manner. Suppose the arch to be cut at the crown C and let the stresses acting upon either half be represented as in Fig. 52. Then let

H_0, V_0, and M_0 = thrust, shear, and bending moment at the crown, considered as positive when acting as indicated;

T, V, and M = thrust, shear, and bending moment at any other section;

M' = bending moment at any section (Fig. 52) due to the external forces only (H_0, V_0, and M_0 being removed);

x, y = coordinates of any point referred to the point C as origin, x to be taken as positive toward the springing line in each half, and y as positive downward.

Then in general

$$M = M' + M_0 + H_0 y \pm V_0 x, \quad . \quad . \quad . \quad . \quad (1)$$

the plus sign being used for the right side and the minus sign for the left side. The values of M' are readily calculated by statics, but the values of M_0, H_0, and V_0 cannot be so determined. They depend upon the form and material of the arch and require the establishing of three additional condition equations. These three equations are obtained by calculating the movement of point C of the left half (Fig. 52), and placing it equal to the deflection of the same point for the right half.

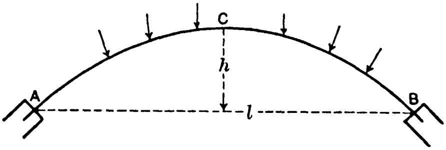


FIG. 51.

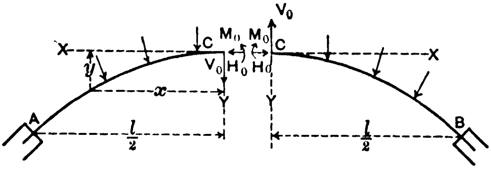


FIG. 52.

If Δx, Δy, and $\Delta \varphi$ represent the horizontal and vertical components of the movement of C, and the change of angle at C, for the left half, and $\Delta' x$, $\Delta' y$, and $\Delta' \varphi$ the same quantities for the right half, we have, with due respect to sign,

$$\left.\begin{aligned} \Delta x &= -\Delta' x \\ \Delta y &= \Delta' y \\ \Delta \varphi &= -\Delta' \varphi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also from Arts. 99–100 we have, for the left half

$$\Delta x = \int \frac{M\,y\,d\,s}{EI};\ \Delta y = -\int \frac{M\,x\,d\,s}{EI};\ \Delta \varphi = \int \frac{M\,d\,s}{EI}. \quad \cdot \quad (3)$$

The expressions for the right half are similar. Substituting in (2) and using the subscript l to represent left half and r to represent right half, we have

$$\left.\begin{aligned} \int \frac{M_l\,y\,d\,s}{EI} + \int \frac{M_r\,y\,d\,s}{EI} &= 0 \\ \int \frac{M_l\,x\,d\,s}{EI} - \int \frac{M_r\,x\,d\,s}{EI} &= 0 \\ \int \frac{M_l\,d\,s}{EI} + \int \frac{M_r\,d\,s}{EI} &= 0 \end{aligned}\right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (4)$$

Substituting the value of M from (1) and collecting terms we have the three general equations:

$$\left.\begin{aligned} \int \frac{M'\,y\,d\,s}{EI} + M_0 \int \frac{y\,d\,s}{EI} + H_0 \int \frac{y^2\,d\,s}{EI} &= 0 \\ \int \frac{M_l'\,x\,d\,s}{EI} - \int \frac{M_r'\,x\,d\,s}{EI} - V_0 \int \frac{x^2\,d\,s}{EI} &= 0 \\ \int \frac{M'\,d\,s}{EI} + M_0 \int \frac{d\,s}{EI} + H_0 \int \frac{y\,d\,s}{EI} &= 0 \end{aligned}\right\} \quad \cdot \quad \cdot \quad (5)$$

Solving for H_0, V_0, and M_0, and omitting E we derive the equations:

$$H_0 = \frac{\int \frac{d\,s}{I} \int \frac{M'\,y\,d\,s}{I} - \int \frac{M'\,d\,s}{I} \int \frac{y\,d\,s}{I}}{\left(\int \frac{y\,d\,s}{I}\right)^2 - \int \frac{d\,s}{I} \int \frac{y^2\,d\,s}{I}}. \quad \cdot \quad \cdot \quad \cdot \quad (6)$$

$$V_0 = \frac{\int \frac{M_l'\,x\,d\,s}{I} - \int \frac{M_r'\,x\,d\,s}{I}}{\int \frac{x^2\,d\,s}{I}}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7)$$

$$M_0 = \frac{\int \frac{M'\,d\,s}{I} \int \frac{y^2\,d\,s}{I} - \int \frac{M'\,y\,d\,s}{I} \int \frac{y\,d\,s}{I}}{\left(\int \frac{y\,d\,s}{I}\right)^2 - \int \frac{d\,s}{I} \int \frac{y^2\,d\,s}{I}} = -\frac{H_0 \int \frac{y\,d\,s}{I} + \int \frac{M'\,d\,s}{I}}{\int \frac{d\,s}{I}}. \quad \cdot \quad \cdot \quad (8)$$

In these expressions all integrals excepting as noted by the subscripts l and r are to be taken for the entire arch, and the terms containing two integral signs indicate simply the product of the two separate integrals.

The foregoing expressions for H_0, V_0, and M_0 are applicable to any form of arch rib or frame composed of a solid beam, but no account has here been taken of the effect of direct stresses upon the deformations. (See Section III, Chapter VI, for application of these formulas to rectanglar frames.)

To apply equations (6), (7), and (8) to any case where the integrations cannot be readily performed, the arch should be divided into numerous divisions of equal or unequal lengths and the process of summation used. All quantities are readily calculated for any given loading. The values of H_0, V_0, and M_0 can thus be determined for single loads and influence lines plotted for fibre stress or shear.*

155. Temperature Stresses.—For a rise of temperature of t degrees, Δx of eq. (3) becomes equal to $\frac{\omega t l}{2}$, and $\Delta \varphi = 0$. Hence,

$$\int \frac{M y \, d s}{E I} = \omega t l, \text{ and } \int \frac{M \, d s}{E I} = 0. \quad . \quad . \quad . \quad (9)$$

In this case $M' = 0$ and $V_0 = 0$; hence by substitution from (1) we have

$$M_0 \int \frac{y \, d s}{E I} + H_0 \int \frac{y^2 \, d s}{E I} = \omega t l, \quad . \quad . \quad . \quad (10)$$

$$M_0 \int \frac{d s}{E I} + H_0 \int \frac{y \, d s}{E I} = 0, \quad . \quad . \quad . \quad . \quad (11)$$

from which we derive

$$H_0 = \frac{\omega t l \int \frac{d s}{E I}}{\int \frac{d s}{E I} \int \frac{y^2 \, d s}{E I} - \left(\int \frac{y \, d s}{E I} \right)^2}, \quad . \quad . \quad . \quad (12)$$

and

$$M_0 = - \frac{H_0 \int \frac{y \, d s}{I}}{\int \frac{d s}{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

---

* For application of this method to reinforced concrete or masonry arches see Turneaure and Maurer, "Principles of Reinforced Concrete Construction."

The bending moment at any point is

$$M = M_0 + H_0 y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

156. Stresses Due to Shortening of Arch from Thrust.—As in Art. 119, eq. (7), the shortening due to thrust may be placed as equal approximately to $\frac{H_0 \cos \alpha_1 L}{E A_0}$. To take account of this element as in Art. 119, the first of eqs. (4) is placed equals to $\frac{H_0 \cos \alpha_1 L}{E A_0}$ instead of zero. This results in adding the term $+\frac{L \cos \alpha_1}{A_0} \cdot \int \frac{d s}{I}$ to the denominator of eq. (6) for H, and $+\frac{L \cos \alpha_1}{A_0} \int \frac{y\, d s}{I}$ to the denominator of the first expression for M_0 in eq. (8). The effect of thrust may, however, be conveniently taken care of as a correction or supplementary effect, in the same manner as temperature changes, in which case it is sufficiently accurate to use eqs. (12) and (13) for temperature effect by substituting $\frac{H_0 \cos \alpha_1 L}{E A_0}$ in place of $\omega t l$, giving, for effect of thrust alone,

$$H_s = -\frac{\frac{H \cos \alpha_1 L}{A_0} \int \frac{d s}{I}}{\int \frac{d s}{I} \int \frac{y^2 d s}{I} - \left(\int \frac{y\, d s}{I}\right)^2}, \quad . \quad . \quad (15)$$

in which H is the value of H_0 for the loaded condition, L = length of arch axis, A_0 = section at the crown, and α_1 = inclination at the springing line.

157. The Parabolic Arch with Variable Moment of Inertia.—As in Art. 123 formulas will be developed for a parabolic arch in which I varies with sec α so that $\frac{d s}{I} = \frac{d x}{I_0}$, I_0 being the moment of inertia at the crown.

158. *Stresses Due to a Single Vertical Load P.*—Let P, Fig. 53, be a single vertical load applied a distance b from the centre. The components of the reactions are H and V_1 at A, and H and V_2 at B. The moments at the supports are, respectively, M_1 and M_2. Let H_0, V_0, and M_0 refer to the crown C as before. The values of H_0, V_0, and M_0 will be found by substituting in the general equations of Art. 154. The equation of the parabolic axis, origin at C, is $y = \frac{4hx^2}{l^2}$. For a single load P, arch cut at C, the moment M' exists only in the section AD.

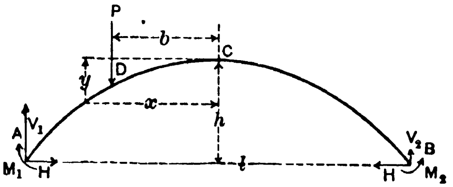


FIG. 53.

Here $M' = -P(x-b)$. Substituting the values of $\frac{ds}{I}$, y, and M' in the several integrals of eqs. (6), (7), and (8), there result the following values:

$$\int \frac{ds}{I} = \frac{l}{I_0}; \quad \int \frac{y\,ds}{I} = \frac{8h}{l^2 I_0}\int_0^{\frac{l}{2}} x^2\,dx = \frac{hl}{3I_0};$$

$$\int \frac{y^2\,dx}{I} = \frac{2 \times 16h^2}{l^4 I_0}\int_0^{\frac{l}{2}} x^4\,dx = \frac{h^2 l}{5I_0};$$

$$\int \frac{x^2\,ds}{I} = \frac{2}{I_0}\int_0^{\frac{l}{2}} x^2\,dx = \frac{l^3}{12I_0};$$

$$\int \frac{M'\,ds}{I} = -\frac{P}{I_0}\int_b^{\frac{l}{2}} (x-b)\,dx = -\frac{P}{2I_0}\left(\frac{l}{2} - b\right)^2;$$

$$\int \frac{M'y\,ds}{I} = -\frac{4Ph}{l^2 I_0}\int_b^{\frac{l}{2}} (x-b)\,x^2\,dx = -\frac{Ph}{48l^2 I_0}(3l^4 - 8bl^3 + 16b^4);$$

$$\int \frac{M_l'\,x\,ds}{I} = -\frac{P}{I_0}\int_b^{\frac{l}{2}} (x-b)\,x\,dx = -\frac{P}{24I_0}(l^3 - 3bl^2 + 4b^3).$$

Substituting these values in eqs. (6), (7), and (8) and reducing, there result the following equations:

$$H_0 = \frac{15l}{64h}\left(1 - \frac{4b^2}{l^2}\right)^2 P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$V_0 = -\tfrac{1}{2}\left(1 + \frac{b}{l}\right)\left(1 - \frac{2b}{l}\right)^2 P \quad . \quad . \quad . \quad . \quad (17)$$

$$M_0 = l\left[\tfrac{1}{8}\left(1 - \frac{2b}{l}\right)^2 - \frac{5}{64}\left(1 - 4\frac{b^2}{l^2}\right)^2\right]P. \quad . \quad (18)$$

Having H_0, V_0, and M_0, the value of M at any point is given by (1). The horizontal components of the reactions at A and B are, for vertical loads, equal to H_0. For a load on the left half

$$V_1 = V_0 + P = \tfrac{1}{2}\left(1 - \frac{b}{l}\right)\left(1 + \frac{2b}{l}\right)^2 P \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$V_2 = - V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$M_1 = M_0 - P\left(\frac{l}{2} - b\right) + H_0 h - V_0 \frac{l}{2}$$

$$= \frac{l}{32}\left(1 + \frac{2b}{l}\right)^2\left[5\left(1 - \frac{2b}{l}\right)^2 - 4\left(1 - \frac{2b}{l}\right)\right]P \quad . \quad (21)$$

$$M_2 = M_0 + H_0 h + V_0 \frac{l}{2}$$

$$= \frac{l}{32}\left(1 - \frac{2b}{l}\right)^2\left[5\left(1 + \frac{2b}{l}\right)^2 - 4\left(1 + \frac{2b}{l}\right)\right]P. \quad . \quad (22)$$

159. *Stresses Due to Changes of Temperature.*—Substituting in (12) we have, for a rise of temperature,

$$H_0 = H_t = \frac{\omega t l \dfrac{l}{E I_0}}{\dfrac{l}{E I_0} \cdot \dfrac{h^2 l}{5 E I_0} - \dfrac{h^2 l^2}{9 E^2 I_0^2}} \quad . \quad . \quad . \quad . \quad (23)$$

$$M_0 = - \frac{H_t \cdot \dfrac{h l}{3 I_0}}{\dfrac{l}{I_0}} = - \frac{H_t h}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$M_1 = M_2 = M_0 + H_t h = \frac{2 H_t h}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

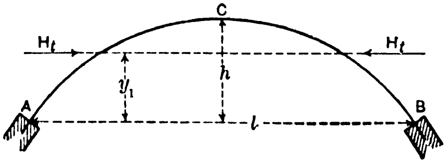


Fig. 54.

The reactions are equivalent to two forces equal to H_t, Fig. 54, applied a distance $y_1 = \frac{2}{3}h$, above the springing lines.

160. *Stresses Due to Shortening of Arch from Thrust.*—As in Art. 156 these may be calculated as a correction, by substituting $\frac{H \cos \alpha_1 L}{E A_0}$ for $\omega t l$ in (23), giving

$$H_s = -\frac{45 H I_0 L \cos \alpha_1}{4 h^2 l A_0}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

in which H = thrust due to the given loading. The moments are given by (24) and (25).

161. *Reaction Lines for a Single Load.*—Fig. 55 gives the graphic analysis of forces for a single load P. $A' D'$ and $D' B'$ are the reaction lines or true equilibrium polygon. Fig. (a) is the force polygon, V_1, V_2, and H being the components of the reactions. There being a moment M_1 at A, the equilibrium polygon does not pass through A but some distance y_1 below, such that $H\,y_1 = M_1$. Likewise $H\,y_2 = M_2$. In a graphical treatment the ordinates y_1, y_0, and y_2 are conveniently treated as the unknowns, but the reaction lines $A' D'$ and $D' B'$ can readily be drawn from the values of H_0, V_0, and M_0 as found in the preceding articles, as follows: Construct the force polygon by laying off $V_2 = -V_0$ and $H = H_0$. Then draw $D' B'$ parallel to $d\,b$ and at a distance y' above C (or below), such that $H_0\,y' = M_0$, or

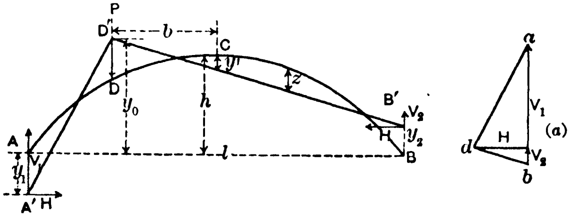


FIG. 55.

$$y' = \frac{M_0}{H_0} = h\left[\frac{8}{15\left(1 + \frac{2b}{l}\right)^2} - \tfrac{1}{3}\right]. \quad . \quad . \quad (27)$$

The line $A' D'$ is then drawn parallel to $a\,d$. The bending moment at any point is then given by the ordinate between the arch and the equilibrium polygon, multiplied by H.

The values of y_1, y_0, and y_2 are determined as follows:

$$y_1 = \frac{M_1}{H_0} = \frac{2h}{15}\left(\frac{l - 10b}{l - 2b}\right) \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

$$y_2 = \frac{M_2}{H_0} = \frac{2h}{15}\left(\frac{l + 10b}{l + 2b}\right). \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

By similar triangles $\dfrac{y_0 - y_1}{\dfrac{l}{2} - b} = \dfrac{V_1}{H_0}$, from which is derived, by substitution from (16), (19), and (28)

$$y_0 = \frac{6h}{5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

The locus of the point D' is therefore a straight line whose ordinate $= \frac{6}{5}h$. Equations (28) to (30) enable the reaction lines to be readily drawn for a load at any point, and thence the moments, shears, and thrusts, if desired, due to this single load, from the equations:

$$\left.\begin{aligned} M &= Hz \\ V &= V_1 \cos\alpha - H \sin\alpha \\ T &= V_1 \sin\alpha + H \cos\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

The reaction lines on each side are all tangent to certain envelopes, *d e* and *f g* (Fig. 56). These envelopes are hyperbolas, tangent to each other at the centre and having vertical asymptotes through the springing lines.

162. *Position of Loads for Maximum Stress.*—In using reaction lines to find position of loads for a maximum stress it is convenient to construct the hyperbolic envelopes and the straight line locus *m n* with $y_0 = \dfrac{6}{5}h$. This being done, the nature of the stress or moment caused by a load at any point is determined as in Art. 140. Thus in Fig. 56, the lines *G i* and *G i'*, drawn tangent to *d e* and *f g* respectively, will determine the position of the loads for a maximum stress of either kind in *D E*, since loads between *i* and *i'* cause positive moment at *G*, while all other loads cause negative moment. Likewise a line drawn tangent to *d e* and parallel to *F G* and *D E*, or toward their intersection, will determine the position of loads for maximum stress of either kind in *D G*, as in Art. 140.

For an arch rib, the kern point, or flange centre, should be taken as the moment centre for maximum fibre stress.

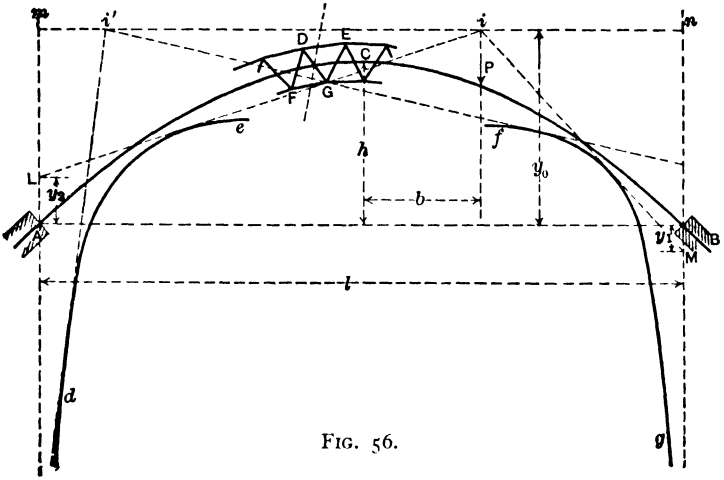


FIG. 56.

The following table, giving values of y_1/h and y_2/h for various load points, will enable enough reaction lines to be drawn to locate the envelopes with sufficient accuracy for the purpose.

VALUES OF $\frac{y_1}{h}$ AND $\frac{y_2}{h}$ FOR VARIOUS VALUES OF $\frac{b}{l}$.

(For loads on the right.)

$\frac{b}{l}$	$\frac{y_1}{h}$	$\frac{y_2}{h}$	$\frac{b}{l}$	$\frac{y_1}{h}$	$\frac{y_2}{h}$
0.50	$-\infty$	+0.400	0.25	−0.400	+0.311
0.45	−4.667	+0.386	0.20	−0.222	+0.286
0.40	−2.000	+0.370	0.15	−0.095	+0.256
0.35	−1.067	+0.353	0.10	0.0	+0.222
0.30	−0.667	+0.333	0.0	+0.133	+0.133

Having found the position of loads for a maximum stress the stresses may be found either graphically or algebraically. A convenient method is to make use of the influence lines of H_0, V_0, and M_0, as described in the next Article.

163. *Influence Lines.*—If the method of influence lines is to be used it is convenient to first construct the influence lines for H_0, V_0, and M_0 from eqs. (16), (17), and (18), Art. 158. It will be seen that the value of V_0 depends only upon the position of the load $\left(\frac{b}{l}\right)$, not upon the rise or span length; M_0 depends also upon the span length; while H_0 depends upon both the span and rise. These quantities are, therefore, conveniently plotted for h and l = unity. The resulting curves are given in Fig. 57. These curves, therefore, serve as influence lines for these functions, for any arch of the type here considered, the true values of the functions being obtained by multiplying the ordinates of the curves by $\frac{l}{h}$ for H_0, and by l for M_0.

These influence lines may now be used to construct influence lines for moment at any other point by use of the general formula,

$$M = M_0 + H_0 y - V_0 x - P(x - b). \quad . \quad . \quad . \quad (32)$$

where x and y are the coordinates of the point or moment centre in question. For values of $x < b$ the term $P(x - b)$ drops out.

If uniform loads are to be employed the position for a maximum is more readily found from the reaction lines. The position being known, the desired moment may then be found from the influence areas of Fig. 57.

164. Deflection of an Arch Rib.—The vertical deflection of any point is given by the general equation,

$$\Delta y = \int \frac{M x\, ds}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where M is the true bending moment at any section due to the given loads. The abscissa x is to be measured from the point where the deflection is desired. The ends of the arch being fixed the deflection of any point is calculated by performing the integration (or summation) between either of the abutments and the point in question.

165. The Braced Arch Without Hinges.—Braced arches of small depth and parallel chords can be satisfactorily analyzed by treating

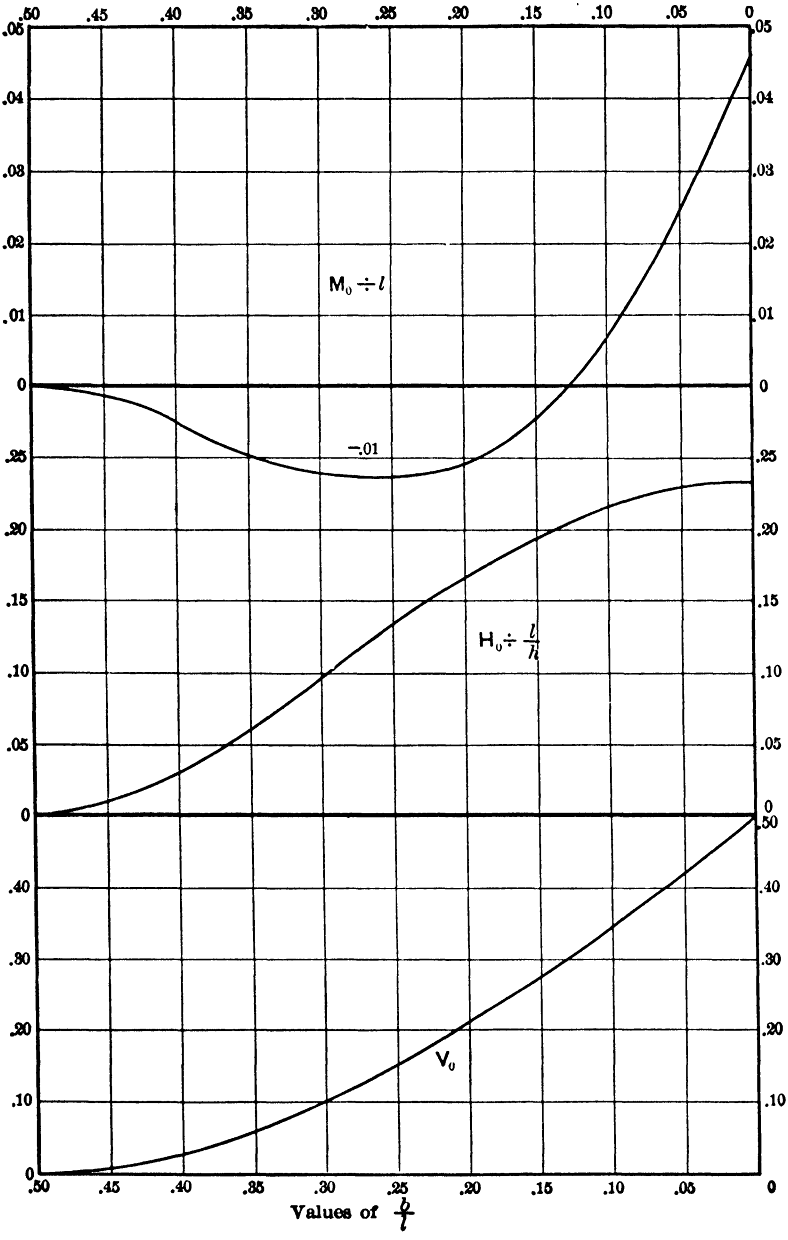

.50
.45
.40
.35
.30
.25
.20
.15
.10
.05
0
.05
.04
.03
.02
.01
0
.25
.20
.15
.10
.05
0
.50
.40
.30
.20
.10
0
$M_0 \div l$
−.01
$H_0 \div \frac{l}{h}$
V_0
Values of $\frac{b}{l}$

FIG. 57.

them as beams and using the methods already explained. Where the truss does not conform to these conditions the analysis must be made by the method of redundant members. If the truss is of the single intersection type the number of redundant members will be three, and the three equations of condition are found by placing equal to zero the relative displacement of the two cut ends of each redundant member, as explained in Part I, Art. 222. Any three members may be taken as the redundant members, provided they are so related that the remaining members form a structure that is stable and at the same time statically determinate.

Let Fig. 58 represent any braced arch without hinges. If members *B D*, *J L*, and *F H* be omitted the structure becomes a three-hinged arch and therefore statically determinate. If three members at one end, as *E G*, *F G*, and *F H*, be omitted it becomes a cantilever supported at the left end and is again statically determinate. Or, again, three members at the crown or some intermediate section may be omitted, as *I K*, *J K*, and *J L*, dividing the structure into two cantilevers, as was done in the case of the arch rib (Art. 154). If the arch is a symmetrical one the last-named arrangement is likely to be the simplest. If a joint comes at the centre it is better for the sake of preserving symmetry to imagine the arch cut through the centre of the joint, and if a vertical member occurs at the centre to consider it as separated into two equal parts. Each half then may take the form shown in Fig. 59, and the problem is similar to that of the arch rib, Art. 154. Instead, however, of denoting the unknowns as a moment, shear and thrust, it will be more convenient to assume them as a shear V, and two horizontal forces, H_1 and H_2, these two forces being equivalent to the moment and thrust of Art. 154. The redundant forces are then V,

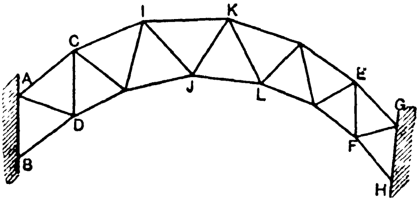


Fig. 58.

H_1, and H_2, and these are to be found by equating to zero the relative deflections of a and a' and b and b'. We proceed as in Art. 222, Part I.

Let S = stress in any member due to any given load, arch in normal condition;

S' = stress in any member due to the external loads, arch cut as shown in Fig. 59;

u_1 = stress in any member of either half due to a force $H_1 = 1$;

u_2 = same for a force of $H_2 = 1$;

u_3 = same for $V = 1$.

Then $$S = S' + H_1 u_1 + H_2 u_2 + V u_3. \qquad (34)$$

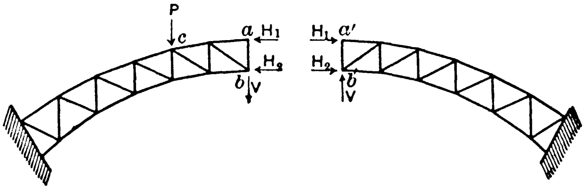


FIG. 59.

Now the horizontal deflection of b with respect to b' must be zero, and therefore we have

$$\Sigma \frac{S u_1 l}{A} = 0. \qquad (35)$$

Likewise for the horizontal movement of a and a'

$$\Sigma \frac{S u_2 l}{A} = 0, \qquad (36)$$

and for the vertical movement of b and b'

$$\Sigma \frac{S u_3 l}{A} = 0. \qquad (37)$$

Substituting from (34) we derive the three linear equations

$$\left.\begin{aligned} \Sigma \frac{S' u_1 l}{A} + H_1 \Sigma \frac{u_1^2 l}{A} + H_2 \Sigma \frac{u_1 u_2 l}{A} + V \Sigma \frac{u_1 u_3 l}{A} &= 0 \\ \Sigma \frac{S' u_2 l}{A} + H_1 \Sigma \frac{u_1 u_2 l}{A} + H_2 \Sigma \frac{u_2^2 l}{A} + V \Sigma \frac{u_2 u_3 l}{A} &= 0 \\ \Sigma \frac{S' u_3 l}{A} + H_1 \Sigma \frac{u_1 u_3 l}{A} + H_2 \Sigma \frac{u_2 u_3 l}{A} + V \Sigma \frac{u_3^2 l}{A} &= 0 \end{aligned}\right\} . \qquad (38)$$

The various summations are to be calculated, and their values being substituted the equations are readily solved for H_1, H_2, and V, whence all stresses become determinate.

166. *Graphical Method.*—The displacement diagram can be used to advantage in the calculation of the various summations above given, especially if the stresses are desired for a unit load at any joint, the same method being employed as in the example of Art. 151. In this case, however, three displacement diagrams are required. These are the following:

(1) A diagram for a one-pound load acting horizontally at a ($H_1 = 1$).

(2) A diagram for a one-pound load acting horizontally at b ($H_2 = 1$), and

(3) A diagram for a one-pound load acting vertically at b ($V = 1$).

These diagrams being drawn, measure thereon the following deflections with notation as given:

δ_a = horizontal deflection of a due to $H_1 = 1$, $\left(= \Sigma \frac{u_1^2 l}{A}\right)$;

δ_{bh} = horizontal deflection of b due to $H_2 = 1$, $\left(= \Sigma \frac{u_2^2 l}{A}\right)$;

δ_{bv} = vertical deflection of b due to $V = 1$, $\left(= \Sigma \frac{u_3^2 l}{A}\right)$;

δ'_a = horizontal deflection of a due to $H_2 = 1$, $\left(= \Sigma \frac{u_1 u_2 l}{A}\right)$;

δ''_a = horizontal deflection of a due to $V = 1$, $\left(= \Sigma \frac{u_1 u_3 l}{A}\right)$;

δ'_{bh} = horizontal deflection of b due to $V = 1$, $\left(= \Sigma \frac{u_2 u_3 l}{A}\right)$.

Consider a vertical load of unity at joint c. The various summations involving S' will then be as follows:

δ_c = vertical deflection of c for $H_1 = 1$, $\left(= \Sigma \frac{S' u_1 l}{A}\right)$;

δ'_c = vertical deflection of c for $H_2 = 1$, $\left(= \Sigma \frac{S' u_2 l}{A}\right)$;

δ''_c = vertical deflection of c for $V = 1$, $\left(= \Sigma \frac{S' u_3 l}{A}\right)$.

Substituting these several deflections in (38) we have

$$\left.\begin{aligned} \delta_c + 2\,\delta_a\,H_1 + 2\,\delta'_a\,H_2 + 2\,\delta''_a\,V &= 0 \\ \delta'_c + 2\,\delta'_a\,H_1 + 2\,\delta_{bh}\,H_2 + 2\,\delta'_{bh}\,V &= 0 \\ \delta''_c + 2\,\delta''_a\,H_1 + 2\,\delta'_{bh}\,H_2 + 2\,\delta_{bv}\,V &= 0 \end{aligned}\right\} \quad . \quad . \quad (39)$$

For a unit load at any other joint the equations are the same, excepting the first term of each. Tne three displacement diagrams thus enable all desired values to be obtained for a load at any joint.

The values at H_1, H_2, and V being determined, the stress in any member is found from eq. (34), the values of S', u_1, u_2, and u_3 being already calculated.

167. *Temperature Stresses.*—For a change of temperature the horizontal deflections of a and $b = \frac{1}{2}\,\omega\,t\,l$. Hence the first and second of eq. (39) should be placed equal to $\omega\,t\,l$, δ_c being omitted. The value of V is also zero, since there is no shear at the centre, hence we have

$$\delta_a\,H_1 + \delta'_a\,H_2 = \omega\,t\,l$$

$$\delta'_a\,H_1 + \delta_{bh}\,H_2 = \omega\,t\,l$$

whence

$$H_1 = \omega\,t\,l\,\frac{\delta'_a - \delta_{bh}}{(\delta'_a)^2 - \delta_a\,\delta_{bh}} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and

$$H_2 = \omega\,t\,l\,\frac{\delta'_a - \delta_a}{(\delta'_a)^2 - \delta_a\,\delta_{bh}}. \quad . \quad . \quad . \quad . \quad . \quad (41)$$

168. Wind Stresses.—The arch of no hinges is always constructed with chords parallel or nearly so, thus requiring the roadway to be supported on vertical struts, as in Figs. 1 and 5. Lateral bracing is therefore usually arranged as described in Art. 153, namely a lateral system along the roadway, vertical transverse bracing at each panel, and lower laterals along the arch. Inasmuch as both the upper and lower chords are fixed at the ends, the lower laterals may be placed along one or both chords and extend as a complete lateral system to the abutments. If two lines of laterals are used the upper one is preferably made of a sufficient size to carry the loads brought down from the roadway by means of the transverse bracing.

The shears in the laterals are determined as for the arch of two hinges, and the resulting vertical and horizontal forces acting upon the main arch also in like manner. These having been found, the stresses in the main arch are determined as for any other loads in the same plane.

169. Relative Advantages of the Arch with Fixed Ends.—Comparatively few arches have been constructed with fixed ends. As compared to the hinged types with parallel chords, the arch without hinges is somewhat more rigid and slightly more economical, especially where the dead load is relatively large. The difficulties of calculation are of some consequence, but where the chords are nearly parallel the use of the beam formulas for preliminary design will greatly facilitate the work.

Where a horizontal upper chord is desired the hingeless arch is not suitable, both on account of the difficulties of end supports and on account of the fact that with such a variable depth of truss the process of successive approximations in its design and analysis does not lead readily to consistent results. As pointed out more fully in Art. 251, a redundant system theoretically cannot be designed so that all members shall be stressed at the desired value. This is a case where the departure from assigned working stresses would be large in some members and hence to that extent result in an uneconomical form. The example of Art. 151 illustrates the difficulties of design and analysis for a two-hinged arch of this type. These would be much greater in the arch of no hinges. With chords nearly parallel, however, the structure approaches a beam in proportions, with much more uniform stresses in the chords and a more ready adjustment between stress and sectional areas.

CHAPTER V

SUSPENSION BRIDGES

170. Introduction.—Suspension Bridges may be classed under two main heads: (*a*) Those composed of a light platform suspended from a cable, the loads passing directly from the floor to the cable; (*b*) those consisting of a roadway supported by a truss which is hung from the cable by means of hangers.

Structures of the first class are called *Unstiffened Suspension Bridges.* Because of their lack of rigidity, structures of this class are limited to short spans and light loads. Heavy concentrations or unsymmetrical loads would distort the cable to such an extent as to cause excessive variation of grade in the roadway near the point of loading.

Structures of the second class are called *Stiffened Suspension Bridges.* The applied loads are taken up by the stiffening trusses and distributed to the cables by means of hangers. Due to the rigidity of the trusses, heavy concentrations or unsymmetrical loads are distributed over the cable approximately as a uniform load, so that it does not vary greatly from its original shape. Stiffened suspension bridges can be constructed rigid enough to carry railway or heavy city traffic.

In this chapter the cable as a structure by itself will be first considered. The unstiffened and the stiffened suspension bridge will then be taken up, and, for the usual conditions of loading, methods will be developed for the determination of the quantities necessary for the computation of stresses in such structures.

SECTION I.—THE CABLE

171. Form of the Cable.—*Any Loading.*—The form of the cable of a suspension bridge will depend upon the distribution of the applied loads. Any portion of the cable must be in equilibrium under the tensions at its ends and the loads. In any case, the form can be determined by drawing an equilibrium polygon for the applied loads, subject to the condition that the polygon must pass through the points of attachment of the cable and some other fixed point, such as the lowest point of the curve.

When the law of variation of the loading is known, the differential equation of the cable curve may be derived from statics, subject to the condition that any portion of the cable is in equilibrium under the applied loads and the internal stresses.

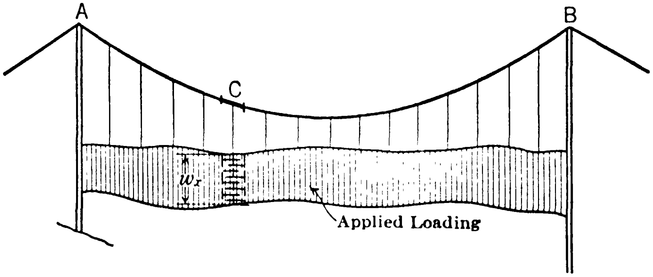


FIG. 1.

Let $A B$, Fig. 1, represent a cable under a variable load of intensity w_x per unit of length at point C. Fig. 2 shows an element of the cable whose horizontal projection is $d x$ in length. Let T and $T - d T$ represent the cable stress on the two faces of the element, and let $w_x d x$ represent the applied loading. Since the element is in equilibrium, we have from $\Sigma V = 0$.

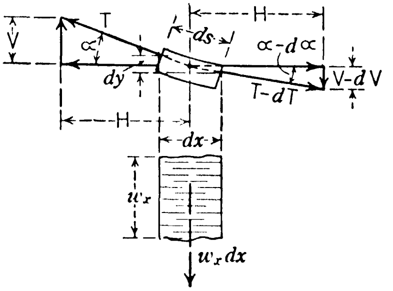


FIG. 2.

$$V - w_x d x - (V - d V) = 0,$$

from which

$$\frac{d V}{d x} = w_x. \qquad \qquad (a)$$

From Fig. 2, $V = H \tan \alpha = H \dfrac{dy}{dx}$.

Then $\dfrac{dV}{dx} = H \dfrac{d^2 y}{dx^2}$, and from eq. (*a*)

$$\frac{d^2 y}{dx^2} = \frac{w_x}{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Equation (1) is the general differential equation of the cable curve.

172. *Load Uniform Along the Horizontal.*—In Fig. 3, let *w* per unit of length be a load, uniform along the horizontal, supported by a cable passing through points *A*, *B*, and *C*. The span of the cable is *l* and the sag is *f*. Assume the load *w* to be transferred to the cable by means of closely spaced hangers. Required the form of the cable.

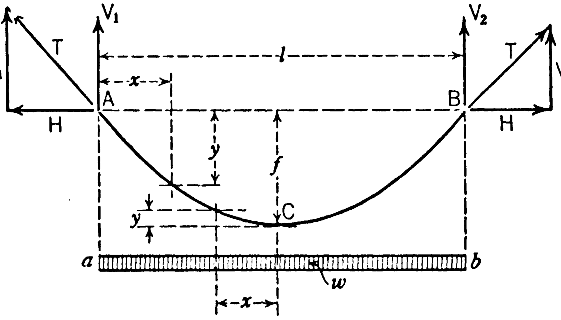


FIG. 3.

As the applied load is generally much greater than the weight of the cable and the hangers, these loads may be considered as uniformly distributed and included in the load *w*. Then in eq. (1), $w_x = w$, a constant, and

$$\frac{d^2 y}{dx^2} = \frac{w}{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Integrating eq. (2) twice, we have

$$y = \frac{w x^2}{2H} + C_1 x + C_2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

For an origin at *C*, Fig. 3, the constants of integration C_1 and C_2 are determined by noting that for $x = 0$, $dy/dx = 0$, and also when $x = 0$, $y = 0$. Hence, $C_1 = 0$ and $C_2 = 0$, and eq. (3) becomes

$$y = \frac{w x^2}{2H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The value of H in eq. (4) may be determined by noting that in Fig. 3, $y = f$ when $x = l/2$. Hence

$$H = \frac{w\,l^2}{8f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Placing this value of H in eq. (4), we have

$$y = \frac{4\,f\,x^2}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is the equation of the cable curve referred to C as an origin. This is readily seen to be the equation of a parabola. Therefore a cable carrying a load uniformly distributed along the horizontal hangs in a curve which is a parabola. This also follows from the principle of graphics which states that the equilibrium polygon for uniform loads is a parabola.

For use in the work to follow it will be convenient to refer the equation of the cable curve to coordinate axes with an origin at one of the supports, as A, Fig. 3, and with the horizontal line $A\,B$ as the X-axis. The equation then becomes

$$y = \frac{4\,f\,x}{l^2}(l - x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Here y is the ordinate to the curve measured downward from the X-axis at a point distance x from the origin, f being the sag of the cable.

173. *Load Uniform Along the Cable.*—When the load on the cable is due to its own weight, in which case the load is uniform per unit of length along the cable, the term w_x in eq. (1) becomes $w\,d\,s/d\,x$ where w = weight of the cable per unit of length and $d\,s$ = length of cable whose horizontal projection is $d\,x$, as shown in Fig. 2. Equation (1) then becomes

$$\frac{d^2\,y}{d\,x^2} = \frac{w}{H}\,\frac{d\,s}{d\,x} = \frac{w}{H}\left[1 + \left(\frac{d\,y}{d\,x}\right)^2\right]^{1/2}.$$

from which we obtain finally as the equation of the cable curve,

$$y = \frac{1}{2c}(e^{cx} + e^{-cx} - 2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which $c = w/H$ ($1/c$ is the length of cable of weight w per unit length, whose total weight is equal to H). In this equation y is the ordinate to the curve from a horizontal line through the low point, or center, of the cable. Equation (8) is the equation of a catenary. Therefore a cable under its own weight hangs in a catenary. As c in eq. (8) depends on H and w for its value, it can be seen that unless these are known before hand, the equation can be solved only by successive approximations.

Equation (8) may also be expressed in terms of hyperbolic functions. Thus we have

$$y = \frac{1}{c}\left(\frac{e^{cx} + e^{-cx}}{2} - 1\right) = \frac{1}{c}[(\cosh cx) - 1]. \quad . \quad . \quad . \qquad (9)$$

174. *Side Spans.*—When the structure has side spans which are also suspended from the cable, as shown in Fig. 4, the loads again

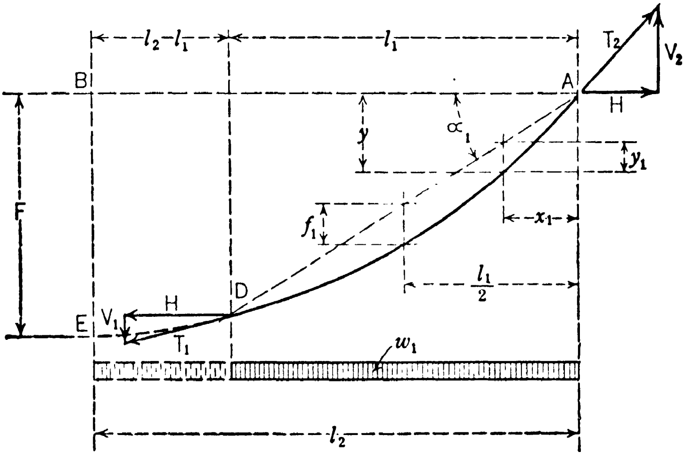


FIG. 4.

being taken as uniformly distributed along the horizontal, we proceed as in Art. 172, substituting $w_x = w$, in eq. (1) and integrating twice, deriving

$$y_1 = \frac{w_1 x_1^2}{2H} + C_1 x_1 + C_2. \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Assuming an origin at A, Fig. 4, and measuring ordinates vertically from the line $A\ D$, the constants of integration are determined from the conditions that $y_1 = 0$ when $x_1 = 0$ and $y_1 = 0$ when $x = l_1$. Hence

$$y_1 = \frac{w_1}{2H}(x_1^2 - l_1 x_1). \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

To determine H, note from Fig. 4 that $y_1 = f_1$ when $x_1 = l_1/2$. Equation (11) then becomes

$$y_1 = \frac{4 f_1 x_1}{l_1^2}(l_1 - x_1). \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

Referred to a horizontal axis $A\,B$, Fig. 4, the equation of the cable curve may be written in the form

$$y = \frac{4 f_1 x_1}{l_1^2}(l_1 - x_1) + x_1 \tan \alpha_1. \quad . \; . \; . \quad (13)$$

The notation is fully shown on Fig. 4.

As shown on Fig. 4, the low point of the curve is at E, a point to the left of D. If l_1, α_1, and f_1 are known, it can be shown that

$$\left.\begin{aligned} l_2 &= \frac{l_1}{2}\left(1 + \frac{1}{4 n_1} \tan \alpha_1\right) \\ \text{and} \qquad F &= f_1 \left(1 + \frac{1}{4 n_1} \tan \alpha_1\right)^2 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (14)$$

in which $f_1/l_1 = n_1$, the sag ratio for the side span cable, and F is the vertical distance from the horizontal axis $A\,B$ to the low point of the curve. Equations (14) were obtained by placing $d\,y/d\,x$, as obtained from eq. (13), equal to zero and solving for l_2, the value of x at the low point of the cable curve. This value of l_2 was then substituted in eq. (13) to determine the value of F.

It can also be shown that when $\tan \alpha_1$ is greater than $4\,n_1$, the low point of the curve lies to the left of D, as shown in Fig. 4. When $\tan \alpha_1$ is less than $4\,n_1$, the low point of the curve will lie to the right of point D, or between points D and A. We then have a special case which will be considered in the next article as an unsymmetrical span.

175. *Unsymmetrical Spans.*—When the tops of the towers are not on the same elevation, as shown in Fig. 5, we have as noted in the preceding article, a special case of side span cables, which may be treated by the method used in Art. 174.

Referred to an origin at A, Fig. 5, and a line $A\ B$ joining the tops of the towers, the equation of the cable curve is

$$y = \frac{4 f x}{l^2}(l - x).$$

Referred to a horizontal line $A\ D$ making an angle α with $A\ B$, Fig. 5, the equation of the curve is

$$y' = y + x \tan \alpha = \frac{4 f x}{l^2}(l - x) + x \tan \alpha.$$

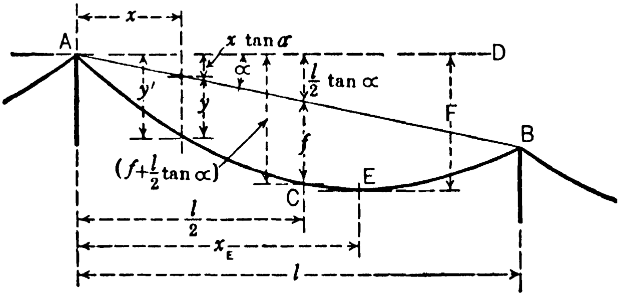


FIG. 5.

The low point of the curve is at E, near the span center. From eq. (14) we may write, using the notation shown on Fig. 5,

$$x_E = \frac{l}{2}\left(1 + \frac{1}{4n} \tan \alpha\right)$$

in which $f/l = n$, the sag ratio for span $A\ B$. The distance F from the horizontal axis to the low point E is given by eq. (14) as

$$F = f\left(1 + \frac{1}{4n} \tan \alpha\right)^2$$

176. Length of the Cable.—From Fig. 2,

$$d s = (d x^2 + d y^2)^{1/2} = \left[1 + \left(\frac{d y}{d x}\right)^2\right]^{1/2} d x.$$

If L is the length of the cable, we have

$$L = \int_0^l \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Main Span, Parabolic Cable.—From eq. (6), $dy/dx = 8fx/l^2$ and

$$L = 2\int_0^{\frac{l}{2}} \left[1 + \frac{64f^2x^2}{l^4}\right]^{\frac{1}{2}} dx$$

$$= \frac{l}{2}\left\{\left(1 + 16\frac{f^2}{l^2}\right)^{\frac{1}{2}} + \frac{l}{4f}\log_e\left[\frac{4f}{l} + \left(1 + \frac{16f^2}{l^2}\right)^{\frac{1}{2}}\right]\right\}. \quad . \quad . \quad (16)$$

This equation gives the exact length of the cable curve from A to B of Fig. 3.

An approximate value of the length of curve may be obtained by expanding the term $\left[1 + \frac{64f^2x^2}{l^4}\right]^{\frac{1}{2}}$ and integrating. We then have

$$L = 2\int_0^{\frac{l}{2}} \left(1 + \frac{32f^2x^2}{l^4} - \frac{512f^4x^4}{l^8} + \ldots\right) dx$$

$$= l\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4 + \ldots\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

in which $n = f/l$, the sag ratio of the cable curve. The value of L given by eq. (17) is sufficiently accurate for most cases. When the sag ratio is small, say not greater than $\frac{1}{10}$, eq. (17) may be written

$$L = \left(1 + \tfrac{8}{3}n^2\right)l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Main Span, Catenary.—From eqs. (8) and (15),

$$L = 2\int_0^{\frac{l}{2}} \frac{1}{2}\left(e^{cx} + e^{-cx}\right) dx = \frac{1}{c}\left(e^{\frac{cl}{2}} - e^{-\frac{cl}{2}}\right). \quad . \quad . \quad (19)$$

Expressed in hyperbolic functions

$$L = \frac{2}{c}\left[\frac{1}{2}\left(e^{\frac{cl}{2}} - e^{-\frac{cl}{2}}\right)\right] = \frac{2}{c}\sinh\frac{cl}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Side Spans, Parabolic.—Referred to point E, Fig. 6, the low point of a side span cable curve, the equation of the curve is $y = \frac{F x^2}{l_2^2}$. From eq. (15), the length of the curve from A to D, Fig. 6, is

$$L_{AD} = \int_{(l_2 - l_1)}^{l_2} \left[1 + \frac{4 F^2 x^2}{l_2^4}\right]^{½} d x. \quad . \quad . \quad . \quad . \quad (21)$$

This distance may also be determined from eq. (16), noting that cable length $A D = E A - E D$.

An approximate expression for length of curve from A to D, Fig. 6, is given by the equation

$$L_{AD} = l_1 \left(\sec \alpha_1 + \frac{8}{3} \frac{n_1^2}{\sec^3 \alpha_1}\right), \quad . \quad . \quad . \quad . \quad . \quad (22)$$

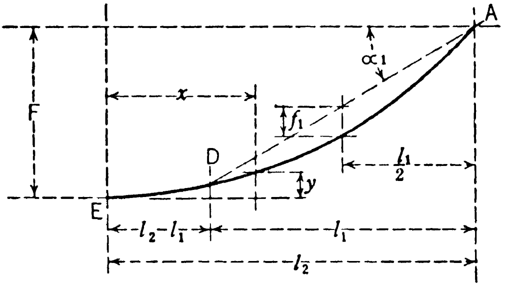


FIG. 6.

in which $n = f_1/l_1$, the side span sag ratio.

177. Stresses in the Cable for Uniform Loads.—*Main Span, Parabolic Cable.*—The stress in a cable carrying a uniformly distributed load due to the weight of the suspended platform may be determined by taking a section at any point in the cable and equating moments of forces on either side of the section equal to zero.

Figure 8 shows a section cut at a distance x from the left end of Fig. 7. All forces are shown in position. A moment equation about D, Fig. 8, gives $- H y + \frac{1}{2} w x (l - x) = 0$. Substituting the value of y from eq. (7), the horizontal component of the cable stress is

$$H = \frac{w l^2}{8 f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This is the same value as given by eq. (5). When the cable carries a uniform live load of p lb. per unit of length over the entire span in addition to the dead load w, eq. (23) becomes

$$H = (w + p)\frac{l^2}{8f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

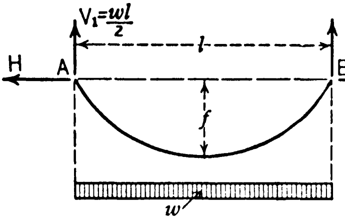


Fig. 7.

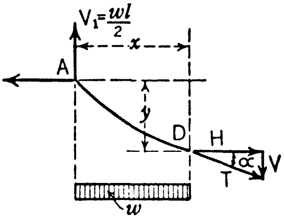


Fig. 8.

From Fig. 8, the stress at any point in the cable is

$$T = H \sec \alpha = H \frac{d\,s}{d\,x} = H \left[1 + \left(\frac{d\,y}{d\,x}\right)^2\right]^{\frac{1}{2}}.$$

Then from eq. (6)

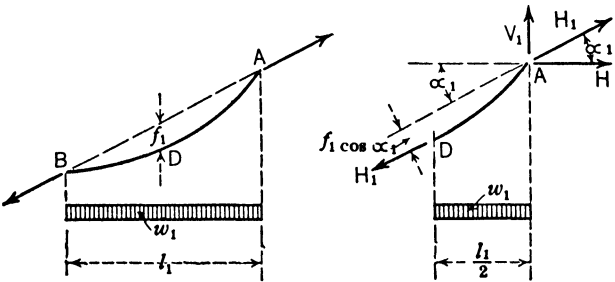
$$T = H\left(1 + \frac{64 f^2 x^2}{l^4}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

At the span center, where $x = 0$, $T = H$, and at the support, where $x = l/2$, the cable stress is a maximum and its value is

$$T = H\left(1 + \frac{16 f^2}{l^2}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

The direction of this maximum stress is given by

$$\tan \alpha = d\,y/d\,x, \text{ in eq. (6)}, = 4f/l. \quad . \quad . \quad . \qquad (27)$$



(a) (b)

Fig. 9.

Side Spans, Parabolic Cable.—From moments about A, Fig. 9 (b),

$$H_1 = \frac{w_1 l_1^2}{8 f_1} \sec \alpha^2$$

and

$$H = H_1 \cos \alpha_1 = \frac{w_1 l_1^2}{8 f_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

The stress at any point in the cable is

$$T = H \frac{d s}{d x} = H\left[1 + \left(\frac{d y}{d x}\right)^2\right]^{1/2}.$$

From eq. (13)

$$T = H\left\{1 + \left[\tan \alpha_1 + \frac{4 f_1}{l_1^2}(l_1 - 2x)\right]^2\right\}^{1/2}. \quad . \quad . \qquad (29)$$

At the span center, where $x = l_1/2$

$$T = H(1 + \tan^2 \alpha_1)^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

and at the support, where $x = 0$,

$$T = H\left[1 + \left(\tan \alpha_1 + \frac{4 f_1}{l_1}\right)^2\right]^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

which is the maximum cable stress.

From eqs. (23) and (28), the relation between sags in the main and side span cables for equal values of H at all points is

$$\frac{w l^2}{8 f} = \frac{w_1 l_1^2}{8 f_1} \quad \text{or} \quad \frac{f_1}{f} = \frac{w_1 l_1^2}{w l^2}. \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

Main Span, Catenary Cable.—The stress at any point in the cable is

$$T = H \frac{d s}{d x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

Here $H = w/c$ and $ds = \left[1 + \left(\frac{d y}{d x}\right)^2\right]^{1/2} d x$. From eq. 8), $\frac{d y}{d x} = \frac{1}{2}(e^{cx} - e^{-cx})$. Hence

$$T = \frac{w}{2 c}(e^{cx} + e^{-cx}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

At the span center, where $x = 0$, $T = H$, and at the supports, where $x = l/2$, the cable stress is a maximum and its value is

$$T = \frac{w}{2c}\left(e^{\frac{cl}{2}} + e^{-\frac{cl}{2}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Expressed in terms of hyperbolic functions, eqs. (34) and (35) may be written

$$T = H \cosh cx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and

$$T = H \cosh \frac{cl}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Equation (34) may also be written

$$T = \frac{w}{2c}\left(e^{cx} + e^{-cx}\right) = w\left(y + \frac{1}{c}\right) = wy + H. \quad . \quad . \quad . \quad (38)$$

That is, the stress in the cable at any point is equal to the horizontal component of the cable stress plus the weight of a section of cable equal in length to the ordinate of the cable curve at that point. At the span center, where $y = 0$, $T = H$, and at the support, where $y = f$, $T = H + wf$.

178. Deformation of the Cable.—*Parabolic Cable.*—Stress and temperature changes alter the span length, the cable length, and the cable sag.

Due to stress, the elongation of the cable is

$$e_s = \int_0^l T\frac{ds}{AE} = \frac{2H}{AE}\int_0^{\frac{l}{2}}\left(1 + \frac{64f^2x^2}{l^4}\right)^{\frac{1}{2}} ds$$

$$= \frac{2H}{AE}\int_0^{\frac{l}{2}}\left(1 + \frac{64f^2x^2}{l^4}\right) dx$$

$$e_s = \frac{Hl}{AE}\left(1 + \frac{16n^2}{3}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

in which $n = f/l$, the sag ratio, and A = area of cross section of cable.

For a change of t degrees in temperature, the elongation of the cable is

$$e_t = \omega t L = \omega t l\left(1 + \frac{8}{3}n^2\right), \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in which ω = coefficient of linear expansion for the material composing the cable.

The effect of these changes in length of cable on the span length and cable sag may be determined by differentiating the equations of the preceding articles. Thus from eq. (16), for an increase in sag, df, the change in the cable length is

$$dL = \frac{1}{2n}\left\{(1 + 16n^2)^{1/2} - \frac{1}{4n}\log_e\left[4n + (1 + 16n^2)^{1/2}\right]\right\}df. \quad (41)$$

and for a change in span length, dl, the change in cable length is

$$dL = \frac{1}{4n}\log_e\left[4n + (1 + 16n^2)^{1/2}\right]dl. \quad \dots \dots \dots \quad (42)$$

From eqs. (41) and (42)

$$dl = \left\{\frac{2(1 + 16n^2)^{1/2}}{\log_e\left[4n + (1 + 16n^2)^{1/2}\right]} - \frac{1}{2n}\right\}df. \quad \dots \dots \dots \quad (43)$$

Approximate values for these quantities, derived from eq. (17), are as follows:

$$dL = \frac{1}{15}(15 - 40n^2 + 288n^4)\,dl. \quad \dots \quad (44)$$

$$dL = \frac{16}{15}n(5 - 24n^2)\,df, \quad \dots \dots \dots \quad (45)$$

and

$$dl = -\frac{16n(5 - 24n^2)}{(15 - 40n^2 + 288n^4)}df. \quad \dots \dots \quad (46)$$

Substituting values of e_s or e_t from eqs. (39) and (40) in eqs. (41) to (46), the changes in form of the cable due to stress or temperature changes are readily determined.

To determine the effect of a small change in cable stress on the sag, differentiate eq. (26) with respect to f. We then have

$$df = -\frac{f}{H}\left(1 + \frac{64f^2x^2}{l^4}\right)^{1/2}dT = -\frac{f}{H}dH. \quad . \quad (47)$$

SECTION II.—UNSTIFFENED SUSPENSION BRIDGES.

179. Introduction.—Unstiffened Suspension Bridges consist of a platform or roadway suspended from cables which pass over towers and are anchored by backstays to a firm foundation, as shown in Fig. 10. Structures of this type are not used for large and important bridges because of their lack of rigidity.* In early structures of this type, the cable consisted of a built-up chain. After iron and steel wire came into use, wire cables were used in place of chains.

180. Stresses in Cable and Towers.—The load carried by the cable consists of the weight of the cable, the hangers, the roadway platform, and the live load. As the cable and hangers are light, compared to the weight of the roadway, the total dead load may be considered as uniformly distributed. We then have the same conditions as for the cable considered by itself in Sec. I.

From eqs. (23) and (26), Art. 177, the maximum dead load stress in the cable, which occurs at the towers, is $T_w = \frac{w\,l^2}{8f}\left(1 + \frac{16f^2}{l^2}\right)^{1/2}$, and the maximum cable stress due to dead load and a live load p covering the entire span is $T = (w + p)\,\frac{l^2}{8f}\left(1 + \frac{16f^2}{l^2}\right)^{1/2}$. In these equations l = span length and f = sag of the cable.

The stress in the tower $A\,E$, Fig. 10, is equal to the sum of the vertical components of the cable stress on both sides of the tower. Thus

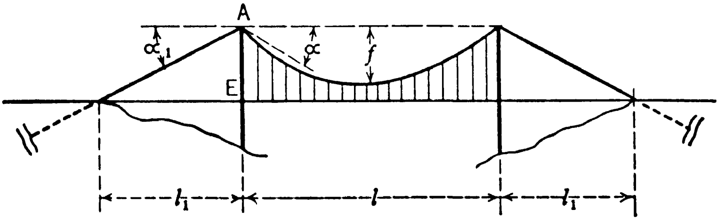


FIG. 10.

$$\text{Tower Stress} = H\,(\tan\alpha + \tan\alpha_1). \quad . \quad . \quad . \quad . \quad (1)$$

---

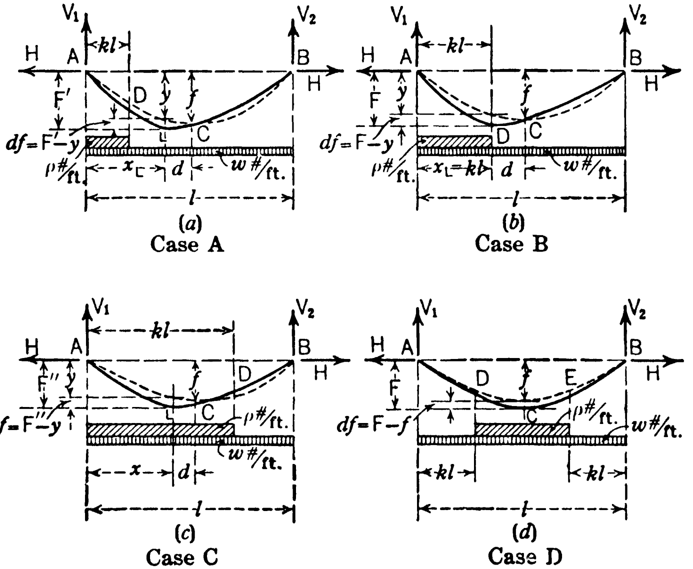
* The Hudson River Bridge, which contains a 3,500-foot center span, will be erected with an unstiffened center span. Later, when the traffic increases, a stiffening truss will be added.

For dead and live load on the center span, and for equal slope of the cable on both sides of the tower, when $\tan \alpha = \tan \alpha_1 = 4f/l$,

$$\text{Tower Stress} = (w + p)\, l. \quad \ldots \ldots \quad (2)$$

181. Deformations of the Cable.—In unstiffened suspension bridges, the deflection of the roadway is due partly to the deviation of the cable from a parabolic curve under unsymmetrical loading by reason of the flexibility of the structure and partly to the elongation of the cable and backstays under stress and temperature changes. Deflections due to these causes may be determined separately and combined for total deflection.

182. *Deflection Due to Deviation from Parabolic Curve.*—Fig. 11 shows the principal conditions of loading which cause extreme deviation of the cable from its normal position, shown by the dotted lines in the figures. For Cases A to C, the equation of the cable curve for the portion of the cable $A\,D$, which supports the live and dead load, where $x < k\,l$, is



(*a*) Case A (*b*) Case B

(*c*) Case C (*d*) Case D

FIG. 11.

$$y = \frac{w}{2}\,\frac{\left\{\left(\frac{p}{w} + 1\right)\left[x\,(l - x)\right] - \frac{p}{w} l\,x\,(1 - k)^2\right\}}{H}, \quad \ldots \quad (3)$$

and for the portion of the cable from D to B, which supports dead load only, where $x > k\,l$, the equation is

$$y = \frac{w}{2}\,\frac{\left(\frac{p}{w}\,k^2\,l + x\right)(l - x)}{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In these equations,

$$H = \frac{l^2\,w}{8f}\left[1 + 2\,\frac{p}{w}\,(3 - 2\,k)\,k^2 + \frac{p^2}{w^2}\,k^3\,(4 - 3\,k)\right]^{\frac{1}{2}} \quad . \quad . \quad (5)$$

Equations (3) to (5) are derived by the methods given in Sec. I on the assumption that the cable does not change length under stress.

The low point of the cable curve shifts its position from C, Fig. 11, to a point L, the location of this low point depending upon the position of the live load and the relative values of dead and live load. To locate the low point L, differentiate eqs. (3) and (4) with respect to x and solve for $x = x_L$. Thus we find for *Case C*, Fig. 11 (*c*),

$$x_L = \frac{l}{2}\,\frac{1 + \frac{p}{w}\,k\,(2 - k)}{\left(1 + \frac{p}{w}\right)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and for *Case A*, Fig. 11 (*a*),

$$x_L = \frac{l}{2}\left(1 - \frac{p}{w}\,k^2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Values of the ordinates to the low points of the curves are found by substituting values of x_L in eqs. (3) and (4).

For *Case A*, Fig. 11 (*a*)

$$F' = \frac{\left(1 + \frac{p}{w}\,k^2\right)^2}{\left[1 + 2\,\frac{p}{w}\,k^2\,(3 - 2\,k) + \frac{p^2}{w^2}\,k^3\,(4 - 3\,k)\right]^{\frac{1}{2}}}\cdot f. \quad . \quad . \quad . \quad (8)$$

For *Case C*, Fig, 11 (*c*)

$$F'' = \frac{\left[1 + \frac{p}{w}\,k\,(2 - k)\right]^2}{\left[1 + 2\,\frac{p}{w}\,k^2\,(3 - 2\,k) + \frac{p^2}{w^2}\,k^3\,(4 - 3\,k)\right]^{\frac{1}{2}}\left(1 + \frac{p}{w}\right)}\cdot f. \quad (9)$$

Maximum values of F' and F'' occur for the live load in the positions indicated by eqs. (10) and (11).

For *Case A*, Fig. 11 (*a*)

$$3\frac{p^2}{w^2}k^4 - 5\frac{p}{w}k^3 - 12\frac{p}{w}k^2 + 3k\left(\frac{p}{w} - 1\right) + 1 = 0. \quad . \quad (10)$$

For *Case C*, Fig. 11 (*c*)

$$3\frac{p^2}{w^2}k^4 - \left(5 - 2\frac{p}{w}\right)\frac{p}{w}k^3 - 3\frac{p}{w}k^2 + 3k - 2 = 0. \quad . \quad (11)$$

To determine the proper position of the live load, values of p and w must be substituted in eqs. (10) and (11) and the resulting equations solved for k.

183. *Coefficients for Various Cases.*—Values of x_L from eqs. (6) and (7), k from eqs. (10) and (11), and maximum F' and F'' from eqs. (8) and (9) are given in Tables A and C. These tables give also values of $d = l/2 - x_L$, the side swing of the low point in the distorted cable; values of $df = F'$; and values of $F' - y$, the change in cable curve ordinates, y being the ordinate to the initial parabolic cable curve.

When x_L from eqs. (6) and (7) are equal, the low point in the cable curve is at the head of the live load, as shown in Fig. 11 (*b*) and *Case B*, and

$$x_L = kl = l\left(\sqrt{\frac{w^2}{p^2} + \frac{w}{p}} - \frac{w}{p}\right). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From eqs. (4) and (5), the ordinate at the low point of the cable is:

$$F = \frac{4k\left(\frac{p}{w}k + 1\right)(1 - k)}{\left[1 + 2\frac{p}{w}(3 - 2k)k^2 + \frac{p^2}{w^2}k^3(4 - 3k)\right]^{\frac{1}{2}}} \cdot f \quad . \quad . \quad (13)$$

Values of F, x_L, d and df for Case B are given in Table B.

For Case D, shown in Fig. 11 (*d*), the live load covers the center portion of the span. The center sag F is found to be

$$F = \frac{\left[\left(\frac{p}{w} + 1\right) - 4\frac{p}{w}k^2\right]}{\left[\left(\frac{p}{w} + 1\right)^2 - 4\frac{p^2}{w^2}k^2(3 - 4k) - 4\frac{p}{w}k^2(3 - 2k)\right]^{1/2}}f. \quad (14)$$

Maximum F occurs for k as given by eq. (15).

$$\left.\begin{aligned}&\frac{p}{w}\left(2\frac{p}{w} + 1\right)k^3 - 3\frac{p}{w}\left(\frac{p}{w} + 1\right)k^2 + \\ &\frac{3}{4}\left(2\frac{p}{w} + 1\right)\left(\frac{p}{w} + 1\right)k - \frac{1}{4}\left(\frac{p}{w} + 1\right)^2 = 0.\end{aligned}\right\} \quad . \; . \; . \quad (15)$$

Values of F, k, and df are given in Table D.

PROPERTIES OF THE CABLE CURVE

CASES A, B, C, AND D, FIG. 11

TABLE A

CASE A FIG. 11 (a)

Values of $\frac{p}{w}$	⅓	½	1	2	3
k	0.300	0.290	0.275	0.265	0.250l
F'	0.988	0.978	0.964	0.936	0.911f
x_L	0.485	0.479	0.462	0.430	0.406l
y	0.999	0.998	0.995	0.980	0.965f
df	0.011	0.020	0.031	0.044	0.054f

TABLE B

CASE B, FIG. 11 (b)

Values of $\frac{p}{w}$	3	2	1	½	⅓
x_L	0.333	0.366	0.414	0.449	0.464l
d	0.167	0.134	0.086	0.051	0.036l
F	0.989	0.984	0.985	1.035	1.151f
y	0.888	0.928	0.968	0.988	0.996f
df	0.101	0.056	0.017	0.047	0.155f

TABLE C

CASE C, FIG. 11 (c)

Values of $\frac{p}{w}$	⅛	½	1	2	3
k	0.655	0.650	0.645	0.640	$0.630l$
F''	1.002	1.012	1.018	1.022	$1.024f$
x_L	0.486	0.479	0.468	0.457	$0.449l$
y	0.999	0.998	0.996	0.994	$0.992f$
df	0.003	0.014	0.022	0.028	$0.032f$

TABLE D

CASE D, FIG. 11 (d)

Values of $\frac{p}{w}$	⅛	½	0	2	3
k	0.352	0.357	0.375	0.402	$0.410l$
F	1.022	1.028	1.045	1.066	$1.097f$
df	0.022	0.028	0.045	0.066	$0.097f$

184. *Deflection Due to Stress and Temperature Changes.*—In addition to the deflections shown on Fig. 11, elongation of the cable and backstays, due to stress and temperature changes, cause further deflection of the cable. Assuming straight backstays, as shown in Fig. 10, and rigid anchorages, the horizontal movement of the top of the tower at A, due to stress and a rise in temperature of t degrees, is

$$\Delta l = \left(\frac{H \sec \alpha_1}{A E} + \omega t\right) l_1 \sec^2 \alpha_1,$$

in which A = area of the cable cross section.

To determine the effect of this tower movement on the cable sag, substitute $2\,\Delta l$ in eq. (43) or (46), Art. 178. The elongation of the main span cable due to stress and a rise in temperature of t degrees is

$$\Delta L = \frac{H l}{A E}\left(1 + 16\frac{n^2}{3}\right) + \omega t l\left(1 + \frac{8}{3}n^2\right)$$

The effect of this cable elongation on the cable sag is determined by substituting ΔL in eq. (41) or (45), Art. 178.

SECTION III.—STIFFENED SUSPENSION BRIDGES

185. Introduction.—In suspension bridges which are to be used for heavy traffic, the roadway must be kept as nearly as possible at the same grade under all conditions of loading. To accomplish this, the cable must be prevented from swinging into a new equilibrium curve when the span is only partially loaded with live load. The desired resistance to distortion is usually obtained by one of the two following methods: 1. By a horizontal stiffening truss suspended from the cable; 2. By making the cables themselves rigid enough to resist the distortion.

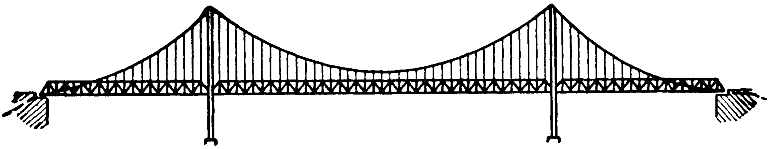


FIG. 12.

Fig. 12 shows a structure of the first class mentioned above. A truss carrying the roadway or track is suspended from the cable. Due to the stiffness of the truss, a load applied at any point, or a uniform load covering a part of the structure, will be distributed to the cable approximately as a uniform load. The stiffer the truss the more nearly uniform will be the load on the cable and therefore the less the distortion of the cable.

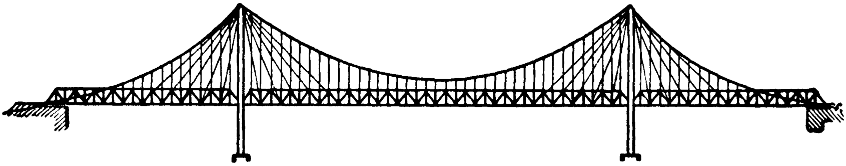


FIG. 13.

In the Brooklyn Bridge, which is of this type, inclined stays were added, which ran out to about the quarter-point of the span as shown in Fig. 13. Due to the uncertainty of just how these stays act under loading, and also because they have not been found to act efficiently as a means of stiffening the roadway, their use has been abandoned.

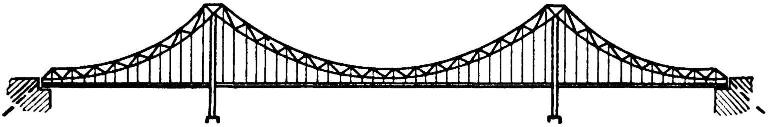


FIG. 14.

In structures of the second class, the cable itself is made rigid. This may be done in several different ways. One arrangement is shown in Fig. 14. Two eye-bar chains are hung one above the other and braced to form a rigid system which is similar in nature to a hingeless arch. Due to difficulties in erection and uncertainty of stress calculation, examples of this system are rare.

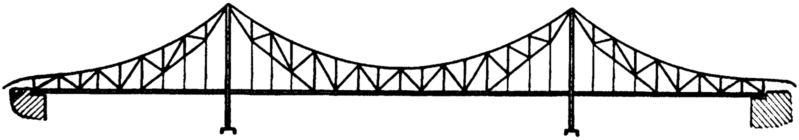


FIG. 15.

The structure shown in Fig. 15 is similar to a two-hinged arch, inverted. This form was used in one of the proposed designs for the Manhattan Suspension Bridge in New York City.

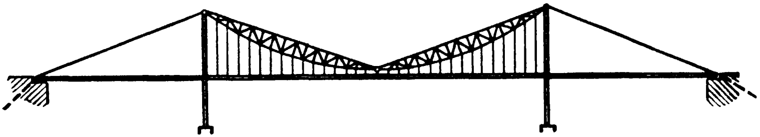


FIG. 16.

In Fig. 16 is shown a structure built on the three-hinge-arch principle. The Point Suspension Bridge at Pittsburg is a notable example of this type. Here the stresses are all statically determinate, and are, therefore, not subject to any of the uncertainties which exist in the types shown in Figs. 14 and 15.

In the following articles the structures here shown will be taken up in detail and methods for stress calculation developed. This work will be divided into two parts—one part referring to structures in which a horizontal stiffening truss is used, the other to structures in which the cable itself is made rigid. The treatment, however, will refer mainly to structures with horizontal stiffening trusses. In the case of structures with stiffened cables the methods used in arch analysis can be applied with but few modifications, which will be taken up in detail.

186. General Method of Procedure.—In the determination of stresses in structures with horizontal stiffening trusses the following assumptions are made:

1. The initial curve of the cable is parabolic.

2. The entire dead load is carried by the cable and causes no stress in the stiffening trusses. The trusses are stressed by changes of temperature, and by live load.

The first assumption is based upon the facts brought out in Section I of this chapter, concerning the cable as a structure by itself.

The second depends upon the field methods used in the erection of such structures. The trusses are erected with all joints field bolted. After the erection has been completed and the trusses brought to the desired grade by adjustment of the hangers, the joints are riveted up at the proper temperature, realizing the assumed conditions.

In developing the methods of calculation, the discussion will be divided into Approximate and Exact Methods. In the Approximate Method, the trusses are considered to be very stiff, and the loads comparatively light. Under these conditions the effect of the deformations due to stress, upon the general dimensions of the structure will be neglected. The cable is therefore assumed to retain its original parabolic form, and the hanger loads acting thereon are assumed to be uniformly distributed under all conditions of loading. This method is sufficiently accurate for most cases.

For structures in which the trusses are not very stiff and the span long, the deformations of truss and cable can no longer be neglected. These conditions are considered in the exact method of calculation.

(A) APPROXIMATE METHODS OF CALCULATION

187. General Expressions for Horizontal Component of Cable Stress and for Moments and Shears in the Stiffening Truss.—A cable over a single opening is stiffened by a truss suspended in any manner at R_1 and R_2 as shown in Fig. 17. Required the horizontal component of the cable stress.

Since the stiffening truss, acting alone, is capable of supporting loads, the truss and cable constitute a redundant system. The cable may be conveniently regarded as the redundant member. The same general method of analysis may be employed as explained in Part I. In proceeding with this method we will imagine the cable cut at a point near the tower. This being done the stresses in all members will be determinate, and will correspond to the stresses S' and the moments M' of Art. 217, Part I. These stresses in the towers, hangers, and cable will be zero, and the moments in the truss will be the same as if the cable was removed. Let H be the horizontal component of the cable stress and let this be taken as the redundant stress. Then from the theory of redundant members, as given in Part I, we have

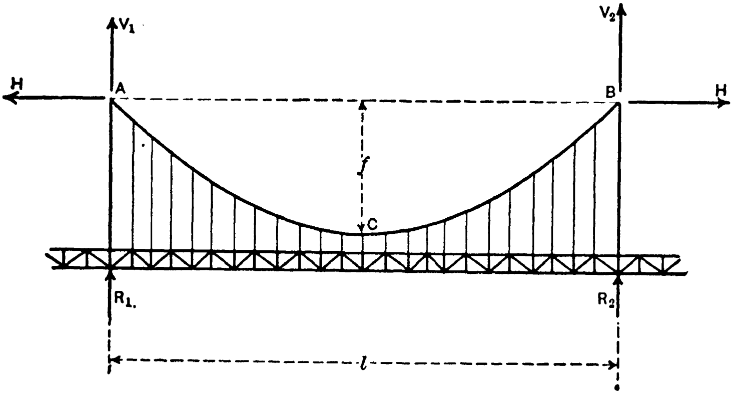


FIG. 17.

for a system made up of members, part of which are acted upon by bending moments, and part by direct, or axial, stresses,

$$S_r = H = -\frac{\Sigma S' \frac{u\,l}{A\,E} + \int_0^l \frac{M'\,m}{E\,I}\,d\,x}{\Sigma \frac{u^2\,l}{A\,E} + \int_0^l \frac{m^2}{E\,I}\,d\,x}. \quad . \quad . \quad . \quad (1)$$

Where u is the direct stress in the hangers, the towers, and at any point in the cable, for a stress in the redundant member such that its horizontal component is 1 lb.; m is the bending moment at any point in the stiffening truss at the same time. The moment of inertia of the stiffening truss, I, and the area of the cable, A, are considered as uniform over their entire length.

Eq. (1) is then a general expression for the horizontal component of cable stress for structures with horizontal stiffening trusses of any form. By calculating the values of M', m, and u, subject to the conditions governing these values for the structure in question, and substituting in eq. (1), an expression for the horizontal component of the cable stress can be obtained for any particular case.

The moment at any point in the stiffening truss after the addition of the cable will be given by the equation

$$M = M' + H\,m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation is obtained in the same manner as the expression for stress in any member of a system with a redundant member.

The shear at the same point is

$$V = \frac{d\,M}{d\,x} = \frac{d\,M'}{d\,x} + H\frac{d\,m}{d\,x} = V' + H\frac{d\,m}{d\,x}. \quad . \quad . \quad . \quad (3)$$

where V' is the shear in the truss before the addition of the cable.

These general expressions will now be applied to particular cases, and values of horizontal component of cable stress, moment, and shears will be derived.

(*a*) *Structure over a Single Opening, Truss Hinged at the Ends*

188. Value of H for Various Cases.—In the structure shown in Fig. 18, the stiffening truss is a simple beam of span l, anchored at the ends, at which points it is free to turn. Required the stresses in the various parts of the structure due to any loading. The horizontal component of the cable stress is given by eq. (1), when M', m, and u are determined for the conditions shown in Fig. 18.

The value of M' depends upon the loading in question and is to be calculated as for a simple beam of span l.

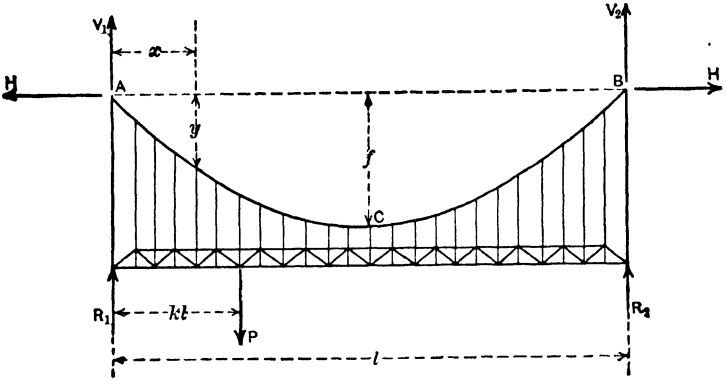


FIG. 18.

The value of m, the bending moment at any point in the stiffening truss for a 1-lb. load at A, acting horizontally, is found by taking moments about the neutral axis of the truss. Fig. 19 shows a portion

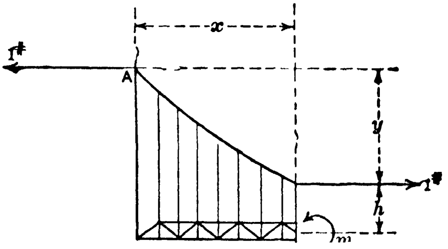


FIG. 19.

of the structure removed with all forces acting, from which we find

$$m = -(h + y) + h = -y. \quad . \; . \; . \; . \; . \quad (4)$$

The same result is obtained by the following method. The 1-lb. load at A will cause an upward pull on the truss through the hangers. Since the cable is assumed to hang in a parabola this pull will be uniform over the whole truss. Its amount will be such as to make

H equal to one pound. From eq. (24) of Section 1, we find that this load must be $\frac{8f}{l^2}$ pounds per foot. The bending moment at any point in a simple beam due to an upward load of $\frac{8f}{l^2}$ lbs. per foot, is $-\frac{4fx}{l^2}(l-x)$. Then we have as before

$$m = -\frac{4fx}{l^2}(l-x) = -y \quad . \quad . \quad . \quad (4a)$$

Also from the equilibrium polygon as drawn for a load of $-\frac{8f}{l^2}$ lbs. per ft., we have in general, $M = -Hy$, as given in Chapter II, Part I. As H is here taken as 1 lb., we have: $m = -y$.

Substituting the values determined above in eq. (1), we have

$$\int_0^l \frac{M'm}{EI}dx = -\frac{1}{EI}\int_0^l M'y\,dx. \quad . \quad . \quad . \quad (5)$$

Also

$$\int_0^l \frac{m^2}{EI}dx = \frac{1}{EI}\int_0^l y^2\,dx = \frac{16f^2}{l^4EI}\int_0^l x^2(l-x)^2\,dx = \frac{8}{15}\frac{f^2l}{EI} \quad (6)$$

The term $\Sigma\frac{u^2l}{AE}$ of eq. (1), must be evaluated for the cable, the hangers, and the towers. It is usual, however, to neglect the parts due to the stresses in the towers and the hangers, as these terms are very small in any case.

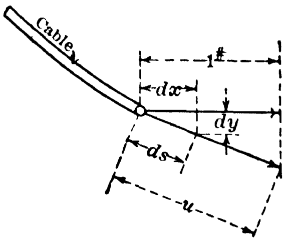


FIG. 20.

For the cable, the value of u for a 1-lb. load at A, Fig. 18, can be seen from Fig. 20 to be $\frac{ds}{dx}$. The element of length l here becomes ds, the total length of the cable from tower to tower being L. Also, A is the area of the cable and E the coefficient of elasticity. Then, for the cable, we have

$$\Sigma\frac{u^2l}{AE} = \frac{1}{AE}\int_0^L \frac{ds^2}{dx^2}ds = \frac{1}{AE}\int_0^L \frac{ds^3}{dx^2}.$$

From the equation of the parabola, origin at the centre,

$$ds = \left[1 + \frac{64\,f^2\,x^2}{l^4}\right]^{\frac{1}{2}} dx.$$

Substituting this value in the above expression, we have

$$\int_0^L \frac{ds^3}{dx^2} = 2\int_0^{\frac{l}{2}} \left[1 + \frac{64\,f^2\,x^2}{l^4}\right]^{\frac{3}{2}} dx$$

$$= l\left\{\frac{1}{4}\left(\frac{5}{2} + \frac{16\,f^2}{l^2}\right)\left(1 + \frac{16\,f^2}{l^2}\right)^{\frac{1}{2}} + \frac{3}{32}\,\frac{l}{f}\log_e\left[\frac{4\,f}{l} + \left(1 + \frac{16\,f^2}{l^2}\right)^{\frac{1}{2}}\right]\right\} \quad (7)$$

For brevity, this integral will hereafter be referred to as L_s, eq. (7).

For the hangers, the value of u is equal to the upward pull of the cable on the truss for a 1-lb. load at A, which has already been found to be $-\frac{8\,f}{l^2}$ lbs. per ft. The length of hanger at any point is seen from Fig. 21 to be

$$F_x = f + F_c - y = F_c + f\left[1 - \frac{4\,x}{l^2}\,(l - x)\right]$$

where F_x = length of hanger at any point and F_c = length of the centre hanger. The area of the hangers will be taken as A_h per foot

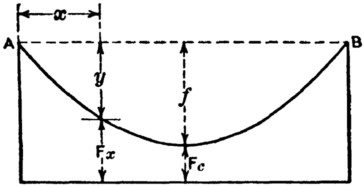


FIG. 21.

of length and E_h will be taken as the coefficient of elasticity of the hanger ropes.

Then for the hangers

$$\Sigma\frac{u^2\,l}{A\,E} = \int_0^l \frac{64\,f^2}{l^4}\,\frac{F_x}{A_h\,E_h} = \frac{64\,f^2}{l^4\,A_h\,E_h}\int_0^l \left\{F_c + f\left[1 - \frac{4\,x}{l^2}\,(l - x)\right]\right\} dx$$

$$= \frac{64\,f^2}{l^3\,A_h\,E_h}\left(F_c + \frac{f}{3}\right). \quad \ldots\ldots\ldots\ldots \quad (8)$$

For the towers, the value of u is equal to the vertical component of the cable stress at point A, Fig. 18, for a 1-lb. horizontal load at the same point, or $u = \frac{dy}{dx}$. For the parabola, origin at A, when $x = 0$ we have $\frac{dy}{dx} = \frac{4f}{l}$. Therefore $u = \frac{4f}{l}$. The length of the tower will be taken as F_T and its area A_T. Then we have for the towers

$$\Sigma \frac{u^2 l}{A E} = \frac{32 f^2}{l^2} \frac{F_T}{A_T E}. \quad \dots\dots \quad (9)$$

Collecting the various terms, as given by eqs. (5) to (9) and inserting in eq. (1), remembering that for this case the term $\Sigma \frac{S' u l}{A E}$ is zero, we have

$$H = \frac{\frac{1}{EI} \int_0^l M' y \, dx}{\frac{8}{15} \frac{f^2 l}{EI} + \frac{L_s}{A E} + \frac{64 f^2}{l^3 A_h E_h} \left(F_c + \frac{f}{3} \right) + \frac{32 f^2 F_T}{l^2 A_T E}} \quad . \quad (10)$$

which is the general expression for the horizontal component of the cable stress in the structure shown in Fig. 18.

In this equation the value of M' depends upon the loading in the particular case. Therefore M' must be expressed as a function of x and the integration performed as indicated in eq. (10). For a single load P, a distance $k\,l$ from the left end of the span, as shown in Fig. 18, the integral must be divided into two parts, one part for x less than $k\,l$, the other part for x greater than $k\,l$. The term then becomes

$$\frac{1}{EI}\int_0^l M' y dx = \frac{1}{EI}\int_0^{kl} [P(1-k)\, x] \left[\frac{4fx}{l^2}(l-x)\right] dx + \frac{1}{EI}\int_{kl}^{l} [P(l-x)k] \left[\frac{4fx}{l^2}(l-x)\right] dx$$

$$= \frac{1}{3EI} P f l^2 k (k^3 - 2k^2 + 1). \quad \dots\dots\dots\dots\dots\dots \quad (11)$$

Substituting this value in eq. (10), neglecting the effect of the hangers and towers, we have

$$H = \frac{\frac{1}{3} P f l^2 k (k^3 - 2 k^2 + 1)}{\frac{8}{15} f^2 l + \frac{I L_s}{A}} \quad \dots\dots \quad (12)$$

which is the value of the horizontal component of the cable stress for a single load P on the main span. The value of H is found to be a maximum when $k = \frac{1}{2}$, or for the load at the centre of the span.

For a uniform load of p pounds per linear foot extending a distance

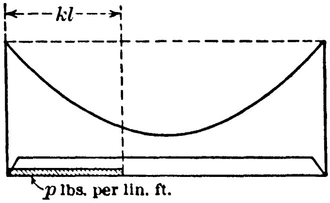


FIG. 22.

kl from the left end as shown in Fig. 22, the term $\int M' y\,dx$ becomes

$$\int_0^l M' y\,dx = \int_0^{kl} \left[\frac{pkl}{2}(2-k)\,x - \frac{px^2}{2}\right]\left[\frac{4fx}{l^2}(l-x)\right]dx$$
$$+ \int_{kl}^{l} \left[\frac{pk^2 l}{2}(l-x)\right]\left[\frac{4fx}{l^2}(l-x)\right]dx$$
$$= \frac{1}{30}\,p f l^3 k^2 (2k^3 - 5k^2 + 5).$$

The same result may be obtained from eq. (11) by replacing P by $p\,l\,d\,k$ and integrating between the limits k and 0. We then have

$$\int_0^l M' y\,dx = \int_0^k \frac{1}{3}\,p f l^3 k\,(k^3 - 2k^2 + 1)\,dk = \frac{1}{30}\,p f l^3 k^2 (2k^3 - 5k^2 + 5).$$

For this case, again neglecting the effect of hangers and towers, eq. (10) becomes

$$H = \frac{\frac{1}{30}\,p f l^3 k^2 (2k^3 - 5k^2 + 5)}{\frac{8}{15} f^2 l + \frac{I L_s}{A}} \quad . \quad . \quad . \quad . \quad (13)$$

which is the value of H for a uniform load extending a distance kl from the left end of the span. The value of H is found to be a maximum when $k = 1$, or for the entire span loaded, when

$$H = \frac{\frac{1}{15}\,p f l^3}{\text{Denominator eq. (10) or (12)}}$$

For a uniform live load p lbs. per lin. ft., covering a portion of the span as shown in Fig. 23, the value of $\int M' y\, d x$ may be found from eq. (11) as explained above. Thus we have

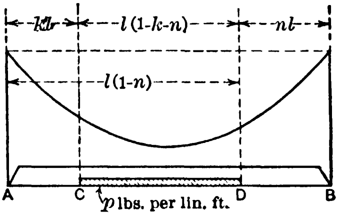


FIG. 23.

$$\int_0^l M' y\, d x = \int_k^{(1-n)} \frac{p f l^3}{3} k(k^3 - 2 k^2 + 1)\, d k$$

$$= \frac{p f l^3}{30} [2 - n^2 (5 - 5 n^2 + 2 n^3) - k^2 (5 - 5 k^2 + 2 k^3)]$$

The value of H for this case is then

$$H = \frac{\frac{1}{30} p f l^3 [2 - n^2 (5 - 5 n^2 + 2 n^3) - k^2 (5 - 5 k^2 + 2 k^3)]}{\text{Denominator eq. (12)}}. \quad . \quad . \quad (14)$$

When the distances $A\,C$ and $D\,B$ are equal, or for $n = k$

$$H = \frac{\frac{1}{15} p f l^3 [1 - k^2 (5 - 5 k^2 + 2 k^3)]}{\text{Denominator eq. (12)}}. \quad . \quad . \quad (15)$$

For opposite loading conditions, that is, for load on $A\,C$ and $D\,B$, Fig. 23, no load on $C\,D$, the value of H may be obtained by subtracting the value of H as given by eq. (14) or (15), from H for full-load conditions, or

$$H = \frac{\frac{1}{15} p f l^3 - \text{Numerator eq. (14) or (15)}}{\text{Denominator eq. (10) or (12)}}.$$

189. Effect of Temperature Changes on H.—The effect of a change of temperature on the stress in the cable is found from the term $\Sigma \frac{S' u l}{A E}$ of eq. (1). Here $\frac{S' l}{A E}$ is to be replaced by $\omega\, t\, d s$, the change in length of an element of the cable, $d s$, due to a change in temperature of t

degrees, the coefficient of linear expansion being denoted by ω. The value of u is again $\frac{d\,s}{d\,x}$. We then have a term of the form $\int_0^L \omega\, t \frac{d\,s^2}{d\,x}$ to be added to the numerator of the equation for H. Inserting the value of $d\,s$, and proceeding as in the derivation of L_s in eq. (7), we have

$$\int_0^L \frac{d\,s^2}{d\,x} = 2\int_0^{\frac{l}{2}} \left[1 + \frac{64 f^2 x^2}{l^4}\right] d\,x = l\left[1 + \frac{16}{3}\frac{f^2}{l^2}\right]. \quad (16)$$

Hereafter this expression will be referred to as L_t, eq. (16).

Inserting this value in eqs. (10) or (11), we have

$$H = \frac{\frac{1}{3} P f l^2 k (k^3 - 2k^2 + 1) - E I \omega t L_t}{\text{Denominator eq. (10) or (12)}}, \quad . \quad . \quad . \quad (17)$$

which is the value of the horizontal component of the cable stress for a single load P, temperature effect included. In the same way we have for a uniform load of p pounds per foot

$$H = \frac{\frac{1}{30} p f l^3 k^2 (2k^3 - 5k^2 + 5) - E I \omega t L_t}{\text{Denominator eq. (10) or (12)}}. \quad . \quad . \quad (18)$$

From eqs. (17) and (18) it can be seen that a rise in temperature tends to decrease the value of the horizontal component of cable stress, while a fall in temperature causes an increase.

190. Moments and Shears.—The moment at any point in the stiffening truss can be obtained from eq. (2) by substituting for M' and m, their values for this case, M' and $-y$ respectively.

We then have

$$M = M' - H y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

That is, the bending moment at any point in the stiffening truss is equal to the bending moment at the same point in a simple beam of like span, minus the product of the horizontal component of the cable stress and the ordinate to the cable curve at that point.

Likewise from eq. (3), the shear at any point is

$$V = V' - H\frac{d\,y}{d\,x} = V' - H \tan\theta \quad . \quad . \quad . \quad (20)$$

where V' is the shear in a simple beam of the same span, and tan θ, or $\frac{dy}{dx}$, is the slope of the cable at the point.

191. Influence Line for Moment.—The variation in the moment and shear at any point in the stiffening truss due to any condition of loading can best be studied by the use of influence lines. The influence line for moment at any point, as E, Fig. 24 (*a*), will now be drawn. For this purpose eq. (19) may be written

$$M = \left(\frac{M'}{y} - H\right) y. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This equation is best plotted as the difference between the two quantities $\frac{M'}{y}$ and H, as shown in Fig. 24 (*b*). The curved line *a c b* shows the value of H, as calculated from eq. (12). Here P is taken as 1 pound, and the effect of temperature is neglected, as this may best be studied separately. The influence line for $\frac{M'}{y}$ is drawn in the same way as the influence line for a simple beam as explained in Part I. It is only necessary to calculate the maximum ordinate, which occurs at the point for which the influence line is to be drawn. From Fig. 24 (*a*) we see that for the 1-lb. load at E, $M' = n\,l\,(1 - n)$. Also at this same point y, the ordinate to the cable curve, is $4fn\,(1 - n)$. This is obtained by substituting $x = n\,l$ in the equation of the cable curve referred to point A, Fig. 24 (*a*), as origin. Then we have $\frac{M'}{y} = \frac{l}{4f}$, from which we see that the maximum ordinate is the same for all influence lines. The lines *a e* and *e b* give the influence line for $\frac{M'}{y}$ for point E.

In the same way, the influence line for any other point M, Fig. (*a*), can be drawn by locating the point *m* in Fig. (*b*) and drawing the lines *a m* and *b m*. The same curve for H can be used for all cases.

192. Moments from Influence Lines.—The moment at point E for a load at point G will be given by the ordinate to this influence line at G multiplied by y, the cable ordinate at E, or

$$M_E = (\text{ordinate } fg) \times y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

From the influence line it can be seen that this moment is positive for loads on the section $D\,K$, for here the value of $\frac{M'}{y}$ exceeds H. For the section $K\,F$ the moment will be negative.

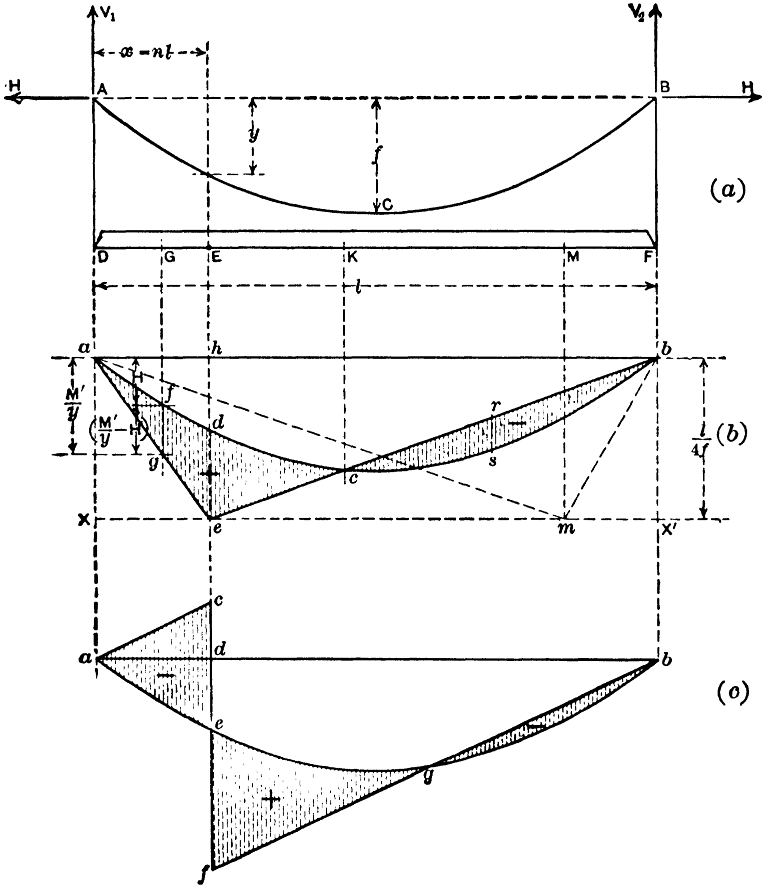


FIG. 24.

The maximum positive moment will occur when the load is at point E. For the load at point K the moment at E will be zero. The maximum negative moment at E will occur when the ordinate $r\,s$ is greatest. The load is then at a point about half way between K and F, the exact position being best determined by trial.

The moment due to a uniform load of p lbs. per ft., extending over any portion of the span, is given by the product of the corresponding area of the influence diagram (shown by the shaded area) and the ordinate y of the given point. Or, in general,

$$M_E = p \times (\text{area influence diagram}) \times y. \quad . \quad . \quad (23)$$

For the maximum positive moment at E, the uniform load would extend from the left end of the span to point K, or

$$M_E = p \times [\text{area } a\,d\,c\,e, \text{Fig. } 24\ (b)] \times y. \quad . \quad . \quad (24)$$

In the same way, the maximum negative moment occurs when the load extends from K to F, and its value is

$$M_E = p \times [\text{area } b\,r\,c\,s, \text{ Fig. } 24\ (b)] \times y. \quad . \quad . \quad (25)$$

193. Effect of Temperature on Moments.—The effect of changes in temperature on the moment at any point will now be considered. Temperature changes affect the value of the horizontal component of the cable stress, but do not affect the value of M' in eq. (19). Then for a change in temperature we have from eq. (19), that the change in moment is

$$M_t = -H\,y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

From eq. (17) we see that for a change in temperature of $+\,t$ degrees, the value of the horizontal component of cable stress is

$$H_t = -\frac{E\,I\,\omega\,t\,L_t}{\text{Denominator eq. (10) or (12)}}. \quad . \quad . \quad (27)$$

Then from eqs. (26) and (27), the moment due to a change in temperature is

$$M_t = \pm\frac{E\,I\,\omega\,t\,L_t\,y}{\text{Denominator eq. (10) or (12)}}. \quad . \quad . \quad (28)$$

Thus a rise in temperature causes a positive moment in the stiffening truss, and a fall in temperature causes a negative moment.

From eq. (28) we see that to obtain maximum moments for live load and temperature effect combined, positive live-load moments and temperature moments due to a rise in temperature are to be taken together. Maximum negative moments are obtained by combining temperature moments due to a fall in temperature with negative live-load moments.

194. Influence Lines for Shear.—To aid in drawing influence lines for shear at any point in the stiffening truss, eq. (20) will be written in the form

$$V = (V' \cot \theta - H) \tan \theta. \quad . \quad . \quad . \quad . \quad (29)$$

The values of tan θ, and cot θ in this expression may be determined from the equation of the cable curve referred to point A, Fig. 24 (a) as origin, or $\tan \theta = \frac{dy}{dx} = \frac{4f}{l^2}(l - 2x)$. For the influence line at point E, where $x = nl$, we have $\tan \theta = \frac{4f}{l}(1 - 2n)$. Eq. (29) then becomes

$$V = \left[\frac{V'l}{4f(1-2n)} - H\right]\frac{4f}{l}(1 - 2n). \quad . \quad . \quad (30)$$

The influence line for the expression in brackets will be drawn. Here the value of H is again determined from eq. (12) as in the case of moments.

The curve for H is shown by $a\,e\,g\,b$ of Fig. 24 (c). The value of $\frac{V'l}{4f(1-2n)}$ is obtained by the same methods as for simple beams. The ordinate $c\,d$, Fig. (c), obtained when the 1-lb. load is just to the left of point E is $c\,d = -\frac{nl}{4f(1-2n)}$. Since this is a negative quantity, it must be plotted so as to add to H, or upward from the base line $a\,b$. For the load just to the right of point E, we have $d\,f = +\frac{(1-n)\,l}{4f(1-2n)}$. This ordinate is positive and is to be plotted downward. The complete influence line is then as shown in Fig. (c).

195. Shears from Influence Lines.—The shear at E, Fig. 24 (a), for a load at any point is obtained by multiplying the ordinate to the influence line under the load by the quantity $\frac{4f}{l}(1 - 2n)$. It can be seen from Fig. (c) that loads on the ends of the span cause negative shears at point E, while loads on a short portion of the span just to the right of E cause positive shears.

For uniform loads, the shear at E is given in terms of the area of the influence diagram under the loaded portion of the span. The shear is given by the expression

$$V = p \times (\text{area influence diagram}) \times \frac{4f}{l}(1 - 2n). \quad . \quad (31)$$

For shear at the span centre, $\tan\theta$ in eq. (20) becomes zero. The cable then has no effect on the shear, which is now equal to the term V' alone. The maximum centre shear is therefore $\pm \frac{pl}{8}$ for all cases.

196. Effect of Temperature on Shear.—Temperature effects are found as before, a rise in temperature causing positive shears, and a fall in temperature causing negative shears. The general expression for shear due to temperature is given by eq. (28) when y is replaced by $\tan\theta$, the slope of the cable curve at the point in question.

197. Deflection of the Stiffening Truss.—To determine the deflection of the stiffening truss, consider the truss as a beam so loaded as to cause a moment $M = M' - Hy$. For a uniform live load of p per unit of length, and a parabolic cable,

$$M = \frac{p}{2}x(l - x) - \frac{4fH}{l^2}x(l - x).$$

From Art. 1, eq. (8), the load which will cause a moment M is

$$w = \frac{d^2M}{dx^2}.$$

Then for the stiffening truss

$$w = \frac{d^2M}{dx^2} = \frac{d^2}{dx^2}\left[\frac{p}{2}x(l - x) - \frac{4fH}{l^2}x(l - x)\right]$$

and
$$w = p - \frac{8fH}{l^2}.$$

From Table I, Art. 7, the deflection of a simple beam uniformly loaded is

$$d = \frac{5}{384}\frac{wl^4}{EI}.$$

Hence for the stiffening truss, the center deflection under a uniform live load of p per unit of length is

$$d = \frac{5}{384}\frac{l^4}{EI}\left(p - \frac{8fH}{l^2}\right). \qquad \cdots\cdots \qquad (32)$$

where H is given in Art. 188.

The deflection of the stiffening truss due to temperature changes may be found in a similar manner. From eq. (26), Art. 193, $M_t = -H_t y$. Then

$$w = \frac{d^2M}{dx^2} = \frac{8fH_t}{l^2}$$

and
$$d_t = \frac{5}{48EI}fH_t l^2, \qquad \cdots\cdots\cdots \qquad (33)$$

where H_t is given by eq. (27), Art. 193.

(B) STRUCTURE OVER THREE OPENINGS. SIDE SPANS SUSPENDED. TRUSSES HINGED

198. Value of H.—When side spans are suspended from the cables, as shown in Fig. 25, eq. (1), Art. 187, must be made up to include these spans as well as the main span.

Values of M', m, and u for the main span are the same as given in Art. 188. For the side spans, $M' = M_1'$ and

$$m = m_1 = -y_1 = -\frac{4f_1x_1}{l_1^2}(l_1 - x_1).$$

Then

$$\left.\begin{aligned}
&\int_0^l \frac{M'm}{EI}dx = -\frac{1}{EI_1}\int_0^l M_1'y_1\,dx\\
&\text{and}\\
&\int_0^l \frac{m^2}{EI}dx = \frac{16f_1^2}{l_1^4EI_1}\int_0^{l_1}x_1^2(l_1 - x_1)^2dx = \frac{8f_1^2l_1}{15EI_1}
\end{aligned}\right\} \quad \cdots \quad (34)$$

For the side span cable $u = d\,s_1/d\,x_1$. Referring the equation of the side span cable to an origin at E, the low point of Fig. 6, Art. 176,

$$\sum \frac{u^2\,l}{A\,E} = \int_0^{L_1} \frac{1}{A\,E}\frac{d\,s_1{}^3}{d\,x_1{}^2} = \int_{(l_2-l_1)}^{l_2} \left(1 + \frac{4\,F^2\,x_1{}^2}{l_2{}^4}\right)^{3/2} d\,x \quad .$$

Then, for the main and side spans

$$\left.\begin{aligned}\sum \frac{u^2\,l}{A\,E} &= \frac{1}{A\,E}\left\{L_s \text{ eq. (7)} + 2\int_{(l_2-l_1)}^{l_2} \left[1 + \frac{4\,F^2\,x_1{}^2}{l_2{}^4}\right]^{3/2} d\,x\right\} \\ &= L_s \text{ eq. (35)}.\end{aligned}\right\} \quad . \; . \qquad (35)$$

The side span temperature term is

$$\int_0^{L_1} \omega\,t\,\frac{d\,s_1{}^2}{d\,x_1} = \omega\,t\int_{(l_2-l_1)}^{l_2} \left[1 + \frac{4\,F^2\,x_1{}^2}{l_2{}^4}\right] d\,x$$

$$= \omega\,t\,l_1\left[1 + \frac{4\,(3\,l_2{}^2 - 3\,l_2\,l_1 + l_1{}^2)\,F^2}{3\,l_2{}^4}\right]$$

Then for the main span and both side spans,

$$\left.\begin{aligned}\sum \frac{S'\,u\,l}{A\,E} &= \omega\,t\left\{l\left[1 + \frac{16 f^2}{3\,l^2}\right] + 2\,l_1\right. \\ &\left.\left[1 + \frac{4\,(3\,l_2{}^2 - 3\,l_2\,l_1 + l_1{}^2)\,F^2}{3\,l_2{}^4}\right]\right\} = \omega\,t\,L_t \text{ eq. (36)}.\end{aligned}\right\} \quad . \qquad (36)$$

Collecting terms and substituting in eq. (1)

$$H = \frac{\displaystyle\int_0^l M'\,y\,d\,x + \frac{I}{I_1}\int_0^l M_1{}'\,y_1\,d\,x - E\,I\,\omega\,t\,L_t \text{ eq. (36)}}{\displaystyle\frac{8}{15} f^2\,l + \frac{16}{15} f_1{}^2\,l_1\,\frac{I}{I_1} + \frac{I}{A}\,L_s \text{ eq. (35)}} \quad . \qquad (37)$$

which is the expression for horizontal component of cable stress for the structure shown in Fig. 25.

For any particular loading, values of $\int_0^l M'\,y\,d\,x$ and $\int_0^l M_1{}'\,y_1\,d\,x$ are found by the methods given in Art. 188.

The horizontal component of cable stress due to temperature change may be determined from eq. (37) by placing M' and M_1', equal to zero. Then

$$H_t = -\frac{E\,I\,\omega\,t\,L_t\,\text{eq. (36)}}{\text{Denominator eq. (37)}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

for a rise in temperature.

199. Influence Lines for Moments and Shears.—Influence lines for moments and shears in the structure of Fig. 25 are shown in Figs. 28 and 29 of Art. 202. These influence lines are constructed by the methods given in Arts. 191 and 194.

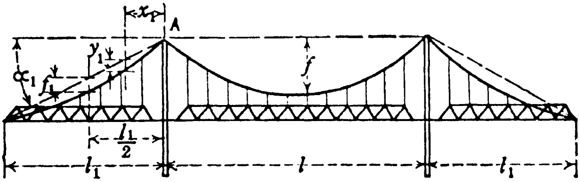


FIG. 25.

General expressions for moments and shears in the main span are the same as given in Art. 190. On substituting side span dimensions these expressions also apply for side span moments and shears.

From the influence lines we see that for positive moments in the main span, the loading conditions are the same as for Art. 192. For side-span positive moments the side span in question is to be fully loaded, no load on the rest of the structure.

For negative moments in the main span the loading conditions are just opposite from those for positive moments; that is, the portions of the main span which were not loaded for positive moment, together with both side spans, are to be loaded for negative moments. In the side spans, the maximum negative moments occur when there is no load in the span in question, the main span and the far side span being fully loaded.

Loading conditions for shear are similar in nature. Positive main-span shears occur for the same loading conditions as described in Art. 195. Negative shears occur under opposite loading conditions. Positive side-span shears occur under partial live load in that span only, negative shears occurring under opposite loading conditions.

200. Deflection of the Stiffening Truss.—The deflection at any point in the main span truss can be found by the same method as used in Art. 197. The resulting equation will be the same in form, the value of H, however, is to be calculated for the loading in question.

For side spans, the same equations for deflection may be used, expressed in terms of side-span dimensions.

From the equations thus derived it will be found that the greatest downward deflection of points in the main span, or side spans, will occur for loads in that span only. For greatest upward deflection of the main span, both side spans are to be fully loaded, the main span to be empty, or in the case of flexible trusses, the ends only being loaded. For the greatest upward deflection of a side span, the main span and the far side span are to be loaded, no load being placed in the side span in question.

201. Deflection of Top of Tower or Movement of Saddle.—In structures over several openings, as shown in Fig. 25, the cable is made continuous from anchorage to anchorage. Two different methods of attachment are used at the tops of the towers. In the first, the cable rests on saddles which are free to slide, while in the second the saddles are rigidly fastened to the towers. In either case the action of moving loads on any span will tend to develop unequal horizontal components in the various parts of the cable, which must be equalized by changes in sag and span length. The point of attachment of the cables at the towers will then move until equilibrium is established. For such changes, the movable saddle slides over the top of the tower, no horizontal force acting on the tower except that

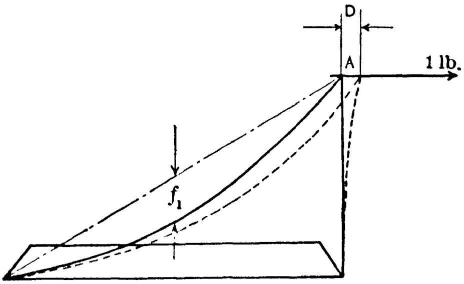


FIG. 26.

due to the friction of the rollers. In the case of the fixed saddle the tower must bend, thereby inducing bending stresses in addition to those due to direct load. If the towers are made comparatively flexible, the required motion can take place without causing heavy bending stresses.

The amount of this movement in any case can be determined by means of the formula $D = \int_0^l \frac{M\,m}{E\,I}\,d\,x$ as given in Part I.

To find the deflection of the point A of the structure shown in Fig. 25 we may remove the left side span, as shown in Fig. 26, and apply the general formula to this portion of the structure. For loads on this side span $M = M_1' - H\,y_1$ and for 1 lb. horizontal load at the top of the tower, acting as shown, $m = -y_1$.

Then

$$D = -\int_0^{l_1} \frac{M_1'\,y_1}{E\,I_1}\,d\,x + \int_0^{l_1} \frac{H\,y_1^2}{E\,I_1}\,d\,x$$

$$= -\int_0^{l_1} \frac{M_1'\,y_1}{E\,I_1}\,d\,x + \frac{8}{15}\,H\,\frac{f_1^2\,l_1}{E\,I_1}. \quad . \quad . \quad . \quad . \qquad (39)$$

Here H and $\int \frac{M_1'\,y_1}{E\,I_1}\,d\,x$ are to be determined as in previous cases. For no load in the side span, but for loads in the main or far side span, $\int M_1'\,y_1\,d\,x = 0$ and H is to be determined for the loading in question. Then eq. (39) becomes

$$D = +\frac{8}{15}\,H\,\frac{f_1^2\,l_1}{E\,I_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

The movement of point A due to temperature changes is found from eq. (40) by substituting H_t as found from eq. (27) of Art. 193, in place of H.

Eqs. 39 and 40 show that loads on the main and far side spans cause point A to move to the right, and that this motion is a maximum for maximum H, or for the main and far side spans completely loaded. Loads on the near side span cause point A to move to the left, and the greatest movement is caused for the side span fully loaded. For temperature changes, a rise in temperature causes point A to move to the left, and a fall causes motion to the right.

202. Example.—The use of the equations given in Arts. 198 to 201 will be illustrated by drawing influence lines for an actual structure and calculating values of horizontal component of cable stress, moment, and shear. The structure used for this purpose is the Manhattan Suspension Bridge in the city of New York.

a. Dimensions of Structure.—The general dimensions of the structure are shown in Fig. 27. The structure consists of four trusses, each suspended from a cable. The dimensions given are for each truss. The stiffening trusses are shown here to be horizontal, while in the actual structure they were built with a rising grade from each side toward the centre. The change was made to simplify some of the calculations relating to hanger and tower stresses.

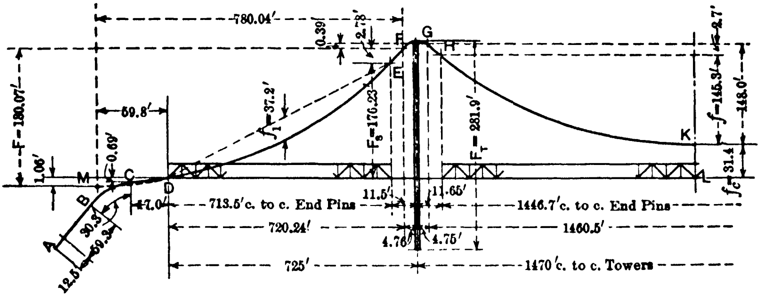


FIG. 27.

The live load for which the structure was designed was 16,000 lbs. per foot, or 4,000 lbs. per foot per truss. In the calculations to follow these same values will be used.

b. Values of H.—As the denominator of the formula for H is the same for all conditions of loading, it may be calculated separately and used in the work to follow as a coefficient in the various equations.

From eq. (37) we have

$$\text{Denominator} = \frac{8}{15} f^2 l + \frac{16}{15}\frac{I}{I_1} f_1^2 l_1 + \frac{I}{A} L_s \text{ Eq. (35).}$$

The values to be used are given in Fig. 27. The span and cable sag to be used are for centre to centre of end pins of the stiffening trusses. These values are

$$l = 1{,}446.7 \text{ ft.} \qquad f = 145.3 \text{ ft.} \qquad I = 43{,}900 \text{ ft.}^2 \times \text{ins.}^2$$

$$l_1 = 713.5 \text{ ft.} \qquad f_1 = 37.2 \text{ ft.} \qquad I_1 = 50{,}860 \text{ ft.}^2 \times \text{ins.}^2$$

$$A = 275 \text{ sq. ins. per cable} \qquad L_s = 3{,}452 \text{ ft.}$$

The value of L_s as given is calculated from eq. (35) in the following manner: The main-span portion of this quantity is to be taken up to point G, Fig. 27, where the cable enters the saddle. From the figure, $l = 1{,}460.5$ ft. and $f = 148$ ft. The side-span cable lies in a parabola whose vertex is at point M, 59.8 feet beyond D, the end of the stiffening truss. From the figure $l_1 = 780.04$ ft. and $F = 180.07$ ft. These values substituted in eq. (35) give for the main span and for each side span, respectively, 1,583.3 and 805.8 feet For the portions of the cable on the tower and anchorage saddles and the portions in the anchorage, the value of $\int \frac{d\,s^3}{d\,x^2}$ was taken as equal to the length of that portion of the cable. The totals are then

Main span	2 (G to K)		1,583.3
Each side span	(F to D)	805.8	
Saddle at tower	(F to G)	9.5	
Saddle at anchorage	(B to C)	30.3	
End of truss to saddle	(C to D)	17.0	
Saddle to anchorage and in anchorage	(A to B)	59.3 12.5	
		934.4	
		2	1,868.8
			3,452.1 ft.

Use $L_s = 3{,}452$ ft.

The various members of the denominator then have the following values:

$$\frac{8}{15} f^2 l = \frac{8}{15} \times 145.3^2 \times 1{,}446.7 = 16{,}290{,}000$$

$$\frac{16}{15} f_1^2 l \frac{I}{I_1} = \frac{16 \times 37.2^2 \times 713.5 \times 43{,}900}{15 \times 50{,}860} = 909{,}000$$

$$\frac{I}{A} L_s = \frac{43{,}900 \times 3{,}452}{275} = \frac{551{,}000}{17{,}750{,}000}$$

which is the value of "Denominator eq. (37)." The equations for H take the form

$$H = \frac{\int M\,y\,d\,x}{17{,}750{,}000}.$$

c. Influence Lines for Moment.—The influence lines are drawn by the methods explained in Art. 199 from the equation $\left(\frac{M'}{y} - H\right) y$. The curve for H is calculated from eq. (37), which, with temperature neglected and the denominator as calculated above substituted, becomes

$$H = \frac{\frac{1}{3} P f l^2 k (k^3 - 2 k^2 + 1)}{17{,}750{,}000}.$$

Substituting main-span dimensions for f and l, and taking $P = 1$ lb.

$$H = \frac{\frac{1}{3} \times 145.3 \times 1{,}446.7^2 \times k (k^3 - 2 k^2 + 1)}{17{,}750{,}000} = 5.711\, k (k^3 - 2 k^2 + 1).$$

The curve for H for loads on the side spans is the same in form and is given by the equation

$$H = \frac{\frac{1}{3} \times 37.2 \times 713.5^2 \times k_1 (k_1^3 - 2 k_1^2 + 1)}{17{,}750{,}000} = 0.307\, k_1 (k_1^3 - 2 k_1^2 + 1).$$

For values of k from 0 to 1, these expressions for H have the following values:

k	Main span H	Side span H
0	0	0
0.1	0.5602	0.0301
0.2	1.0599	0.0569
0.3	1.4511	0.0780
0.4	1.6996	0.0913
0.5	1.7848	0.0959

(Symmetrical about centre of span.)

These values are plotted in Fig. 28. From these curves we see that loads on the side spans have a very small effect on the cable stress.

The value of $\frac{M'}{y}$ was shown in Art. 191 to be a constant and to have a value of $\frac{l}{4f}$ for the main span. Substituting main-span values we have

$$\frac{M'}{y} = \frac{l}{4f} = \frac{1{,}446.7}{4 \times 145.3} = 2.489.$$

The corresponding side-span quantity is of the same form. Its value is

$$\frac{M_1'}{y_1} = \frac{l_1}{4\,f_1} = \frac{713.5}{4 \times 37.2} = 4.795.$$

These values are plotted in Fig. 28 and influence lines drawn for moment at the quarter point and at the centre of the main and side spans.

d. Moments from Influence Lines.—The influence line shows that for maximum positive moment at the quarter point the left end of the main span is to be covered with live load, which will be taken as 4,000 lbs. per lin. ft. The area of the portion of the influence lines between the curves for H and $\frac{M'}{y}$ was calculated by dividing the area into vertical strips each $\frac{1}{20}$ of the span in length. Then by scaling these ordinates the area was easily calculated. It was found to be 375.35 units, which represents the quantity $\left(\frac{M'}{y} - H\right)$ in the general equation. The moment is then $M = 4{,}000 \times (375.35) \times 109 = 163{,}650{,}000$ ft.-lbs. In this equation the term 109 is the cable ordinate, in feet, at the quarter point, as found from the equation $y = \frac{4\,f\,x}{l^2}\,(l - x)$ with $x = \frac{l}{4}$, $l = 1{,}446.7$ and $f = 145.3$. The term 4,000 is the load per foot in pounds.

The maximum negative moment at the quarter point is given by loading the right-hand portion of the main span and both of the side spans. The influence line shows that the effect of loads on the side spans is such as to cause negative moments at points in the main span, so that these spans must also be loaded to obtain maximum negative moments. The area of the main span portion of the influence line is found to be 226.7 and that of each side span 43.4. The moment is then given by the expression

$$M = -\ 4{,}000\ (226.7 + 2 \times 43.4)\ 109 = -\ 137{,}035{,}000 \text{ ft.-lbs.}$$

In the same way the maximum positive moment at the centre is found to be 107,800,000 ft.-lbs., and the negative moment is found to be 76,400,000 ft.-lbs.

These moments may also be calculated directly from the equation $M' - H\,y$. For positive moment at the quarter point we find by scale from the influence line that the load is to extend 675 feet from the left end of the span. The value of k is then $\frac{675}{1{,}446.7} = 0.466$. Substituting this value of k in eq. (37), neglecting the effect of temperature, we have

$$H = \frac{4{,}000 \times 145.3 \times 1446.7^3 \times 0.217 \times 4.117}{30 \times 17{,}750{,}000} = 2{,}954{,}000 \text{ lbs.}$$

which is the value of H for the given loading.

The value of M', the moment at the quarter point of a simple beam of span l for a load extending a distance $k\,l$ from the end of the beam, is given by the equation

$$M' = \frac{p\,(2 - k)\,k\,l\,x}{2} - \frac{p\,x^2}{2}.$$

Substituting $k = 0.466$, $x = \frac{l}{4}$ and $l = 1{,}446.7$, we find $M' = 486{,}600{,}000$ ft.-lbs. As before $y = 109$ ft. The expression for moment is then

$$M = M' - H\,y = 486{,}600{,}000 - 2{,}954{,}000 \times 109 = 164{,}800{,}000 \text{ ft.-lbs.}$$

By comparing this value with the one calculated above from influence-line areas we see that they agree within 0.7 per cent.

The variation in moments in the main span is shown by the moment curves of Fig. 28 which are drawn for maximum positive and negative moments.

In the side spans, the influence line shows that for maximum positive moments the side span in question should be fully loaded, no load on the rest of the structure. The positive moment is then given by an expression of the form,

$$(\text{load per ft.}) \times \left[\left(\text{Area curve for } \frac{M'}{y'}\right) - (\text{area of curve for } H)\right] \times \text{ordinate to cable curve.}$$

It can be seen here that the only variable is the cable ordinate. As this follows the parabolic law, the moment also follows this law. For the centre point the moment is

$$4{,}000\left[\frac{713.5 \times 4.795}{2} - 43.4\right]37.2 = +\,242{,}300{,}000 \text{ ft.-lbs.}$$

For maximum negative moments the influence line shows that the main span and the far side span only are to be loaded, no load on the side span in question. Then, in the above expression, the area of the curve for $\frac{M_1'}{y_1}$ is zero. The area of the curve for H for the main span is found to be 1,649.5. The expression for centre moment is then

$$\begin{aligned}-\,M_c &= -\,4{,}000\,(\text{Area } H \text{ curves for main and one side span})\,y\\ &= -\,4{,}000\,(1{,}649.5 + 43.4)\,37.2 = -\,251{,}900{,}000 \text{ ft. lbs.}\end{aligned}$$

The negative moments are seen to be a little greater than the positive moments.

e. Influence Lines for Shear.—The influence lines for shear are drawn from the equation $V = V' - H\tan\theta = (V'\cot\theta - H)\tan\theta$. From Art. 194, $\tan\theta = \frac{4f}{l}(1 - 2n)$, where $n\,l$ is the distance from the left end of the span to the point for which the influence line is to be drawn. For a 1-lb. load just to the left of the point in question, the ordinate to the line for $V'\cot\theta$ is given by the expression $-\frac{n\,l}{4f(1 - 2n)}$, and for the load just to the right of the point the value is $+\frac{(1 - n)\,l}{4f(1 - 2n)}$.

The influence lines for shear at the end of the main span and at the quarter

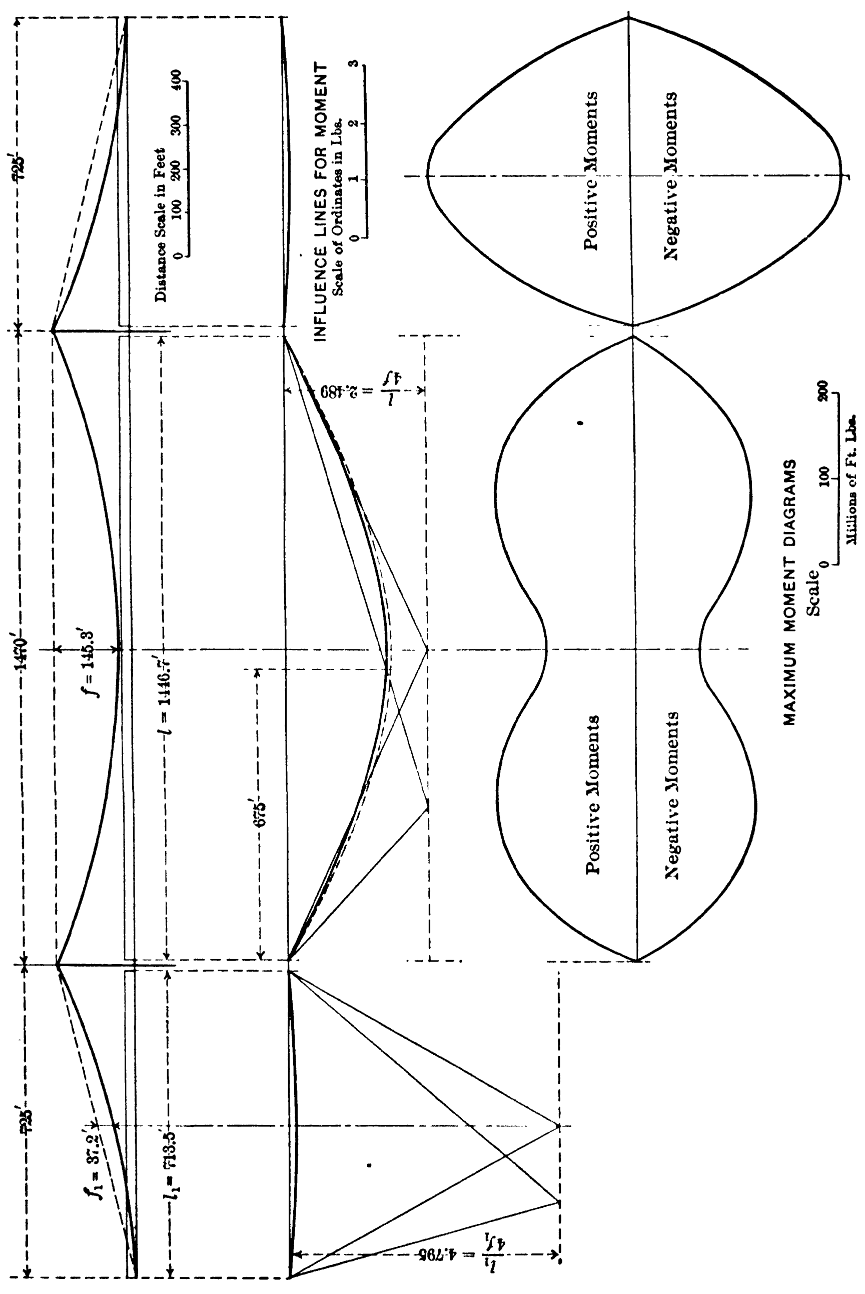

725'
1470'
725'
f = 145.3'
l = 1446.7'
f₁ = 37.2'
l₁ = 713.5
675'
Distance Scale in Feet
0 100 200 300 400
INFLUENCE LINES FOR MOMENT
Scale of Ordinates in Lbs.
0 1 2 3
l/4f = 2.489
l₁/4f₁ = 4.795
Positive Moments
Negative Moments
Positive Moments
Negative Moments
MAXIMUM MOMENT DIAGRAMS
Scale
0 100 200
Millions of Ft. Lbs.

FIG. 28.

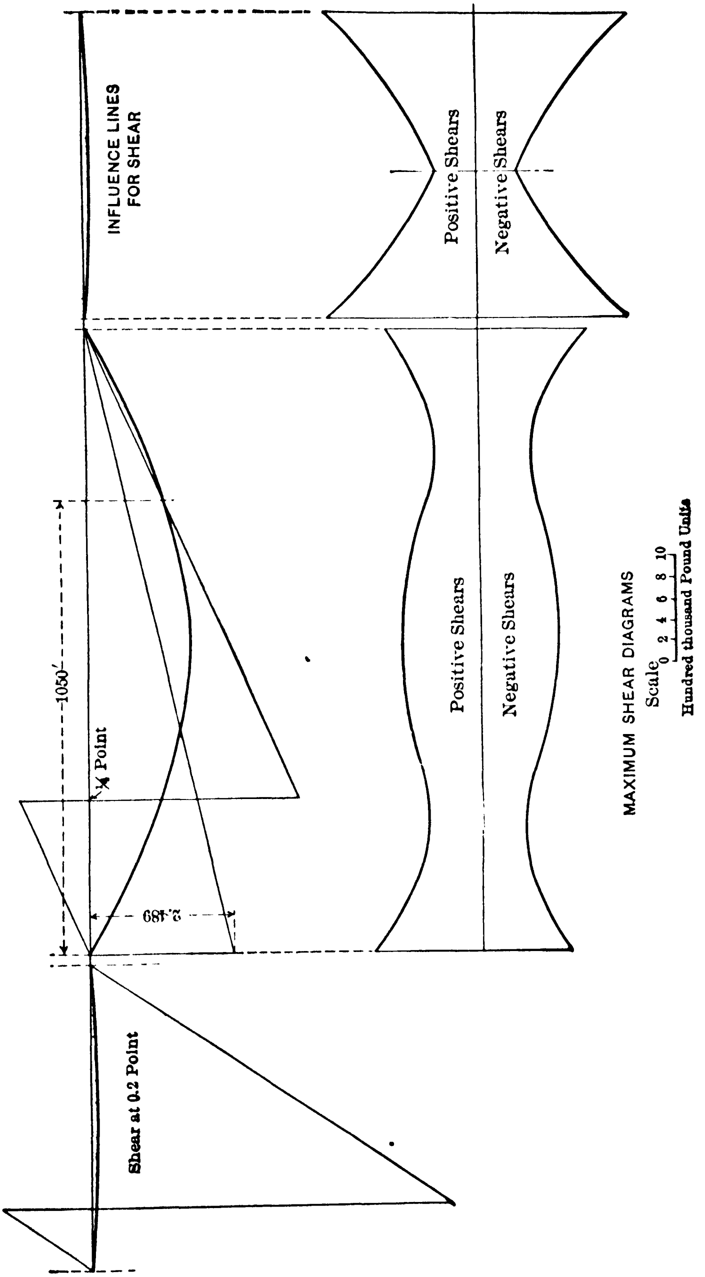

INFLUENCE LINES FOR SHEAR
Positive Shears
Negative Shears
1050'
¼ Point
2.489
Shear at 0.2 Point
Positive Shears
Negative Shears
MAXIMUM SHEAR DIAGRAMS
Scale 0 2 4 6 8 10
Hundred thousand Pound Units

FIG. 29.

point are shown in Fig. 29. The ordinates to the influence line at the quarter point are found from the above values to be

$$+\frac{(1-n)\,l}{4f\,(1-2n)} = +3.734 - \frac{n\,l}{4f\,(1-2n)} = -1.245.$$

f. Shears from Influence Lines.—From the influence line for shear at the quarter point the loading for maximum positive shear is found to extend from the quarter point to a point 1,050 feet from the left end of the main span. Proceeding as in the case of moments, the area between the lines for V' cot θ and H is found to be 635.2 units. The shear is then

$$+V = 4{,}000 \times 635.2 \times \tan\theta = +510{,}350 \text{ lbs.}$$

where $\tan\theta = \frac{4f}{l}(1-2n)$. Here $f = 145.3$ $l = 1446.7$ $n = \frac{1}{4}$.

The maximum negative shear at the quarter point is found for loading conditions which are opposite to those for positive shear; that is, all the structure is loaded except that part which was loaded for positive shear. The main-span influence-line area is found to be 489.36 and the area for both side spans is 86.80. The value of tan θ is the same as before.

Then

$$-V = 4{,}000\,(489.36 + 86.80)\tan\theta = -462{,}940 \text{ lbs.}$$

For the influence line for end shear, the value of n in the expressions for ordinates as given above is zero.

Then

$$\frac{(1-n)\,l}{4f\,(1-2n)} = \frac{l}{4f} = 2.489.$$

The influence line shows that for maximum positive shear, the main span is to be loaded from the left end out to a point 512.5 ft. from the end. For this loading the shear is found to be + 1,010,000 lbs.

For maximum negative shear, opposite loading conditions occur for which the shear is found to be 845,500 lbs.

The curves given in Fig. 29 show the variation in positive and negative main-span shears.

The shears for points in the side spans are determined in exactly the same way.

g. Deflection.—From eq. (32) we find the centre deflection to be

$$d = \frac{5}{384}\,\frac{l^4}{EI}\left(p - \frac{8fH}{l^2}\right).$$

The greatest downward deflection in the stiffening truss occurs when the main span is fully loaded, no load on the side span. The value of H for this loading is found from eq. (37) to be 6,606,000 lbs. Then in the above equation $\frac{8fH}{l^2} = 3{,}664.5$ lbs. This quantity is the hanger pull per foot, and therefore shows the amount of the live load which is taken up by the cable when the truss is fully loaded, the proportion being $\frac{3{,}664.5}{4{,}000} = 91.6$ per cent. The stiffening truss takes $4{,}000 - 3{,}664.5 = 335.5$ lbs. or 8.4 per cent of the live load. Substituting these values in the above equation

$$\eta = \frac{5 \times (1{,}446.7)^4 \times 335.5}{384 \times 29{,}000{,}000 \times 43{,}900} = 15.03 \text{ feet, downward.}$$

The greatest upward deflection of the centre point occurs when the side spans only are loaded, no load on the centre span. From eq. (37) the value of H is 347,300 lbs., and $\frac{8fH}{l^2} = 192.9$ lbs. per ft. In this case, p in the equation for deflection is zero, as there is no load in the main span. The deflection is found to be 8.64 ft. upward.

h. Effect of Temperature.—In the expression for temperature, the term L_t is to be calculated from eq. (36). Using the same general dimensions as for L_s we have for the various parts of the cable

Main span	(G to K)		1,540.5
Each side span	(F to D)	775.6	
Saddle on tower	(F to G)	9.5	
Saddle at anchorage	(B to C)	30.3	
End of truss to saddle	(C to D)	17.0	
Saddle to anchorage	(H to B)	59.3	
		891.7	
		2	1,783.4
			3,323.9 ft.

Use $L_t = 3{,}324$ ft.

For the portions of the cable on the saddle and towers the value of $\int \frac{ds^2}{dx}$ is taken as equal to the length of that portion of the cable. The portion of the cable in the anchorage is not considered here, as it is protected by the heavy masses of masonry and is not much affected by temperature changes.

From eq. (37), Art. 198, neglecting the terms for loads and considering temperature effect alone, we have

$$H_t = -\frac{E\,I\,\omega\,t\,L_t \text{ eq. (36)}}{\text{Denominator eq. (37)}}$$

which gives the effect of temperature on the value of H. For $\omega = 0.0000065$ and $t = 55°$ the value of H_t is

$$H_t = \pm\frac{29{,}000{,}000 \times 43{,}900 \times 0.0000065 \times 55 \times 3{,}324}{17{,}750{,}000} = \pm 85{,}230 \text{ lbs.}$$

This value is negative for a rise and positive for a fall in temperature.

The effect of the change in temperature on the moments is given by eq. (26) of Art. 193, as

$$M_t = \pm H_t y = \pm 85{,}230 \times y.$$

For moment at the quarter-point, we have

$$M_t = \pm 85{,}230 \times 109 = \pm 9{,}290{,}000 \text{ ft.-lbs.}$$

For a rise in temperature M_t is positive.

Then the total positive moment at the quarter point, live load and temperature combined, is 163,650,000 + 9,290,000 = 172,940,000 ft.-lbs. For a fall in temperature M_t is negative and the total negative moment is − 137,035,000 − 9,290,000 = 146,325,000 ft.-lbs.

The effect of temperature on shear is given by the expression

$$V_t = \pm H_t \tan \theta.$$

For shear at the quarter point this becomes

$$V_t = + 85{,}230 \times 0.200875 = + 17{,}120 \text{ lbs.}$$

The shears then become

$$+ V_{\frac{1}{4}} = + 510{,}350 + 17{,}120 = + 527{,}470$$
$$- V_{\frac{1}{4}} = - 462{,}940 - 17{,}120 = - 480{,}060.$$

The effect of temperature on the deflection is given by eq. (33) of Art. 197 as

$$\eta_t = \frac{\frac{5}{48} f\, l^2\, \omega\, t\, L_t}{\text{Denominator eq. (37)}} = \frac{5 \times 145.3 \times (1446.7)^2 \times 0.0000065 \times 55 \times 3{,}324}{48 \times 17{,}750{,}000}$$
$$= 2.12 \text{ ft.}$$

which is downward for a rise and upward for a fall in temperature.

(B) EXACT METHODS OF CALCULATION*

(*a*) *Structures with Trusses Hinged or Continuous at Towers*

203. Method of Procedure.—In structures in which the span is long and the trusses comparatively flexible, the effect of deflection under loads must be taken into account. In the general formula of Art. 187, the moment at any point in the stiffening truss was given as $M = M' - H y$. In this formula the deflection was neglected. If taken into account, the equation would read $M = M' - H(y + \eta)$, where η is the deflection at the point in question. From this equation it can be seen that the effect of this deflection is to reduce the amount of the moment, thus effecting a saving in material in the stiffening trusses. In the case of long spans this saving is considerable, as, for example, in the problem of Art. 202, the deflection was found to be almost 10 per cent of the centre sag. In the analysis to follow, values of moment, shear, and horizontal component of cable stress will be determined, taking into account the effect of deflection.

---

* Based on the work of Melan, "Eiserne Bogenbrücken und Haengebrücken," Leipsic, 1888 and 1906. This method was fully developed by Mr. L. S. Moisseiff of the Department of Bridges of the City of New York, and used in the calculations of the Manhattan bridge, and again by the author, with various modifications, in the recalculations of this structure for the report of Mr. Ralph Modjeska to the City of New York. See also Max am Ende, in Proceedings Institution of Civil Engineers, 1898.

In deriving the equations the same general assumptions will be made as before, that is:

The initial curve of the cable will be assumed as parabolic.

The entire dead load will be assumed as carried by the cable, causing no stress in the stiffening trusses. The truss is stressed by live load, and by changes in temperature from the normal.

The notation used in the work to follow will, in general, be the same as in the preceding articles. For convenience this notation is repeated below:

H_w = horizontal component of cable stress due to dead load and mean temperature.

H = additional horizontal component of cable stress due to any cause, such as live load or temperature change.

η = deflection of truss and cable at any point from the normal position, due to H and any given live load (the effect of the stretch of the hangers is neglected.)

p = live load per unit length on part or all of the main or side span trusses.

w = dead load per unit length = load on cable, including its own weight.

M' = bending moment at any point in the truss due to the given live load, assuming the truss to be a simply supported beam.

M = actual bending in the truss at any point due to assumed conditions.

V = actual shear at any point.

I = moment of inertia of stiffening truss.

l = length of span.

f = centre sag of cable.

Subscripts will be used to indicate corresponding side span dimensions.

The general form of the structure to be considered will be as shown in Fig. 25, of Art. 198.

The main and side spans are suspended from the cable by hangers.

The trusses may or may not be continuous over the towers. However, the formulas will be developed on the assumption of non-continuous trusses. Continuity of trusses will modify only the values of the constants of Art. 205.

204. Derivation of Formulas for Deflection, Bending Moment, and Shear.—Under all conditions of loading the shape of the cable constitutes an equilibrium polygon for the hanger loads. If the horizontal component of the cable stress is $H_w + H$, the moment of the hanger stresses on the truss at any point A', Fig. 30, is therefore equal to $(H_w + H)\ (y + \eta)$. But under the assumed conditions the portion $H_w y$ just balances the dead load; therefore the moment tending to balance the effect of the live load, or temperature change, is $(H + H_w)\ \eta + H\ y$. Hence the resultant moment in the stiffening truss due to this live load, or temperature change, is

$$M = M' - (H + H_w)\ \eta - H\ y. \quad (1)$$

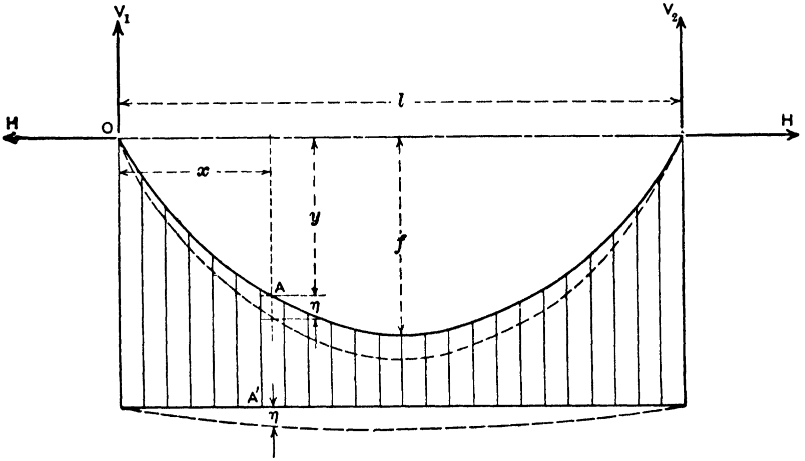


Fig. 30.

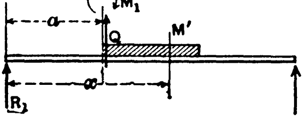


Fig. 31.

Consider any portion of the truss along which p and I may be assumed as constant, and suppose that the left end of this portion is at a distance a from the origin, Fig. 31. The value of M' at any point in this section can be written

$$M' = M_1 + Q\,(x - a) - \frac{p}{2}\,(x - a)^2,$$

in which M_1 is the bending moment and Q the shear at the left end of this section. Also, from the initial parabola, origin at O, Fig. 30, $y = \frac{4fx}{l^2}\,(l - x)$. Substituting these values in eq. (1), we have

$$M = M_1 + Q\,(x-a) - \frac{p}{2}(x-a)^2 - (H + H_w)\eta - H\left[\frac{4fx}{l^2}(l-x)\right] \quad . \quad (2)$$

With respect to the truss, the general relation of deflection to bending moment is

$$EI\,\frac{d^2\eta}{dx^2} = -M.$$

Substituting the value of M from eq. (2) in this expression, placing $\frac{H + H_w}{EI} = c^2$, we have

$$\frac{d^2\eta}{dx^2} = c^2\eta - \frac{c^2}{(H + H_w)}\left[M_1 + Q\,(x - a) - \frac{p}{2}(x - a)^2\right] + \frac{Hc^2}{(H + H_w)}\left[\frac{4fx}{l^2}(l-x)\right] \quad (3)$$

Integrating eq. (3), again substituting M' for brevity, we have*

$$\eta = \frac{H}{(H + H_w)}\left[C_1 e^{cx} + C_2 e^{-cx} + \frac{M'}{H} + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right) - \frac{4fx}{l^2}\,(l-x)\right] \quad (4)$$

which is the deflection at any point in the stiffening truss. C_1 and C_2 are integration constants to be determined from the conditions of the problem (Art. 205). Then from eq. (1), the bending moment at any point is

$$M = -H\left[C_1 e^{cx} + C_2 e^{-cx} + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right)\right]. \quad . \quad (5)$$

The shear at any point can be obtained by differentiating eq. (5), which gives

$$V = -Hc\,[C_1 e^{cx} - C_2 e^{-cx}]. \quad . \quad . \quad . \quad . \quad (6)$$

---

*Equation (3) may be written in the form $\frac{d^2\eta}{dx^2} = c^2\eta + c^2(a + bx + dx^2) = c^2\eta + c^2 F(x)$. The integral of this is $\eta = A_1 e^{cx} + A_2 e^{-cx} - F(x) - \frac{1}{c^2}F''(x)$, in which A_1 and A_2 are integration constants and $F''(x)$ is the second derivative of $F(x)$. From this is written eq. (4), substituting $\frac{C_1 H}{H + H_w}$ and $\frac{C_2 H}{H + H_w}$ for A_1 and A_2.

Eqs. (4), (5), and (6) enable the deflections, moments, and shears to be calculated at any point in the main span for any given value of cable stress H and of load p on this span. The value of H to be used in these equations is given in Art. 206. For cases in which the deformation of the cable is known to be slight, the value of H as given by the Approximate Method, Art. 188, may be used.

From eqs. (4), (5), and (6), it can be seen that the deflection, moment, and shear at any point is no longer proportional to the load p. This is due to the fact that the horizontal component of the dead load cable stress, H_w, which existed before the advent of the live load, is involved in the quantity c, whose value is given by $c^2 = \frac{H + H_w}{EI}$. For this reason influence lines cannot be drawn for the above equations. In any case, however, the influence lines as drawn by the Approximate Method can be used as a guide in determining maximum values. Then by several trials with about the same loading as required by the Approximate Method, the maximum is easily found.

The formulas for deflection, moment, and shear are applicable to side spans by using the span length l_1, and the centre sag f_1, for these spans in place of the corresponding main span values l and f. The value of c is also to be used in terms of side-span quantities. This value is then $c^2{}_1 = \frac{H + H_w}{EI_1}$.

Eqs. (4), (5), and (6) may also be applied to structures in which the trusses are continuous over the towers. This may be done by making proper provision for this continuity in the term M' of eq. (4).

In the previous work it has been assumed that the hanger pull due to live load is uniform over the entire length of the cable for the span in question. The exact amount of live load taken by the stiffening truss in any case may now be found by taking the second derivative of the moment for the given loading; that is, the load per unit length $= -\frac{d^2 M}{d x^2}$. From eq. (5), the load taken by the stiffening truss is, therefore,

$$-\frac{d^2 M}{d x^2} = c^2 H \left[C_1 e^{-cx} + C_2 e^{-cx}\right]. \quad . \quad . \quad . \quad (7)$$

If p is the live load per unit length, the amount of load taken by the hangers is

$$\text{Hanger Pull} = s_1 = p - c^2 H \left[C_1 e^{cx} + C_2 e^{-cx}\right]. \quad . \quad (8)$$

Eq. (8) is the general formula for hanger pull. For any particular condition of loading, the constants C_1 and C_2 must be determined for that case.

205. Constants of Integration.—The quantities C_1 and C_2 in eqs. (4), (5), (6), and (8) are constants of integration which must be determined for each different case of loading. Thus for each different value of p or I, there is a corresponding set of constants C_1 and C_2.

For a uniform load p per unit length throughout, and constant I, C_1 and C_2 are obtained from the two conditions that for $x = 0$ and $x = l$ in eq. (5), $M = 0$. Substituting these values, we have

$$C_1 + C_2 + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right) = 0.$$

and

$$C_1 e^{cl} + C_2 e^{-cl} + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right) = 0.$$

We have here two independent equations involving the unknowns C_1 and C_2, from which we find

$$C_1 = - \frac{1}{c^2 (e^{cl} + 1)}\left(\frac{8f}{l^2} - \frac{p}{H}\right) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$C_2 = C_1 e^{cl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which are the constants of integration for the main span fully loaded.

For a uniform load of p per unit of length extending a distance $k\,l$ from the left end of the span, as shown in Fig. 32, the constants of integration for the loaded section $A\,B$ and for the unloaded section $B\,C$, may be determined at the same time. The constants for the loaded section will be denoted by C_1 and C_2; those for the unloaded portion by C_3 and C_4. The additional equations now required for the four constants are obtained from the condition that the moments and shears at the right end of section $A\,B$ of Fig. 32 are equal to those at the left end of section $B\,C$. The values of these constants will now be determined.

From eqs. (5) and (6) the moments and shears for the various sections are given by the following expressions:

$$M_{AB} = -H\left[C_1 e^{cx} + C_2 e^{-cx} + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right)\right] \quad . \quad (A)$$

$$M_{BC} = -H\left[C_3 e^{cx} + C_4 e^{-cx} + \frac{8f}{c^2 l^2}\right] \quad . \ . \ . \ . \quad (B)$$

$$V_{AB} = -Hc\,[C_1 e^{cx} - C_2 e^{-cx}] \quad . \ . \ . \ . \ . \ . \ . \quad (C)$$

$$V_{BC} = -Hc\,[C_3 e^{cx} - C_4 e^{-cx}] \quad . \ . \ . \ . \ . \ . \ . \quad (D)$$

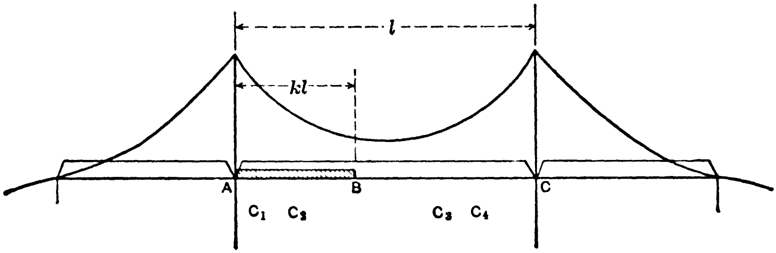


FIG. 32.

The conditions from which the independent equations are to be made up. are as follows:

For $x = 0$, $M = 0$ in eq. (A)
$x = l$, $M = 0$ in eq. (B)
$x = kl$, eq. (A) = eq. (B)
$x = kl$, eq. (C) = eq. (D).

Substituting these conditions in eqs. (A) to (D), the independent equations are found to be

$$C_1 + C_2 + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right) = 0$$

$$C_3 e^{cl} + C_4 e^{-cl} + \frac{8f}{c^2 l^2} = 0$$

$$C_1 e^{kcl} + C_2 e^{-kcl} + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right) = C_3 e^{kcl} + C_4 e^{-kcl} + \frac{8f}{c^2 l^2}$$

$$C_1 e^{kcl} - C_2 e^{-kcl} = C_3 e^{kcl} - C_4 e^{-kcl}$$

From these equations, the values of the various constants are given by

$$C_1 = \frac{\frac{p}{2Hc^2}\left[e^{cl(l-k)} + e^{-cl(l-k)} - 2e^{-cl}\right] - \frac{8f}{c^2 l^2}\left(1 - e^{-cl}\right)}{\left(e^{cl} - e^{-cl}\right)} \quad . \quad (11)$$

$$C_2 = -C_1 - \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

$$C_3 = C_1 - \frac{p}{2Hc^2}e^{-kcl} = \frac{\frac{p}{2Hc^2}\left[e^{-cl}\left(e^{kcl} + e^{-kcl} - 2\right)\right] - \frac{8f}{c^2 l^2}\left(1 - e^{-cl}\right)}{\left(e^{cl} - e^{-cl}\right)} \quad (13)$$

$$C_4 = -C_3 e^{2cl} - \frac{8f}{c^2 l^2}e^{cl}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

When eqs. (4), (5), and (6) are used to calculate deflection, moment, or shear, the proper values of C_1 and C_2 are to be substituted as determined for the loading in question. Thus for the loading shown in Fig. 27, the deflection, moment, or shear at any point in section $A\,B$ is found by substituting values of C_1 and C_2 as given by eqs. (11) and (12). For corresponding values in the unloaded portion $B\,C$, the values of the constants are given by C_3 and C_4 of eqs. (13) and (14), C_3 and C_4 replacing C_1 and C_2 respectively. For this section, the value of p in eqs. (5) and (6) is to be taken as zero, as no live load exists on this section.

For other conditions of loading, such as are apt to arise in the calculation of stresses in any structure, the values of the constants of integration are found by a process similar to that given above. Values of these constants for various cases have been worked out and are given on Plates I to IV, Art. 207.

The value of the constants for side-span conditions are exactly similar to those for corresponding mid-span conditions.

Trusses Continuous over Towers.—For this case the constants of integration are determined by equating the moments at the towers for main span with those for side span, the deflections at the towers equal to zero, and equal slopes for the tangents to the elastic curves on each side of the towers.

206. Formula for H.—In most cases it is desirable to use a formula for horizontal component of cable stress which will give results more exact than those given by the Approximate Method. For this purpose a formula for H will be derived in which the deflection of the truss and cable are taken into account.

The cable stress, H, may be caused by a load on either main or side spans, or by temperature change. In deriving an expression for H the entire cable, anchorage to anchorage, must be considered. Imagine the hangers to be cut just below the cable and their stresses replaced by external forces. The total load per unit of length acting on the cable after advent of live load, or temperature change, will be equal to s_1. The application of this live load, or temperature change, will cause the cable to stretch, and will also cause vertical displacements along the cable, some downward and some upward. The total work performed in this vertical movement must be equal to the total work done in stretching the cable, which may be divided into two parts, one part due to changes in the stress in the cable, and the other part due to temperature changes.

Work in the Cable Due to Stress.—Let A be the constant area of the cable, and $d\,s$ an increment of length. The stress at any point in the cable due to H is then $H\frac{d\,s}{d\,x}$. The stretch of the element is $\frac{H}{A\,E}\frac{d\,s}{d\,x}d\,s$. The average total stress in the cable at the same point during the application of the load is $\left(\frac{H}{2}+H_w\right)\frac{d\,s}{d\,x}$, and hence the work done on this element is $\left(\frac{H}{2}+H_w\right)\frac{H}{A\,E}\frac{d\,s^3}{d\,x^2}$. The total work on the cable may then be written

$$\left(\frac{H}{2}+H_w\right)\frac{H}{A\,E}\int_0^l\frac{d\,s^3}{d\,x^2}. \qquad . \quad . \quad . \quad . \quad . \qquad (A)$$

In this equation the expression $\int_0^l\frac{d\,s^3}{d\,x^2}$ is the same as already worked out in Art. 188 and referred to as L_s. This notation will again be used.

Work in the Cable Due to Temperature Change.—Let ω be the coefficient of expansion, and t the change in temperature. The change in length of any element $d\,s$ is $\omega\,t\,d\,s$, and the average stress is again $\left(\frac{H}{2} + H_w\right)\frac{d\,s}{d\,x}$, hence the total work in the whole cable is

$$\left(\frac{H}{2} + H_w\right)\omega\,t\int_0^l \frac{d\,s^2}{d\,x}. \quad . \quad . \quad . \quad . \quad . \quad (B)$$

The expression $\int_0^l \frac{d\,s^2}{d\,x}$ is again the same as given in Art. 189, and there called L_t.

Total Work in the Cable.—From eqs. (A) and (B) the total work in the cable is given by

$$\left(\frac{H}{2} + H_w\right)\frac{H\,L_s}{A\,E} + \left(\frac{H}{2} + H_w\right)\omega\,t\,L_t. \quad . \quad . \quad (15)$$

Work Done by Vertical Displacement.—The load per unit length of the cable is s_1 and the movement is η, as given by eq. (4). Hence the work done is $\int s_1\,\eta\,d\,x$. The dead load is uniformly distributed along the cable, and for the purpose of calculating $\int s_1\,\eta\,d\,x$, the additional load may also be assumed as uniformly distributed. Then s_1 may be found from the fact that the load per unit length is equal to the second derivative of the moment caused by the load. In this case the moment in terms of H caused by the load on the cable (neglecting the increment of deflection due to the additional load) will be $(H + H_w)\,y$, where $y = \frac{4\,f\,x}{l^2}(l - x)$. Then we have

$$M = (H + H_w)\,\frac{4\,f\,x}{l^2}(l - x),$$

and

$$s_1 = -\frac{d^2\,M}{d\,x^2} = +(H + H_w)\,\frac{8\,f}{l^2} \text{ (approximately).}$$

The average load on the cable will then be $\left(\frac{H}{2} + H_w\right)\frac{8\,f}{l^2}$. The work in the cable of the main span is therefore given by the expression

$$\frac{8\,f}{l^2}\left(\frac{H}{2} + H_w\right)\int_0^l \eta\,d\,x. \quad . \quad . \quad . \quad . \quad . \quad (C)$$

A similar expression must be derived for the side spans. This expression will be of the same form as that for the main span, given in terms of side-span dimensions. It is therefore

$$\frac{8 f_1}{l_1^2}\left(\frac{H}{2} + H_w\right)\int_0^{l_1} \eta_1 \, d\, x. \quad . \quad . \quad . \quad . \qquad (D)$$

The total work in the main span, and both side spans is then

$$\frac{8 f}{l^2}\left(\frac{H}{2} + H_w\right)\left[\int_0^{l} \eta \, d\, x + \frac{2 f_1 l^2}{f l_1^2}\int_0^{l_1} \eta_1 \, d\, x\right]. \quad . \qquad (16)$$

Expression for H.—Equating the expression for work in the cable, as given by (15), with that of vertical displacement, as given by (16), placing $\frac{f_1 l^2}{f l_1^2} = K$, we have

$$\left(\frac{H}{2} + H_w\right)\frac{H\, L_s}{A\, E} + \left(\frac{H}{2} + H_w\right)\omega\, t\, L_t = \frac{8 f}{l^2}\left(\frac{H}{2} + H_w\right)\left[\int_0^{l} \eta\, d\, x + 2\, K\int_0^{l_1} \eta_1\, d\, x\right] (17)$$

Substituting the value of η from eq. (4), integrating known values of functions containing x, and solving for H, we derive the general expression

$$H = \frac{\int_0^l \left(M' - \frac{p}{c^2}\right) d\, x + K\begin{bmatrix}\text{Term for}\\ \text{Side Spans}\end{bmatrix} - \frac{l^2\, \omega\, t}{8\, f} H_w\, L_t}{-\int_0^l (C_1 e^{c\, x} + C_2 e^{-c\, x})\, dx - \frac{8f}{c^2 l^2} + \frac{2}{3} f l + 2K\begin{bmatrix}\text{Terms for}\\ \text{Side Spans}\end{bmatrix} + \frac{c^2 I l^2}{8 f A} L_s + \frac{l^2 \omega t}{8 f} L_t} \qquad (18)$$

which is the horizontal component of the cable stress for a general case.

The terms in brackets in eq. (18) indicate that terms for each side span are to be written out exactly similar to the preceding terms for the main span. If there are no loads in any span, then the values of M' and p for that span are zero. It is to be noted that in eq. (18), the quantities C_1 and C_2, and also c, contain the quantity H so that the value of H can be obtained only by successive approximations.

Simplified Formula for H.—The expression for H given in eq. (18) can be simplified somewhat, for any particular form of loading, as the constants C_1 and C_2, and also any similar constants which may be involved, can be expressed in detail and some of the terms containing H transferred to the first member of the equation. Then by solving this equation for H, an expression somewhat more convenient of application that eq. (18) can be obtained. This simplified formula for H will now be worked out for the condition of loading shown in Fig. 50.

The terms in eq. (18) which depend upon the conditions of loading for their value are $\int_0^l \left(M' - \frac{p}{c^2}\right) d\,x$, $-\int_0^l (C_1 e^{cx} + C_2 e^{-cx})\,d\,x$, and the corresponding side-span terms.

For the loading shown in Fig. 32. the term $\int_0^l \left(M' - \frac{p}{c^2}\right) d\,x$ takes the form

$$\int_0^l \left(M' - \frac{p}{c^2}\right) d\,x = \int_0^{kl} \left[\frac{p\,k\,l}{2}(2 - k)\,x - \frac{p\,x^2}{2} - \frac{p}{c^2}\right] d\,x$$

$$+ \int_{kl}^{l} \left[\frac{p\,k^2\,l}{2}(l - x)\right] d\,x = p\,k\,l \left[\frac{k\,l^2}{12}(3 - 2\,k) - \frac{1}{c^2}\right]. \quad . \quad (19)$$

Since, for the assumed loading, the side spans have no load, the corresponding side span terms are zero.

The term $-\int_0^l (C_1 e^{cx} + C_2 e^{-cx})\,d\,x$ is to be made up for the main and side spans. For the main span, the values of these constants are given in eqs. (11) to (14) of Art. 205. Performing the required integrations, substituting the values of the constants, and reducing, we have

$$-\int_0^l (C_1 e^{cx} + C_2 e^{-cx})\,d\,x = -\int_0^{kl} (C_1 e^{cx} + C_2 e^{-cx})\,d\,x$$

$$-\int_{kl}^{l} (C_3 e^{cx} + C_4 e^{-cx})\,d\,x = \frac{1}{c}\Bigg\{C_1\,(2 - e^{cl} - e^{-cl})$$

$$+ \frac{p}{2\,H\,c^2}\left[e^{cl(1-k)} + e^{-cl(1-k)} - 2 + 2\,e^{-cl}\right] + \frac{8\,f}{c^2\,l^2}(1 - e^{-cl})\Bigg\}$$

$$= \frac{1}{c\,(e^{cl} - e^{-cl})}\Bigg\{\frac{p}{H\,c^2}\left[e^{cl(1-k)} + e^{-cl(1-k)} - e^{cl} - e^{-cl} - e^{kcl} - e^{-kcl} + 2\right]$$

$$+ \frac{16\,f}{c^2\,l^2}(e^{cl} - 2 + e^{-cl})\Bigg\}. \quad . \quad . \quad (20)$$

The corresponding side-span terms will be of the form

$$-\,2\,K \int_0^{l_1} (B_1\,e^{c_1 x} + B_2\,e^{-c_1 x})\,d\,x,$$

where B_1 and B_2 are the side-span constants. The values of these constants will be similar to those for the main span given in eqs. (9)

and (10), except that p is zero as there is no load on these spans. Proceeding as for the main span, we have

$$-2K\int_0^{l_1}(B_1e^{c_1x}+B_2e^{-c_1x})\,dx=-\frac{4KB_1}{c_1}(e^{c_1l_1}-1)$$

$$=+\frac{32K}{c_1^3l_1^2}\frac{(e^{c_1l_1}-1)}{(e^{c_1l_1}+1)}. \quad \ldots\ldots \quad (21)$$

Substituting the values of the expressions given in eqs. (19), (20), and (21) in eq. (18), and solving for H, we have finally,

$$H=\frac{pkl\left[\frac{kl^2}{12}(3-2k)-\frac{1}{c^2}\right]-\frac{p}{c^3(e^{cl}-e^{-cl})}\left[e^{cl(1-k)}+e^{-cl(1-k)}-e^{cl}-e^{-cl}-e^{kcl}-e^{-kcl}+2\right]-\frac{l^2\omega t}{8f}H_wL_t}{\frac{16f}{c^3l^2}\frac{(e^{cl}-1)}{(e^{cl}+1)}-\frac{8f}{c^2l}+\frac{2}{3}fl+K\left[\frac{32f_1}{c_1^3l_1^2}\frac{(e^{c_1l_1}-1)}{(e^{c_1l_1}+1)}-\frac{16f_1}{c_1^2l_1}+\frac{4}{3}f_1l_1\right]+\frac{c^2Il^2}{8fA}L_s+\frac{l^2\omega t}{8f}L_t} \quad (22)$$

which is the simplified form of the expression for the horizontal component of cable stress for the loading conditions shown in Fig. 32,

For any other condition of loading, the value of H can be found by a similar process. On Plates I to IV are given such values for various conditions of loading. In these expressions the denominator is abbreviated by the letter "D," with a subscript $+t$ or $-t$, denoting a rise or fall in temperature.

Value of the Denominator of eq. (22).—It will be found on comparison of the various expressions for H, that one result of the above transformation has been to make the denominator of all such expressions identical in form. All changes due to the various conditions of loading now appear in the numerator of the expression for H. The denominator of eq. (22) may then be determined separately, and hereafter used as a coefficient in the various expressions. Its value is

$$D=\frac{16f}{c^3l^2}\frac{(e^{cl}-1)}{(e^{cl}+1)}-\frac{8f}{c^2l}+\frac{2}{3}fl+K\left[\frac{32f_1}{c_1^3l_1^2}\frac{(e^{c_1l_1}-1)}{(e^{c_1l_1}+1)}-\frac{16f_1}{c_1^2l_1}+\frac{4}{3}f_1l_1\right]+\frac{c^2Il^2}{8fA}L_s\pm\frac{l^2\omega t}{8f}L_t. \quad (23)$$

The temperature term, $\frac{l^2\omega t}{8f}L_t$, will be positive for a rise in temperature and negative for a fall. Hereafter, eq. (23) will be referred to as D_{+t} for temperature conditions above the normal, and D_{-t}

for those below the normal. It is to be noted that D_{+t} and D_{-t} are dependent upon H for their values, because of the quantities c and c_1 which appear therein. To facilitate calculation by means of the modified formulas, the denominator as given by eq. (23) may be calculated, and tabulated or plotted for future reference. Such curves of values are given in Fig. 33.

207. Conditions of Loading for Maximum Moments and Shears. Tables of Values.—The effect of the loading conditions on the value of the moment at any point can be determined by a study of eq. (1) of Art. 204.

For positive moments at any point, eq. (1) shows that the value of M' is to be as large as possible, while H is to be small at the same time. In general it will be found that this condition is realized when the centre of moments and the adjacent portions of the span in question are covered with live load, no load on the rest of the structure. The temperature conditions existing at the same time must be such as to reduce the value of H. From eq. (22) we see that a rise in temperature is necessary to realize this condition.

For negative moments at any point, the value of M' is to be as small as possible, while H is to be large at the same time. In general this condition is realized when there is no load on the portion of the structure adjacent to the moment centre, conditions opposite to those for positive moments. The temperature conditions must be such as to increase the value of H. Eq. (22) shows that a fall in temperature is necessary.

The shear at any point is given by the equation

$$\frac{dM}{dx} = V = V' - H \tan \theta$$

where the notation is the same as given in Art. 190.

From this equation we find, by a process similar to that given for moments, that similar loading conditions hold for shears as for moments.

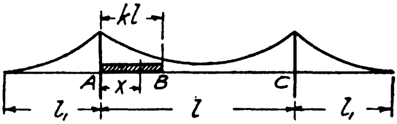
The loading conditions found to exist for the Approximate Method can be used as a guide for corresponding results by the Exact Method. By determining the loading conditions from influence lines as given in Arts. 191 and 194 and then applying exact methods, it is usually possible after one or two trials to arrive at the correct conditions

PLATE I

MAXIMUM POSITIVE MOMENTS AND SHEARS, CENTER SPAN

CASE I.

Maximum Positive Moments; End to near Center.
Maximum Positive Shears; End, and near Center



Conditions of Loading

Main Span;	Partially Loaded
Side Spans;	No Load
Temperature;	Highest.

$$H=\frac{pkl\left[\frac{kl^2}{12}(3-2k)-\frac{1}{c^2}\right]-\frac{p}{c^3(e^{cl}-e^{-cl})}\left[e^{cl(1-k)}+e^{-cl(1-k)}-e^{cl}-e^{-cl}-e^{kcl}-e^{-kcl}+2\right]-\frac{l^2\omega tH_w}{8f}L_t}{D+t}$$

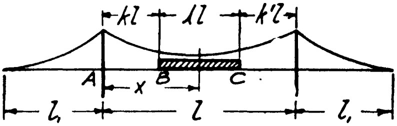
Section A to B

$$C_1=\frac{\frac{p}{2Hc^2}\left[e^{cl(1-k)}+e^{-cl(1-k)}-2e^{-cl}\right]-\frac{8f}{c^2l^2}(1-e^{-cl})}{(e^{cl}-e^{-cl})}$$

$$C_2=-C_1-\frac{1}{c^2}\left(\frac{8f}{l^2}-\frac{p}{H}\right)$$

CASE II

Maximum Positive Moments near Center
Maximum Positive Shears, End to near Center
Maximum Downward Deflection of Center Point.



Conditions of Loading

Main Span;	Partially Loaded
Side Spans;	No Load
Temperature;	Highest.

$$H=\frac{p\lambda l\left\{l^2\frac{[k(1-k)+k'(1-k')]}{4}+\frac{\lambda^2l^2}{12}-\frac{1}{c^2}\right\}+\frac{p}{c^3(e^{cl}-e^{-cl})}\left[e^{cl(1-k)}+e^{-cl(1-k)}+e^{cl(1-k')}+e^{-cl(1-k')}-e^{kcl}-e^{-kcl}-e^{k'cl}-e^{-k'cl}\right]-\frac{l^2\omega tH_w}{8f}L_t}{D+t}$$

Section B to C

$$C_1=\frac{\frac{p}{2Hc^2}\left[e^{k'cl}+e^{-k'cl}-(e^{kcl}+e^{-kcl})e^{-cl}\right]-\frac{8f}{c^2l^2}(1-e^{-cl})}{(e^{cl}-e^{-cl})}$$

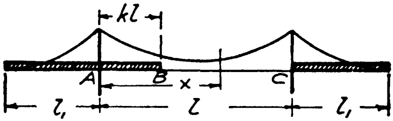
$$C_2=-C_1+\frac{p}{2Hc^2}(e^{kcl}+e^{-kcl})-\frac{8f}{c^2l^2}$$

PLATE II

MAXIMUM NEGATIVE MOMENTS AND SHEARS, CENTER SPAN

CASE III

Maximum Negative Moments; End to near Center.
Maximum Negative Shears; End, and near Center.



Conditions of Loading

Main Span;	Partially Loaded
Side Spans;	Fully Loaded
Temperature;	Lowest.

$$H=\frac{pkl\left[\frac{kl^{2}}{12}(3-2k)-\frac{1}{c^{2}}\right]-\frac{p}{c^{3}(e^{cl}-e^{-cl})}\left[e^{cl(1-k)}+e^{-cl(1-k)}-e^{cl}-e^{-cl}-e^{kcl}-e^{-kcl}+2\right]+2pl_{1}\left[\frac{l_{1}^{2}}{12}-\frac{1}{c_{1}^{2}}\right]+\frac{4p(e^{c_1 l_1}-1)}{c_{1}^{3}(e^{c_1 l_1}+1)}+\frac{l^{2}wtH_{w}}{8f}L_{t}}{D-t}$$

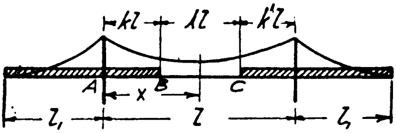
Section B to C

$$C_{1}=\frac{\frac{p}{2Hc^{2}}\left[e^{-cl}(e^{kcl}+e^{-kcl}-2)\right]-\frac{8f}{c^{2}l^{2}}(1-e^{-cl})}{(e^{cl}-e^{-cl})}$$

$$C_{2}=-C_{1}e^{2cl}-\frac{8f}{c^{2}l^{2}}e^{cl}$$

CASE IV

Maximum Negative Moments near Center
Maximum Negative Shears, End to near Center



Conditions of Loading

Main Span;	Partially Loaded
Side Spans;	Fully Loaded
Temperature;	Lowest

$$H=\frac{pl\left[\frac{l^{2}}{12}-\frac{1}{c^{2}}\right]+2pl_{1}\left[\frac{l_{1}^{2}}{12}-\frac{1}{c_{1}^{2}}\right]+\frac{2p(e^{cl}-1)}{c^{3}(e^{cl}+1)}+\frac{4p(e^{c_1 l_1}-1)}{c_{1}^{3}(e^{c_1 l_1}+1)}-\left[\text{Numerator Case II}\right]}{D-t}$$

Section B to C

$$C_{1}=\frac{\frac{p}{2Hc^{2}}\left[(e^{kcl}+e^{-kcl})e^{-cl}-e^{k'cl}-e^{-k'cl}\right]-\frac{1}{c^{2}}\left(\frac{8f}{l^{2}}-\frac{p}{H}\right)(1-e^{-cl})}{(e^{cl}-e^{-cl})}$$

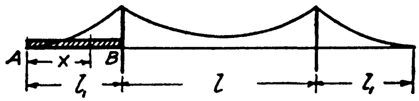
$$C_{2}=-C_{1}-\frac{p}{2Hc^{2}}(e^{kcl}+e^{-kcl})-\frac{1}{c^{2}}\left(\frac{8f}{l^{2}}-\frac{p}{H}\right)$$

PLATE III

MAXIMUM POSITIVE MOMENTS AND SHEARS, SIDE SPANS

CASE V

Maximum Positive Moments; Loaded Side Span.



Conditions of Loading
Main Span; No Load
Side Spans; One, Fully Loaded; Other, No Load.
Temperature; Highest.

$$H = \frac{p l_1\left(\frac{l_1^2}{12} - \frac{1}{c_1^2}\right) + \frac{2p}{c_1^3}\frac{(e^{c_1 l_1} - 1)}{(e^{c_1 l_1} + 1)} - \frac{l^2 \omega t H_w}{8f} L_t}{D + t}$$

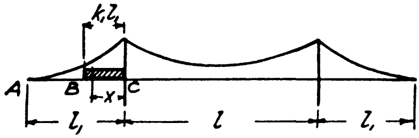
Section A to B

$$C_1 = -\frac{1}{c_1^2(e^{c_1 l_1} + 1)}\left(\frac{8f_1}{l_1^2} - \frac{p}{H}\right)$$

$$C_2 = C_1 e^{c_1 l_1}$$

CASE VI

Maximum Positive Shears; Partially Loaded Side Span



Conditions of Loading
Main Span; No Load
Side Spans; One, Partially Loaded; Other, No Load.
Temperature; Highest

$$H = \frac{p k_1 l_1\left[\frac{k_1 l_1^2}{12}(3 - 2k_1) - \frac{1}{c_1^2}\right] - \frac{p}{c_1^3(e^{c_1 l_1} - e^{-c_1 l_1})}\left[e^{c_1 l_1(1-k_1)} + e^{-c_1 l_1(1-k_1)} - e^{c_1 l_1} - e^{-c_1 l_1} - e^{k_1 c_1 l_1} - e^{-k_1 c_1 l_1} + 2\right] - \frac{l^2 \omega t H_w}{8f} L_t}{D + t}$$

Section B to C

$$C_1 = \frac{\frac{p}{2Hc_1^2}\left[e^{c_1 l_1(1-k_1)} + e^{-c_1 l_1(1-k_1)} - 2e^{-c_1 l_1}\right] - \frac{8f_1}{c_1^2 l_1^2}(1 - e^{c_1 l_1})}{(e^{c_1 l_1} - e^{-c_1 l_1})}$$

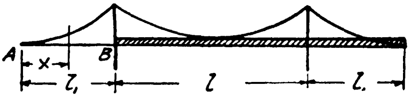
$$C_2 = -C_1 - \frac{1}{c_1^2}\left(\frac{8f_1}{l_1^2} - \frac{p}{H}\right)$$

PLATE IV

MAXIMUM NEGATIVE MOMENTS AND SHEARS, SIDE SPANS

CASE VII

Maximum Negative Moments; Unloaded Side Span



Conditions of Loading

Main Span; Fully Loaded
Side Spans; One, No Load; Other, Fully Loaded
Temperature; Lowest

$$H=\frac{pl\left(\frac{l^2}{12}-\frac{1}{c^2}\right)+pl_1\left(\frac{l_1^2}{12}-\frac{1}{c_1^2}\right)+\frac{2p}{c^3}\frac{(e^{cl}-1)}{(e^{cl}+1)}+\frac{2p}{c_1^3}\frac{(e^{c_1l_1}-1)}{(e^{c_1l_1}+1)}+\frac{l^2\omega tH_w}{8f}L_t}{D-t}$$

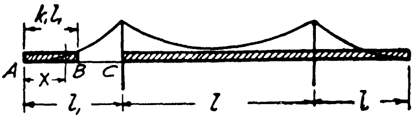
Section A to B

$$C_1=-\frac{1}{c_1^2(e^{c_1l_1}+1)}\,\frac{8f_1}{l_1^2}$$

$$C_2=C_1e^{c_1l_1}$$

CASE VIII

Maximum Negative Shears. Partially Loaded Side Span



Conditions of Loading

Main Span; Fully Loaded
Side Spans; One, Partially Loaded; Other, Fully Loaded
Temperature; Lowest.

$$H=\frac{pk_1l_1\left[\frac{k_1l_1^2}{12}(3-2k_1)-\frac{1}{c_1^2}\right]-\frac{p}{c_1^3(e^{c_1l_1}-e^{-c_1l_1})}\left[e^{c_1l_1(1-k_1)}+e^{-c_1l_1(1-k_1)}-e^{c_1l_1}-e^{-c_1l_1}-e^{k_1c_1l_1}-e^{-k_1c_1l_1}+2\right]+[\text{Numerator Case VII}]}{D-t}$$

Section A to B

$$C_1=\frac{\frac{p}{2Hc_1^2}\left[e^{c_1l_1(1-k_1)}+e^{-c_1l_1(1-k_1)}-2e^{-c_1l_1}\right]-\frac{8f_1}{c_1^2l_1^2}(1-e^{c_1l_1})}{(e^{c_1l_1}-e^{-c_1l_1})}$$

$$C_2=-C_1-\frac{1}{c_1^2}\left(\frac{8f_1}{l_1^2}-\frac{p}{H}\right)$$

of loading. Problems bringing out these points will be given in Art. 208.

To aid in the calculation of moments and shears for a structure of the type of the Manhattan Suspension Bridge, formulas for H have been derived for the various cases of loading which arise. These formulas are given on Plates I to IV. In calculating moments and shears eqs. (5) and (6) are to be used. The values of C_1 and C_2 which are to be used are also given on the Plates, together with the necessary temperature conditions.

208. Example.—The exact methods of Arts. 204 to 207 will now be applied to the calculation of horizontal component of cable stress, moment, shear, and deflection in the case of the Manhattan Suspension Bridge. The dimensions of the structure are as given in Fig. 27 of Art. 202. The live load in all cases will be taken as 4,000 lbs. per lin. ft. per truss.

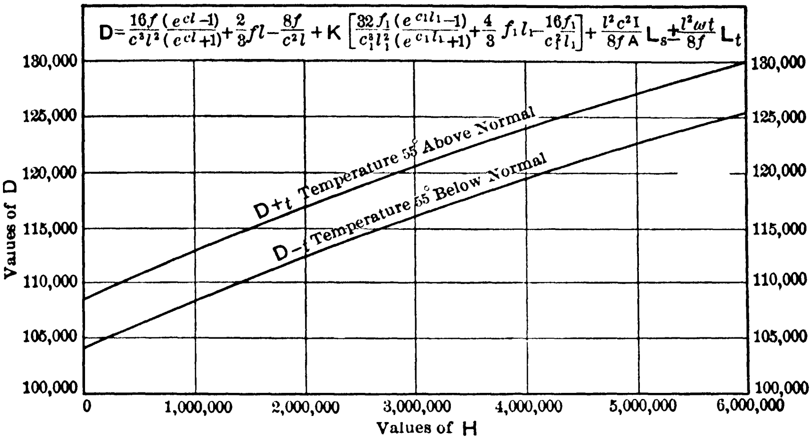
Values of Denominator D.—As stated in Art. 206, the value of the denominator is constant for all cases. The values of D_{+t} and D_{-t} as given by eq. (23), were calculated for values of H ranging from 0 to 6,000,000, and for temperature conditions 55 degrees above and below the normal. The curves plotted from these results are shown in Fig. 33. From these curves the value of the denominator for any particular value of H can be determined, thus reducing the labor of calculation.

Values of Horizontal Component of Cable Stress.—The value of H for a uniform load of 4,000 lbs. per ft., extending from the left end of the main span to the quarter point, will be calculated to illustrate the methods employed. From Case I, Plate I, the formula for H for this case, temperature considered as above normal, is,

$$H = \frac{pkl\left[\frac{kl^2}{12}(3-2k) - \frac{1}{c^2}\right] - \frac{p}{c^3(e^{cl}-e^{-cl})}\left[e^{cl(1-k)} + e^{-cl(1-k)} - e^{cl} - e^{-cl} - e^{kcl} - e^{-kcl} + 2\right] - \frac{l^2 \omega t H_w}{8f} L_t}{D_{+t}}.$$

This equation can be solved only by successive approximations, for the term c, whose value is given by $c^2 = \frac{H + H_w}{EI}$, also contains H. In the solution of this equation, a value of H is assumed, from which the value of c is calculated and substituted above. If the value of H was correctly assumed, the result of the substitution will equal the assumed value. Usually after one or two trials, a result is obtained for which the assumed and calculated values of H differ by only a small amount, which should be limited to 1 per cent.

For the problem in question, assume $H = 665{,}000$ lbs. The value of H_w is given by $\frac{wl^2}{8f}$. For the dimensions given in Fig. 27, with the dead



$$D=\frac{16f}{c^3l^2}\frac{(e^{cl}-1)}{(e^{cl}+1)}+\frac{2}{3}fl-\frac{8f}{c^2l}+K\left[\frac{32f_1}{c_1^3l_1^2}\frac{(e^{c_1l_1}-1)}{(e^{c_1l_1}+1)}+\frac{4}{3}f_1l_1-\frac{16f_1}{c_1^2l_1}\right]+\frac{l^2c^2I}{8fA}L_s\pm\frac{l^2\omega t}{8f}L_t$$

FIG. 33.

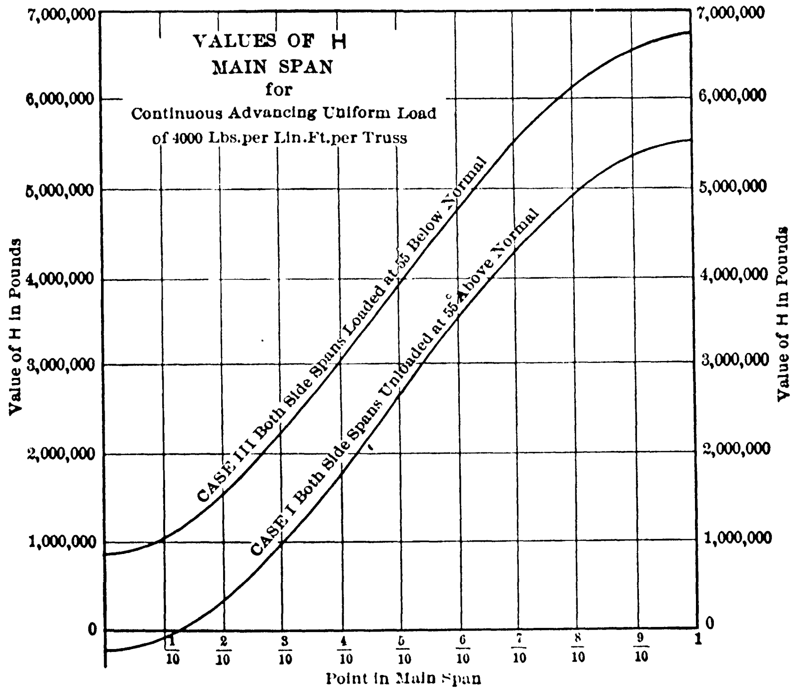


FIG. 34.

load on the main and side spans taken respectively as 5,820 and 6,130 lbs. per ft., we have

$$\frac{w\,l^2}{8f} = \frac{5{,}820 \times (1446.7)^2}{8 \times 145.3} = 10{,}480{,}000 \text{ lbs.}$$

$$\frac{w_1 l_1^2}{8f_1} = \frac{6{,}130 \times (\ 713.5)^2}{8 \times 37.2} = 10{,}480{,}000 \text{ lbs.}$$

Taking the average of these values, we find $H_w = 10{,}480{,}000$ lbs.

The moment of inertia of the main span is given as 43,900 ft.² ins.² With $E = 29{,}000\,000$ lbs. per sq. in., we find

$$c^2 = \frac{H + H_w}{E\,I} = \frac{10{,}480{,}000 + 665{,}000}{43{,}900 \times 29{,}000{,}000} = 0.0000087 54,$$

from which $c = 0.0029587$. Also, $\frac{1}{c} = 337.98$, $\frac{1}{c^2} = 114{,}230.6$, and $\frac{1}{c^3} = 38{,}607{,}000$.

For $k = 0.25$, $p = 4{,}000$, $l = 1446.7$, we have

$$p\,k\,l\left[\frac{k\,l^2}{12}(3 - 2\,k) - \frac{1}{c^2}\right] = -\ 7{,}556{,}260{,}000.$$

The values of the terms containing e^{cl} and e^{kcl} are best found by logarithms ($\log e = 0.4342945$). With $c = 0.0029587$, we have $c\,l = 4.2805$, from which $\log e^{cl} = 1.858967$. The values of e^{kcl} are found by multiplying $\log e^{cl}$ by k. The values of the various terms are given in the following table:

$$\begin{array}{lll}
\log e^{cl} = 1.858967 & e^{cl} = 72.2714 & e^{-cl} = 0.0138. \\
\log e^{\frac{1}{4}cl} = 0.464742 & e^{\frac{cl}{4}} = 2.9157 & e^{-\frac{cl}{4}} = 0.3430. \\
\log e^{(1-\frac{1}{4})cl} = 1.394225 & e^{\frac{3}{4}cl} = 24.7870 & e^{-\frac{3}{4}cl} = 0.0403.
\end{array}$$

For these values the second term in the numerator of the formula for H becomes

$$-\frac{p}{c^3\,(e^{cl} - e^{-cl})}[-\ 48.7167] = +\ 104{,}118{,}350{,}000.$$

The value of the temperature term in the numerator is found to be

$$\frac{l^2\,\omega\,t\,H_w}{8f}\,L_t = -\ 22{,}720{,}000{,}000$$

where $L_t = 3452$ (from Art. 202), $\omega = 0.0000066$, and $t = +\ 55°$.

The total for the numerator is $+\ 73{,}842{,}100{,}000$.

From the curve of Fig. 33 for D_{+t}, we find the value of the denominator for $H = 665{,}000$ to be, $D_{+t} = 111{,}300$. The resulting value of H is then 663,400 lbs. As this value is within 0.25 per cent of the assumed H, it will be taken as final.

Values of H for a continuous advancing uniform load of 4,000 lbs. per lin. ft. for Cases I and III, as given on Plates I and II, have been calculated, and are plotted in Fig. 34. It will be noted in the curve in Case I, that the effect of temperature above the normal, no load on the structure is such as to decrease the amount of the dead load cable stress. This effect also occurs under loading out to about the $\frac{1}{8}$ point.

Moment at Quarter Point.—The general expression for moment at any point, as given by eq. (5) of Art. 204, is

$$M = -H\left[C_1 e^{cx} + C_2 e^{-cx} + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right)\right].$$

The constants of integration for loading conditions which give maximum positive moments at the quarter point are found under Case I, on Plate I. The values are,

$$C_1 = \frac{\frac{p}{2Hc^2}\left[e^{cl(1-k)} + e^{-cl(1-k)} - 2e^{cl}\right] - \frac{8f}{c^2 l^2}\left(1 - e^{-cl}\right)}{e^{cl} - e^{-cl}}$$

$$C_2 = -C_1 - \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right).$$

In the problem of Art. 202, the loading for maximum positive moment at the quarter point was found to extend from the end of the span to $k = 0.466$. Using this as a guide, the moments were calculated by the above equations for $k = 0.400$, 0.425, 0.450, and 0.475, from which the maximum was found to occur for $k = 0.425$. For the various cases, the value of H was taken from the curves of Fig. 34. For $k = 0.425$ we find $H = 2{,}022{,}000$ lbs. For this value of H we have $c^2 = 0.000009821$ and $c = 0.003134$. The various terms in the above equation then have the following values:

$$e^{cl} = e^{4.5340} = 93.111 \qquad e^{-cl} = 0.0105$$

$$e^{cl(1-k)} = e^{0.575\,cl} = 13.555 \qquad e^{-cl(1-k)} = 0.0738$$

$$\frac{p}{2Hc^2} = 100.710 \qquad \frac{8f}{c^2 l^2} = 56.552$$

With these values we find

$$C_1 = \frac{100.710\,[13.555 + 0.0738 - 0.0210] - 56.552\,(1 - 0.0105)}{(93.111 - 0.0105)} = +14.119$$

$$C_2 = -14.119 - \frac{1}{0.000009821}(0.0005554 - 0.0019782) = +130.750.$$

For moment at the quarter point, we have $x = \frac{l}{4}$. Then $e^{cx} = e^{\frac{cl}{4}} = 3.106$ and $e^{-\frac{cl}{4}} = 0.322$. Substituting these values in the general expression, together with the values of C_1 and C_2, we have

$$\begin{aligned} M = &-2{,}022{,}000\,[14.119 \times 3.106 + 130.75 \times 0.322 \\ &+ \frac{1}{0.000009821}(0.0005554 - 0.0019782)] \\ = &+119{,}110{,}000 \text{ ft.-lbs.} \end{aligned}$$

By the approximate method, the positive moment at the quarter point, temperature effect included, was found to be 172,940,000 (Problem, Art. 202). This result exceeds the value given above by 45 per cent.

Moment at Centre.—The maximum moment at the centre of the main span was found to occur for a uniform load extending from $k = 0.3$ to $k = 0.7$. The formula for H and the values of the constants of integration are given under Case II, on Plate I. In the determination of the values of H, trial values were obtained by taking the difference between the ordinates at the ends of the load, as given by the curves of Fig. 34. The final value of H was usually found on the second substitution in the formula. By a process similar to that used for moment at the quarter point, the maximum positive centre moment was found to be 89,710,000 ft. lbs. The final value of H was 3,155,000 lbs., and the constants of integration were found to be $C_1 = 1.806$, and $C_2 = 205.602$.

The Approximate Method result of Art. 202, temperature effect included, exceeds the moment given above by 33.9 per cent.

Shear at the End of Main Span.—The maximum positive shear at the left end of the main span was found to occur for a uniform load extending from the left end of the span to $k = 0.325$.

The general formula for shear at any point is given by eq. (6) of Art. 204.

$$V = - H c \left[C_1 e^{cx} - C_2 e^{-cx}\right].$$

The formula for H, and the values of the constants of integration are given under Case I, on Plate I. From the curve of Fig. 34, the value of H was found to be 1,200,000 lbs. With this value of H we find $c = 0.003029$. $C_1 = 43.05$, and $C_2 = 259.72$. At the end of the span, where $x = 0$, we have, $e^{cx} = 1$ and $e^{-cx} = 1$. The shear is then

$$V = - 1{,}200{,}000 \times 0.003029\,(+ 43.05 - 259.72) = + 787{,}500 \text{ lbs.}$$

Deflection of Centre Point.—The maximum downward deflection of the centre point will be found to occur for the main span fully loaded, no load on the side spans, temperature highest. The general equation for deflection is given by eq. (4) of Art. 204. Placing $x = \frac{l}{2}$ and $M' = \frac{p\,l^2}{8}$ in this equation, we have

$$\eta = \frac{H}{H + H_w}\left[C_1 e^{\frac{cl}{2}} + C_2 e^{-\frac{cl}{2}} + \frac{p\,l^2}{8H} + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{p}{H}\right) - f\right].$$

From the curves of Fig. 34, the value of H for the main span fully loaded, temperature above normal, is found to be 5,470,000 lbs. The value of c is then 0.003594. The constants of integration are given under Case II, on Plate I, with k and $k' = 0$, or by Case I for $k = 1$. We then have

$$C_1 = - \frac{1}{c^2\,(e^{cl} + 1)}\,\frac{8f}{l^2} \qquad C_2 = C_1\,e^{cl}.$$

For the value of H given above we find

$$C_1 = + 0.07711 \qquad C_2 = + 13.96.$$

Substituting in the above equation, the deflection is found to be

$$\eta = 0.34295\,[0.07711 \times 13.445 + 13.96 \times 0.07432 + 191.31 - 14.04 - 145.3]$$
$$= 11.677 \text{ ft.}$$

SECTION IV.—STRESSES IN LATERAL TRUSSES, TOWERS, AND FLOORBEAMS

209. Wind Stresses in Structures with Horizontal Stiffening Trusses.—The action of wind forces on the side of the stiffening truss causes the truss to deflect laterally as shown in Fig. 35. This lateral deflection is resisted by the lateral trusses and by the pull of the cables.

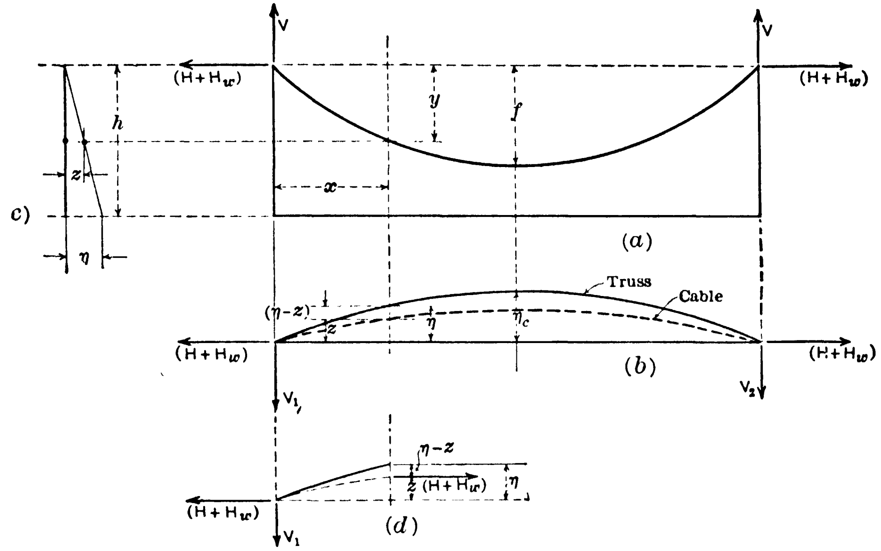


FIG. 35.

From Fig. (*d*), moments about any point, horizontal forces only being considered, we have

$$M = M' - (H + H_w)\,\eta + (H + H_w)\,(\eta - z) = M' - (H + H_w)\,z \quad (24)$$

From Fig. (*c*) we have $z = y\dfrac{\eta}{h}$, assuming that the hanger at any point forms a straight line from the truss to a line joining the tops of the towers. Substituting this value of z in eq. (24), we have

$$M = M' - (H + H_w)\frac{y\,\eta}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

In eq. (25) y is variable as well as η, and to reach an exact solution is difficult. However, as the cable effect is of greatest importance along the central portion where y is nearly equal to f we may for purposes of an approximate solution place $y = f$. The differential equation of the elastic curve of the truss is then given by

$$\frac{d^2 \eta}{d x^2} = c^2 n - \frac{M' h c^2}{f (H + H_u)}. \quad . \; . \; . \; . \; . \; . \quad (26)$$

where $c^2 = \frac{f}{h} \frac{(H + H_u)}{E I}$. Integrating eq. (26), we have

$$\eta = \frac{h}{f (H + H_u)} \left[C_1 e^{c x} + C_2 e^{-c x} + M' - \frac{p}{c^2} \right] \quad . \; . \quad (27)$$

Substituting this value of η in eq. (25), the moment is given by

$$M = - \left[C_1 e^{c x} + C_2 e^{-c x} - \frac{p}{c^2} \right] \quad . \; . \; . \; . \quad (28)$$

where p is the wind load per unit length.

The constants of integration in eq. (28) depend upon the loading conditions. It will be found that the moments are a maximum for a uniform wind load over the whole span.

For uniform lôad, the constants are determined from the condition that for $x = 0$ or $x = l$, $M = 0$. The values are then

$$C_1 = \frac{p}{C_2} \frac{1}{(e^{c l} + 1)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (29)$$

$$C_2 = C_1 e^{c l} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

Substituting the values of these constants in eq. (28), the moment at any point under full load is found to be

$$M = \frac{p}{c^2} \left[1 - \frac{e^{c x} + e^{c(l - x)}}{(e^{c l} + 1)} \right] \quad . \; . \; . \; . \; . \; . \quad (31)$$

The moment is found to be a maximum at the centre, where the value is

$$M_c = \frac{p}{c^2} \frac{\left(1 - e^{\frac{c l}{2}}\right)^2}{(e^{c l} + 1)}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (32)$$

The shear at any point is found by taking the first derivative of eq. (28) with respect to x. We then have

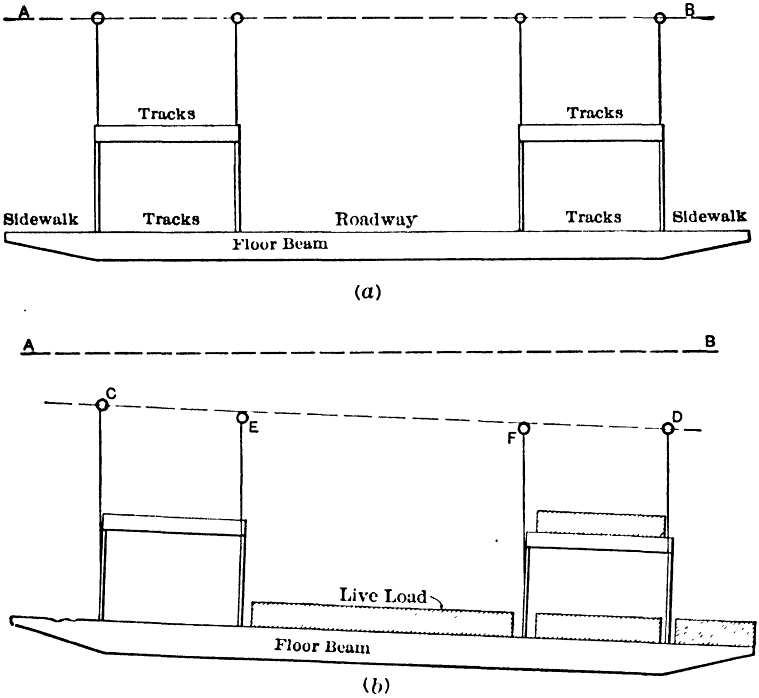
$$V = -c \left[C_1 e^{c x} - C_2 e^{-c x} \right] \quad . \; . \; . \; . \; . \quad (33)$$

The constants of integration for any loading conditions are found by the same methods as given in Art. 205. It will be found that these constants can be obtained from those given on Plates I to IV by placing $f = 0$ and $H = 1$ in the corresponding case.

210. Floorbeam Stresses.—In a suspension bridge with two cables, the stresses in the floorbeams are found in the same way as for a simple beam supported at two points. Where four cables are used, as in the Manhattan Bridge, and the floorbeams are continuous, the analysis requires the consideration of the deflection of the cables and the continuity of the beams. Fig. 36 shows the arrangement used in the Manhattan Bridge.

When all tracks, roadways, and side-walks are fully loaded, the cables will deflect equal distances and the reactions of the floorbeam will be equal.

Under unsymmetrical loading the cables will receive varying proportions of the load, thus causing unequal deflections of the various cables and also of the points of support of the floorbeams. The beams are then continuous on four supports which do not lie in the same straight line. The maximum bending moments in the floorbeams will be found to occur for the loading which will cause the greatest difference in deflection of the cables. This loading is shown in Fig. 36 (*b*), the



(*a*)

(*b*)

FIG. 36.

tracks and sidewalks on one side and the roadway being fully loaded over the entire length of the structure. The load taken by cable *C* is then transmitted entirely by the bending of the floorbeams.

In determining the stresses in the floorbeams for the loading conditions given above, it is sufficiently exact to assume the loads on the cables to be proportional to their deflection; and, as these deflections are large compared with that of the floorbeam, the deformations of the beam may be neglected. The reactions may then be found in the same manner as the fibre stress in a beam subjected to direct compression as well as moment. If P = total live load per panel, e = eccentricity, R = any of the four cable reactions, and a = distance of such cable from the axis, we have $R = \frac{P}{4} \pm \frac{P\,e\,a}{\Sigma\, a^2}$. Here the term $\Sigma\, a^2$ replaces the moment of inertia of the usual beam formula. With these reactions, the bending moments and shears are readily calculated by the usual methods.

211. Tower Stresses.—When the saddles are rigidly fastened to the tops of the towers, as in the Manhattan Suspension Bridge, the movement of the saddles due to changes in loading or temperature conditions will cause bending stresses to be set up in the towers. As this deflection must take place, the bending stresses induced will be proportional to the stiffness of the tower.

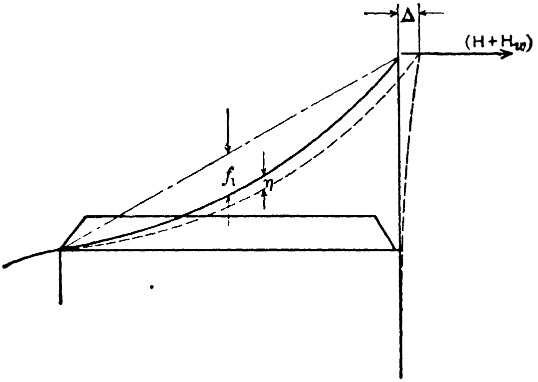


FIG. 37.

The deflection may be determined by the approximate method given in Art. 204, or by a more exact method in which work done by external and internal forces during the deflection are placed equal to zero.

In Fig. 37, let Δ be the deflection of the tower, and $(H + H_w)$ the cable stress for any given loading conditions. Then the average work done by the force $(H + H_w)$ moving through the distance Δ must be equal to the average work in the cable due to stress and temperature, and the work done in the deflection of the side span. The average work done by the force $(H + H_w)$ is given by $\left(\frac{H}{2} + H_w\right)\Delta$. From Art. 206, the average work in the cable due to stress and temperature is respectively $\left(\frac{H}{2} + H_w\right)\int_0^{l_1} \frac{d\,s^3}{d\,x^2}$ and $\left(\frac{H}{2} + H_w\right)\int_0^{l_1} \omega t\frac{d\,s^2}{d\,x^2}$. The average work done in the deflection of the side span is given by $\left(\frac{H}{2} + H_w\right)\frac{8\,f_1}{l_1^2}\int_0^{l_1} \eta_1\,d\,x$ where η_1 is found from eq. (4) of Art. 204. Neglecting the work done in bending the tower, we have

$$\left(\frac{H}{2} + H_w\right)\Delta = \left(\frac{H}{2} + H_w\right)\frac{H}{A\,E}I_s + \left(\frac{H}{2} + H_w\right)\omega\,t\,L_t + \frac{8\,f_1}{l_1^2}\int_0^{l} \eta_1\,d\,x. \quad . \quad (34)$$

where L_s and L_t represent the value of $\int_0^{l_1} \frac{d\,s^3}{d\,x^2}$ and $\int_0^{l_1} \frac{d\,s^2}{d\,x}$ respectively for the side spans only. Then from eq. (34),

$$\Delta = \frac{H}{A\,E}L_s + \omega\,t\,L_t + \frac{8\,f_1}{l_1^2}\int_0^{l_1} \eta_1\,d\,x. \quad . \quad . \quad . \quad (35)$$

The greatest deflection will occur when the main and far side spans are fully loaded, no load on the side span in question, temperature lowest.

Substituting the value of η for this condition of loading, the maximum deflection is found to be

$$\Delta = \frac{H}{AE}L_s + \omega t L_t + \frac{H}{(H + H_w)}\left[\frac{16\,f_1}{c_1^3 l_1^2}\,\frac{(e^{c_1 l_1} - 1)}{(e^{c_1 l_1} + 1)} - \frac{8f_1}{c_1^2 l_1} + \frac{2}{3}f_1 l_1\right]\frac{8f_1}{l_1^2} \quad (36)$$

The bending stresses in the tower are due to a horizontal load Q and a vertical load V as shown in Fig. 38 (*a*).

The load Q is such that if applied at the top of the tower, it would (together with V) cause a deflection equal to Δ of eq. (36). The load V is equal to the sum of the vertical components of the cable stress acting on the tower, as shown in Fig. 38 (*b*). The amount of this force is given by

$$(H + H_w)\,[\tan\alpha + \tan\alpha_1 + \Delta\tan\alpha + \Delta\tan\alpha_1]$$

where $\Delta \tan \alpha$ and $\Delta \tan \alpha_1$ are the changes in $\tan \alpha$ and $\tan \alpha_1$ due to the deflection of the main and side spans. From eq. (4) of Art. 204, $\Delta \tan \alpha = \frac{d\eta}{dx}$. The loads V_1 and V_2 are the end reactions of the main and side spans.

The amount of the load Q may be determined by placing the deflection of the top of the tower, as given by the equation of the elastic curve, equal to Δ as given by eq. (36). As Q is the only unknown, its value is readily determined. From Fig. 38 (a) the bending moment at any point in the tower, neglecting the effect of the loads V_1 and V_2, is given by

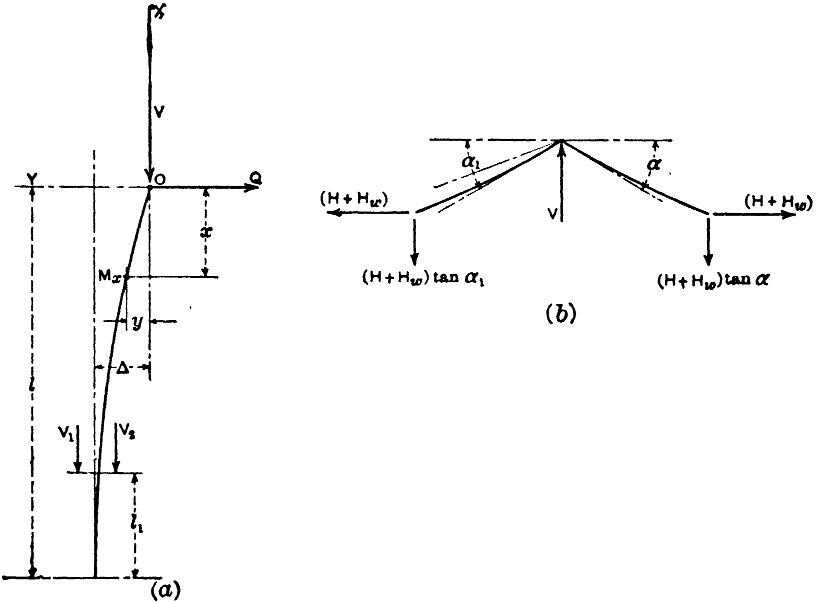


FIG. 38.

$$M_x = Vy + Qx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The differential equation of the elastic curve is

$$\frac{d^2 y}{dx^2} + c^2 y = -\frac{Qc^2}{V}x \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

where $c^2 = \frac{V}{EI}$. Integrating eq. (38) we have

$$y = A \sin cx + B \cos cx - \frac{Q}{V}x \quad . \quad . \quad . \quad . \quad (39)$$

where A and B are constants of integration.

If the moment of inertia of the tower is uniform over its whole length, the values of A and B are determined from the two conditions that for $x = 0$, $y = 0$ and $x = l$, $y = \Delta$, from which

$$A = \left(\frac{Q\,l}{V} + \Delta\right) \text{cosec. } c\,l \text{ and } B = 0.$$

Eq. (39) then becomes

$$y = \left(\frac{Q\,l}{V} + \Delta\right)\frac{\sin c\,x}{\sin c\,l} - \frac{Q}{V}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The values of Q and Δ are found from eq. (40) for the condition that $\frac{d\,y}{d\,x} = 0$ when $x = l$.

We then have

$$\Delta = \frac{Q}{V\,c}(\tan c\,l - c\,l)$$

and

$$Q = \frac{V\,\Delta\,c}{(\tan c\,l - c\,l)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where Δ is given by eq. (36).

Substituting the value of y from eq. (39) in eq. (37), the moment at any point is found to be

$$M_x = (Q\,l + V\,\Delta)\frac{\sin c\,x}{\sin c\,l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

For $x = 0$, or at the top of the tower, $M = 0$; and for $x = l$, or at the foot of the tower, $M = (Q\,l + V\,\Delta)$.

As actually constructed, the sections in such towers are made to fit the stresses, so that the moment of inertia is not uniform. In such cases, the tower is divided into lengths over which the moment of inertia may be taken as uniform. Then by placing the moment and shear at the end of one section equal to that at the beginning of the next, a sufficient number of independent equations are obtained for the determination of the constants of integration. For this purpose the moment at any point, from eqs. (37) and (39), is given by

$$M_x = V\,(A \sin c\,x + B \cos c\,x).$$

The shear at any point is

$$\frac{d\,M_x}{d\,x} = c\,V\,(A \cos c\,x - B \sin \cdot c\,x).$$

The deflection at any point is given by substituting the constants of integration for the section in question, as determined above, in eq. (39).

CHAPTER VI

MISCELLANEOUS PROBLEMS IN STATICALLY INDETERMINATE STRUCTURES

SECTION I.—MULTIPLE INTERSECTION TRUSSES

212. General Formulas.—In Chapter VII of Part I, the general method of analyzing structures with redundant members was given and briefly illustrated. In the preceding chapters of this Part several special cases have been considered. There remain numerous other problems of statically indeterminate structures arising in practice which merit attention, but before taking these up in detail the method explained in Part I will be restated and the equations presented in a somewhat more general form.

Consider a structure containing any number of redundant members and supporting certain loads applied at joints only. Assume, first, that all members are subjected to direct stress only (no members acting as beams).

Let n = number of necessary members;

m = number of redundant members;

S = stress in any one of the n necessary members, due to the given loads;

S_1, S_2, S_3, etc., = stresses in the several redundant members due to the given loads (the stresses S, S_1, S_2, etc., occur simultaneously);

S' = stress in any one of the n necessary members due to the external loads, with all redundant members removed;

u_1 = stress in any of the n members due to a one-pound tension in redundant member No. 1, all other redundant members being removed; u_2 = stress for a one-pound tension in No. 2; etc., etc.;

E, A and l = modulus of elasticity, area of cross-section, and length of any member.

Then for any member

$$S = S' + u_1 S_1 + u_2 S_2 + u_3 S_3 + \text{etc.} \quad . \quad . \quad . \quad (1)$$

Now, as in Art. 227 of Part I, we assume each of the redundant members in turn cut near one end, and place equal to zero the deflection of the cut end with respect to the adjacent joint. This deflection is expressed in terms of the deformations of the *n* necessary members and the redundant member in question. We thus have the general equations

$$\left.\begin{array}{l} \Sigma_0^n \dfrac{S u_1 l}{EA} + \dfrac{S_1 l_1}{EA_1} = 0 \\ \Sigma_0^n \dfrac{S u_2 l}{EA} + \dfrac{S_2 l_2}{EA_2} = 0 \\ \Sigma_0^n \dfrac{S u_3 l}{EA} + \dfrac{S_3 l_3}{EA_3} = 0 \\ \quad \text{etc.} \qquad\quad \text{etc.} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting the value of S from eq. (1) we have the following series of equations between the several redundant stresses and the stresses S':

$$\left.\begin{array}{l} S_1\left(\dfrac{l_1}{EA_1} + \Sigma\dfrac{u_1^2 l}{EA}\right) + S_2\,\Sigma\dfrac{u_1 u_2 l}{EA} + S_3\,\Sigma\dfrac{u_1 u_3 l}{EA} + \ldots \text{etc.} = -\Sigma\dfrac{S' u_1 l}{EA} \\ S_1\Sigma\dfrac{u_2 u_1 l}{EA} + S_2\left(\dfrac{l_2}{EA_2} + \Sigma\dfrac{u_2^2 l}{EA}\right) + S_3\,\Sigma\dfrac{u_2 u_3 l}{EA} + \ldots \text{etc.} = -\Sigma\dfrac{S' u_2 l}{EA} \\ S_1\Sigma\dfrac{u_3 u_1 l}{EA} + S_2\,\Sigma\dfrac{u_3 u_2 l}{EA} + S_3\left(\dfrac{l_3}{EA_3} + \Sigma\dfrac{u_3^2 l}{EA}\right) + \ldots \text{etc.} = -\Sigma\dfrac{S' u_3 l}{EA} \\ \qquad\qquad \text{etc.} \qquad\qquad\quad \text{etc.} \qquad\qquad\quad \text{etc.} \end{array}\right\} . \quad (3)$$

In these equations the summation in all cases extends over the *n* necessary members only, as no other member has a stress *u*. If we consider that the *u* for the redundant member in question is unity, then the two terms in the parentheses may in general be represented by the single summation $\Sigma \dfrac{u^2 l}{EA}$.

In this manner as many equations may be written as there are redundant members. The numerical values of the coefficients of S_1, S_2, S_3, etc., and of the second members of the equations, are readily calculated and the resulting linear equations may then be solved for S_1, S_2, etc. The value of any stress S is then found from eq. (1). It is well to note that the several coefficients of S_1, S_2, etc., of eq. (3) contain products of *u* which are, in each equation, the product of a particular u times each of the other u's in succession, the latter corresponding to the S of the same term.

As already illustrated by the analysis of Art. 166, the displacement diagram may be used advantageously in calculating the various summations in eq. (3). For this purpose all redundant members are to be considered as cut near one end. A displacement diagram is then drawn for a one-pound load applied at the cut ends of No. 1, no external loads acting; another diagram for a one-pound load at the ends of No. 2, etc.; one diagram being drawn for each member. Finally, a like diagram is drawn for the external loads, involving the stresses S'. It will then be found that all the coefficients of S_1, S_2, etc., of the first of equations (3), are given by the first diagram; those of the second equation by the second diagram, etc.; and all the summations of the second members of the equations are given by the last diagram mentioned. Or, in detail, considering the first diagram, the quantity $\left(\frac{l_1}{EA_1} + \Sigma\frac{u_1^2 l}{EA}\right)$ is the relative displacement of the cut ends of redundant member No. 1; the quantity $\Sigma\frac{u_1 u_2 l}{EA}$ is the relative displacement of the cut ends of No. 2, etc. In the diagram constructed for the external loads, $\Sigma\frac{S' u_1 l}{EA}$ is the relative displacement of the cut ends of No. 1, etc. It will be noted that $\Sigma\frac{u_1 u_2 l}{EA}$ is given in both the first and second diagrams, and $\Sigma\frac{u_1 u_3 l}{EA}$ in both the first and third diagrams, etc. The utilization of this principle (of reciprocal deflections) will make it necessary to construct but a portion of the displacement diagrams of the second and succeeding members.

In case the structure includes members acting as beams then the expressions for deflection include terms of the form $\int\frac{M\,dx}{EI}\,.\,m$. If the redundant members themselves are subjected to direct stresses only, then eq. (2) takes the form

$$\Sigma_0^n\frac{S u_1 l}{EA} + \Sigma_0^n\int\frac{M\,dx}{EI}\,.\,m_1 + \frac{S_1 l_1}{EA_1} = 0 \quad . \quad . \quad . \quad (4)$$

in which M = bending moment in any member under normal conditions, and m_1 = bending moment due to a one-pound tension in redundant member No. 1. In the same manner as S is expressed in eq. (1) we may write $M = M' + S_1 m_1 + S_2 m_2 + \ldots$, etc. The values of S and M being substituted, gives a series of linear equations similar to (3), as follows:

$$\begin{aligned} & S_1\left(\frac{l_1}{EA_1} + \Sigma \frac{u_1{}^2 l}{EA} + \Sigma \int \frac{m_1{}^2 dx}{EI}\right) + S_2\left(\Sigma \frac{u_1 u_2 l}{EA} + \Sigma \int \frac{m_1 m_2 dx}{EI}\right) \\ & \quad + S_3\left(\Sigma \frac{u_1 u_3 l}{EA} + \Sigma \int \frac{m_1 m_2 dx}{EI}\right) + \ldots\text{etc.,} = -\Sigma \frac{S' u_1 l}{EA} - \Sigma \int \frac{M' m_1 dx}{EI} \qquad (5) \\ & \qquad\qquad\qquad \text{etc.} \qquad\qquad\qquad \text{etc.} \qquad\qquad\qquad \text{etc.} \end{aligned}$$

Instead of assuming as the unknowns the various direct stresses S_1, S_2, etc., it is sometimes more convenient to take certain of the bending moments as the unknowns, or certain direct stresses together with certain moments. In this case the values u_1 and m_1, corresponding to an unknown bending moment M_1, are respectively the stresses and moments in the various members due to a bending moment of unity in the redundant member. The resulting equations are then the same as (5) excepting that certain unknown moments, M_1, M_2, etc., are used instead of the stresses S_1, S_2, etc.

213. *Equations Derived from the Principle of Least Work.*—The entire internal work of displacement of a structure is given by the expression

$$\text{Work} = \Sigma \frac{S^2 l}{2EA} + \Sigma \int \frac{M^2 dx}{2EI}. \quad . \quad . \quad . \quad . \quad (6)$$

If the structure contains various redundant stresses S_1, S_2, etc., or moments M_1, M_2, etc., the first derivative of the expression for work may be written with respect to each of the unknowns in turn, as explained in Part I, Art. 216, and each of these expressions placed equal to zero. This gives us, for member No. 1,

$$\Sigma \frac{Sl}{EA} \cdot \frac{dS}{dS_1} + \Sigma \int \frac{M\,dx}{EI} \cdot \frac{dM}{dS_1} = 0. \quad . \quad . \quad . \quad (7)$$

But in general $\frac{dS}{dS_1} = u_1$ and $\frac{dM}{dS_1} = m_1$, as here defined. Furthermore, in (7) the expression $\Sigma \frac{Sl}{EA} \cdot \frac{dS}{dS_1}$ includes also member No. 1, in which $\frac{dS}{dS_1} = 1$, hence (7) is identical with (4).

With respect to any unknown moment M_1 the derivative is

$$\Sigma \frac{Sl}{EA} \cdot \frac{dS}{dM_1} + \Sigma \int \frac{M\,dx}{EI} \cdot \frac{dM}{dM_1} = 0 \quad . \quad . \quad . \quad (8)$$

in which $\frac{dS}{dM_1}$ and $\frac{dM}{dM_1}$ is the stress or moment in any member due to a unit bending moment in the redundant member. These quantities may be called u_1 and m_1 as before. The physical interpretation of eq. (8) is that the relative angular change in the two cut ends of the redundant member is zero, corresponding to the longitudinal movement in the case of redundant direct stresses.

214. Forms of Trusses and Methods of Analysis.—The most common types of multiple intersection trusses are the double Warren or triangle truss and the Whipple or double intersection Pratt truss. These forms have been fully treated in Part I by the usual approximate methods, based on the assumption of independent action of the two web systems. This same method of treatment may be applied to triple and higher forms of multiple trusses with generally satisfactory results.

Where the several systems are connected at points other than at supports, as when inclined end-posts are used, some ambiguity arises in the usual approximate treatment (see Art. 101, Part I). Uncertainty of procedure also arises in the case of an odd number of panels or an otherwise unsymmetrical system. In the case of curved chords, also, the web systems cannot act independently to so great an extent as with horizontal chords. In certain other multiple forms it is difficult or impossible to separate the structure into two or more single systems so as to arrive at any satisfactory solution by approximate methods.

It is the purpose of the following articles to explain more fully than was done in Part I, the use of exact methods of calculation and to illustrate by examples the errors involved in certain cases by the application of the usual approximate methods.

215. The Double Triangular Truss with Vertical End-Posts.—The results given in the example of Art. 224, Part I, showed that the usual assumption of independent action of the two web systems gives correct results for the chord stresses under uniform loads, and that the maximum error in web stress when calculated on this basis was, in the truss in question, only 5 per cent, and this for the web member nearest the centre. For all other web members the error was too small to be noted. In the case of an eight-panel truss of similar form, analyzed by Winkler,[*] the corresponding errors in the web stresses ranged from 0 to 1 per cent from end toward centre. The first counter-stress showed an error of 1.1 per cent and the second counter-stress an error of 4 per cent.

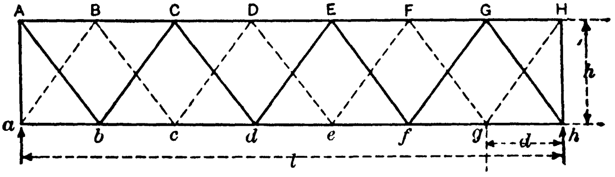


FIG. 1.

From these and other examples it may be concluded that in the form here considered, the number of panels being even, the results by the approximate method are well within the desired limit of accuracy.

216. *Double Triangular Truss With Odd Number of Panels.*—When the number of panels is odd then the symmetry of the systems is interrupted and their independence of action is more in question.

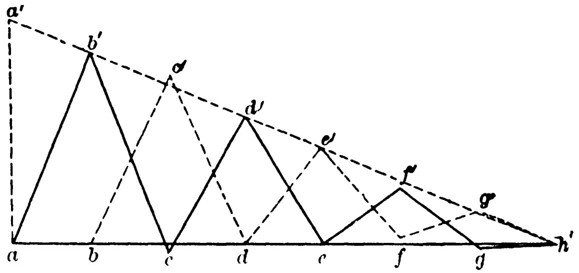


FIG. 2.

A peculiar condition arises in such systems when the structure is fully loaded. In the truss of Fig. 1, for example, under full uniform load, there is no shear in the panel *d e* and hence, apparently, no diagonals need be stressed in this panel. It is obvious that any stress which may exist in one of the diagonals is opposed by an equal stress in the other. Assuming independent systems, however, there will be a tension in each diagonal of $\frac{2}{7}P$, if a through bridge, and an equal compression, if a deck bridge, where P = joint load. It is sometimes assumed that under a uniform load, since there is no shear in the central panel, neither of the members dE or De are stressed, and that the two web systems break up into a different arrangement from that existing under an unsymmetrical load.

---

[*] "Theorie der Brücken," II Heft.

The maximum stresses calculated under the two different assumptions differ considerably. Thus, assuming independent systems, and all joints loaded, we have

$$\text{Vert. comp. } aB = \frac{9}{7}P$$

$$\text{Vert. comp. } Ab = \frac{12}{7}P$$

whereas, with members De and dE out of action, the values are $\frac{7}{7}P$ and $\frac{14}{7}P$ respectively, a difference of $\frac{2}{7}P$. Then in the chord members the assumption of independent systems gives stresses as follows (d = panel length, h = height):

$$DE = \frac{44\,d}{7\,h}P; \qquad de = \frac{40\,d}{7\,h}P;$$

$$CD = \frac{40\,d}{7\,h}P; \qquad cd = \frac{37\,d}{7\,h}P.$$

The assumption of zero stress in the members De and dE gives

$$CD = DE = de = \frac{42\,d}{7\,h}P;$$

$$cd = \frac{35\,d}{7\,h}P.$$

It will be found that, in general, the stresses in each diagonal under full load, obtained by the two different methods, differ by $\frac{2}{7}P$, and the stresses in each chord member differ by $\frac{2\,d}{7\,h}P$.

To determine the correct distribution of stress, and to illustrate further the analysis of multiple intersection trusses, an exact analysis will be made of the seven-panel truss of Fig. 1. Span = 7x24 = 168 ft.; height = 32 ft. The member *A a* is taken as the redundant member and its stress for a concentrated load at each of the joints *g*, *f*, and *e* is determined. To simplify the numerical work each joint load is

Member	A	u	$\frac{ul}{A}$	$\frac{u^2l}{A}$	LOAD AT *g*		LOAD AT *f*		LOAD AT *e*	
					S'	$\frac{S'ul}{A}$	S'	$\frac{S'ul}{A}$	S'	$\frac{S'ul}{A}$
A a	22	+1.00	+.182	.182						
A b	25	−1.25	−.250	.312	0	0	0	0	0	0
a B	25	−1.25	−.250	.312	−1.25	+ .31	−2.50	+ .62	−3.75	+ .93
B c	20	+1.25	+.312	.390	+1.25	+ .39	+2.50	+ .78	+3.75	+1.17
b C	20	+1.25	+.312	.390	0	0	0	0	0	0
C d	14	−1.25	−.446	.558	0	0	0	0	0	0
c D	14	−1.25	−.446	.558	−1.25	+ .56	−2.50	+1.12	−3.75	+1.67
D e	12	+1.25	+.521	.652	+1.25	+ .65	+2.50	+1.30	+3.75	+1.95
d E	12	+1.25	+.521	.652	0	0	0	0	0	0
E f	14	−1.25	−.446	.558	0	0	0	0	0	0
e F	14	−1.25	−.446	.558	−1.25	+ .56	−2.50	+1.12	+5.00	−2.23
F g	20	+1.25	+.312	.390	+1.25	+ .39	+2.50	+ .78	−5.00	−1.56
f G	20	+1.25	+.312	.390	0	0	+8.75	+2.73	0	0
G h	25	−1.25	−.250	.312	0	0	−8.75	+2.18	0	0
g H	25	−1.25	−.250	.312	+7.50	−1.87	−2.50	+ .62	+5.00	−1.25
H h	22	+1.00	+.182	.182	−6.00	−1.09	+2.00	+ .36	−4.00	− .72
A B	20	+ .75	+.112	.084	0	0	0	0	0	0
a b	20	+ .75	+.112	.084	+ .75	+ .08	+1.50	+ 17	+2.25	+ .25
B C	36	− .75	−.062	.047	−1.50	+ .09	−3.00	+ .19	−4.50	+ .28
b c	36	− .75	−.062	.047	+ .75	− .05	+1.50	− .09	+2.25	− .14
C D	50	+ .75	+.045	.034	−1.50	− .07	−3.00	− .14	−4.50	− .20
c d	50	+ .75	+.045	.034	+2.25	+ 10	+4.50	+ .20	+6.75	+ .30
D E	50	− .75	−.045	.034	−3.00	+ .14	−6.00	+ .27	−9.00	+ .40
d e	50	− .75	−.045	.034	+2.25	− .10	+4.50	− .20	+6.75	− .30
E F	50	+ .75	+.045	.034	−3.00	− .14	−6.00	− .27	−9.00	− .40
e f	50	+ .75	+.045	.034	+3.75	+ .17	+7.50	+ .34	+6.00	+ .27
F G	36	− .75	−.062	.047	−4.50	+ .28	−9.00	+ .56	−3.00	+ .19
f g	36	− .75	−.062	.047	+3.75	− .23	+2.25	− .14	+6.00	− .37
G H	20	+ .75	+.112	.084	−4.50	− .50	+1.50	+ .17	−3.00	− .34
g h	20	+ .75	+.112	.084	0	0	+5.25	+ .59	0	0
Σ				7.436		− .33		+13.26		− .10

taken as 7 instead of unity, and the panel length and height are called 3 and 4 units, respectively. The complete calculations are given in the subjoined table. The values of the cross-sections are approximately what would be required for a railroad structure designed for an E-50 loading.

The stresses in $A\ a$ are then as follows:

$$\text{for load at } g,\ A\ a = -\frac{-.33}{7.436} = +\ .045$$

$$\text{for load at } f,\ A\ a = -\frac{13.26}{7.436} = -\ 1.784$$

$$\text{for load at } e,\ A\ a = -\frac{-.\ 10}{7.436} = +\ .013.$$

For loads of unity instead of 7 these stresses become, respectively, + 0.006, − 0.255, and + 0.002.

For loads at b, c, and d the stresses in $A\ a$ are the same as in $H\ h$ for loads at g, f, and e, respectively. Knowing the stresses in $A\ a$ for loads at g, f, and e, the corresponding stresses in $H\ h$ are readily found. They are − 5.96, + 0.216, and − 3.99 respectively; hence, dividing by 7, we have for stress in $A\ a$ for unit loads at b, c, and d, the values − 0.852, + 0.031, and − 0.570, respectively.

These values for stress in $A\ a$ are the shears in the end panel $a\ b$, of the full system as shown in Fig. 1. The shears in the dotted system are found by subtracting the above values from the total shear. The shears in the end panels of the two systems, for unit loads at each joint, are given below, together with the shears calculated on the assumption of independent action of the two web systems, and the percentage of error of the latter values.

The assumption of independent systems therefore gives somewhat too small results for the dotted system and too large results for the full system. The chief reason for this variation is the fact that, as regards shears in the left half of the truss, the dotted system is somewhat more rigid than the full system. In the right half the full system is the more rigid and the maximum in $g\ H$ will be the same as the maximum in $A\ b$, etc. Fig. 2 shows the values above given, plotted as influence lines for end shears or vertical components in the diagonals of the two systems. The dotted line $a'\ h$ represents the total shear.

For all joints loaded the true shear in panel *d e* of the full system (vertical component in *d E*), is 1.638 − 2.00 = − 0.362, giving a tensile stress in *d E*. In the dotted system the shear (vertical component in *D e*) is 1.363 − 1.00 = + 0.363, giving a tension in *D e* equal to that in *d E*. The assumption of independent systems gives values of $\frac{2}{7}P$ or .286 for each, instead of .363.

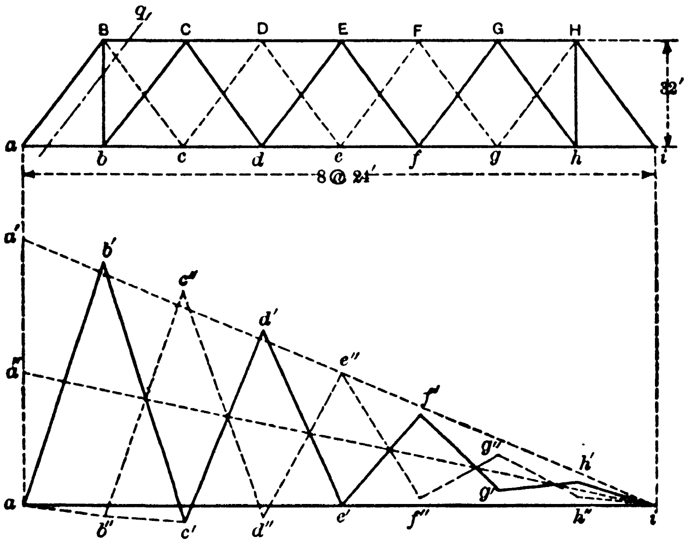
SHEARS IN THE END PANEL (FIG. 1) FOR UNIT JOINT LOADS

Joint Loaded.	Full System			Dotted System		
	True Shear.	Approx. Shear.	% Error.	True Shear.	Approx. Shear.	% Error.
g	−.006	0		+.149	+.143	−4.0
f	+.255	+.286	+12.2	+.031	0	
e	−.002	0		+.431	+.429	−0.5
d	+.570	+.572	+0.4	+.002	0	
c	−.031	0		+.745	+.714	−4.2
b	+.852	+.857	+0.6	+.005	0	
All Joints	+1.638	+1.715	+4.7	+1.363	+1.286	−5.7

The stresses in the centre diagonals being known, the stresses in *D E* and *d e* are readily obtained by moments, and thence the stresses in the other chord members. The resulting value for $DE = (6 + .363)\frac{d}{h}P = 6.363\frac{d}{h}P$, compression, and in *d e* a value of $(6 - .363)\frac{d}{h}P = 5.637\frac{d}{h}P$, tension. By the approximate methods, assuming independent systems and a shear in each system of $\frac{2}{7}P$, the values are $6.286\frac{d}{h}P$ and $5.714\frac{d}{h}P$, respectively. The errors involved are $.077\frac{d}{h}P$ in each case, or 1.2 per cent and 1.4 per cent. For other chord stresses the same absolute errors are involved, but the relative errors increase toward the end of the span.

From these calculations it may be concluded that the assumption of independent web systems, in the case of an odd number of panels, will lead to results substantially correct, but that the assumption of zero stress in the central web members, under a symmetrical loading, is considerably in error. If approximate methods are to be used, the stresses for all conditions of loading should therefore be determined on the basis of separate action of the two web systems.

217. The Double Triangular Truss with Inclined End Posts.—Fig. 3 shows a more common arrangement of end posts than that of the truss analyzed in the preceding article. Such an arrangement, however, makes the determination of stresses by approximate methods



Figs. 3 and 4.

somewhat more uncertain than in the cases already considered. The joints B and H must be considered as belonging to both systems and loads at these joints must be divided in some approximate manner. An equal division is generally assumed. It is to be observed also that a load applied at b, or brought there by the member bC of the full system, is carried up to B where it then exerts an influence on the dotted system BcD. The complication arises from the fact that the two systems are connected at point B, which is itself not rigidly supported, a condition not present in the trusses previously considered. Inasmuch as this arrangement is common in the various forms of multiple intersection trusses, an analysis will be made of the truss of Fig. 3. The same height and panel length will be taken as in Art. 216, and the same values of cross-sections. The values of A, u, and $u\,l/A$ for the various members are as follows, assuming $B\,b$ to be the redundant member:

Member.	A sq. in.	u	$\frac{u\,l}{A}$	Member.	A sq. in.	u	$\frac{u\,l}{A}$
B b	22	+1.00	+.182				
B c	20	−1.25	−.312	*B C*	36	+.75	+.062
b C	20	−1.25	−.312	*b c*	36	+.75	+.062
C d	14	+1.25	+.446	*C D*	50	−.75	−.045
c D	14	+1.25	+.446	*c d*	50	−.75	−.045
D e	12	−1.25	−.521	*D E*	58	+.75	+.039
d E	12	−1.25	−.521	*d e*	58	+.75	+.039
E f	12	+1.25	+.521	*E F*	58	−.75	−.039
e F	12	+1.25	+.521	*e f*	58	−.75	−.039
F g	14	−1.25	−.446	*F G*	50	+.75	+.045
f G	14	−1.25	−.446	*f g*	50	+.75	+.045
G h	20	+1.25	+.312	*G H*	36	−.75	−.062
g H	20	+1.25	+.312	*g h*	36	−.75	−.062
H h	22	−1.00	−.182				

The graphical method will be used in this case. Fig. 5 is a displacement diagram of the structure for the values of $\frac{u\,l}{A}$ given above. It is begun at h and $H\,h$ is assumed to remain vertical. The point b_1 on the diagram represents the cut end of $B\,b$, at a point near b. The vertical distance from b_1 to b in Fig. 5 represents the displacement of b_1 referred to b, and is the quantity $\Sigma\,\frac{u^2\,l}{A}$ of the formula. Call this deflection δ. Then, as in Art. 222, Part I, for a load unity on any joint as h, the quantity $\Sigma\,\frac{S'\,u\,l}{A}$ is equal to the deflection of h, due to forces of unity acting at b and b_1. Call this deflection δ_h. The stress in the member $B\,b$, due to unit load at h, is then equal to this deflection δ_h divided by the deflection δ. Before measuring the deflections of the several joints it will be necessary to draw a correction diagram, the points a and i standing fast. This is shown in dotted lines. From the diagram the following deflections are measured:

$$\begin{array}{ll} \delta_h = -\ 0.61 & \delta_d = -\ 4.72 \\ \delta_g = -\ 0.42 & \delta_c = +\ 0.42 \\ \delta_f = -\ 2.49 & \delta_b = -\ 6.61 \\ \delta_e = \ \ \ 0.0 & \end{array}$$

The deflection of b_1 with respect to $b = \delta = 7.20$. Dividing the above values by 7.20 we derive the stresses in *B b* for unit loads at the several joints. These stresses will be the shears on section q carried by the full system. The shears carried by the dotted system (member *B c*) will be found by subtracting from the total shears the several stresses in *B b*. The resulting values in the two systems are as follows:

Joint Loaded.	Shear on Sec. q.	Stress in *B b*.	Vert. Comp. *B c*.
h	.125	+ .084	+ .041
g	.250	+ .058	+ .192
f	.375	+ .345	+ .030
e	.500	00	+ .500
d	.625	+ .655	− .030
c	.750	− .058	+ .808
b	.875	+ .916	− .041
All loaded	3.500	+ 2.000	+ 1.500

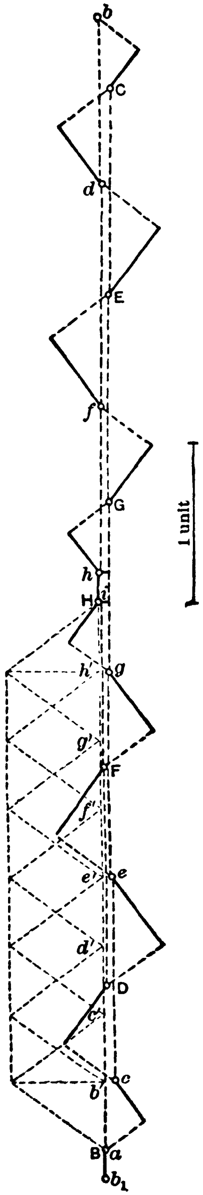


FIG. 5.

In Fig. 4 these results are plotted as influence lines for *B b* (shown by the full zigzag line), and for vertical component in *B c* (shown by the dotted line). The line $a'\,i$ is the line for total shear, and if the systems of bracing were entirely independent the vertices of the influence lines would fall on this line and on the zero line $a\,i$. The variation from this assumption is clearly shown in the figure and also in the table given above. It is noted that the load at *h* divides itself between the two systems, but that the larger part is carried by the full system (68 per cent); the load at *g* goes mainly to the dotted system, but a considerable part (about one-fourth) to the other system. At *f* the proportion taken by the dotted system is still smaller, while at *e* the load goes wholly to the system to which this joint belongs. For a full load, the member *B b* carries exactly the load belonging to the full system, and hence for this condition the two systems may be taken as independent and the maximum chord stresses so found will be correct. The stresses in *d E* and *E f* = 0.

For maximum stress in *b C* and *C d*, load to *d*. Maximum shear in full system = .084 + .058 + .345 + .655 = 1.142 = vertical component in *b C* and *C d*. By the approximate method, assuming the load at *h* to be equally distributed, shear $= \frac{1}{16} + \frac{3}{8} + \frac{5}{8} = 1.06$, an error of 8 per cent. For maximum stress in *d E* and *E f*, load to *f*. Shear = .084 + .058 + .345 = .487. By the approximate method, shear $= \frac{1}{16} + \frac{3}{8} = .437$, an error of 11 per cent. For maximum in *B c* load to *c*. Shear carried by *B c* = 1.500 + .041 = 1.541. Approximate method gives 1.562, an error of 1.4 per cent. For maximum in *c D* and *D e*, load to *e*. Shear = .763. Approximate method gives .812, an error of 6.4 per cent. For maximum in *e F* and *F g* load to *g*. Shear = .233 and approximate method gives .312, an error of 34 per cent.

From these calculations it may be concluded that in a truss of this type, with an even number of panels, the chord stresses are correctly found by assuming independent systems, and the web stresses will be found with a reasonable degree of accuracy on the same basis, it being assumed also that loads at *b* and *h* or *B* and *H* are equally divided between the two systems.

218. The Whipple Truss.—In the Whipple truss, Fig. 6, the approximate method of solution also leads to results very nearly correct. In the type shown in Fig. (*b*) the arrangement is somewhat more favorable than in Fig. (*a*), as in the latter case the loads at *c* or *C* are distributed over both systems in a manner not readily estimated. The two systems are not of equal flexibility, the full system containing six members up to D, and the dotted system eight; the former will therefore carry more than one-half of the shear going to the right support. The entire shear is, however, small, and it may be assumed without material error, as in the triangular truss, that one-half is carried by each system.

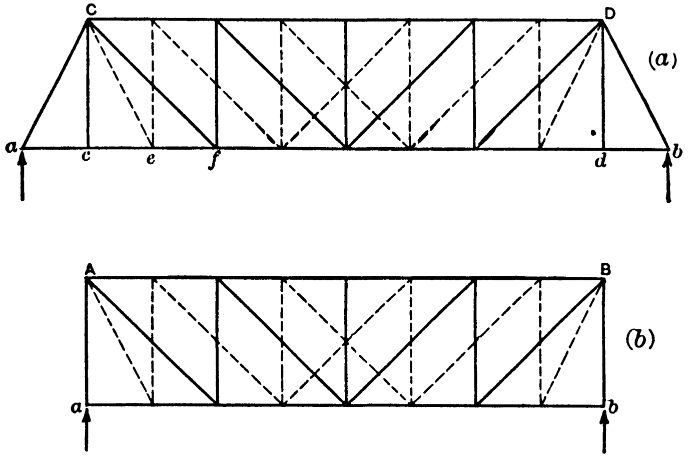


FIG. 6.

Loads at other joints will affect to a small extent the systems not belonging to the joint in question, but except for point c, independent systems may be assumed. The chord stresses will be correctly obtained for full load on the basis of independent systems. Winkler finds in a truss of 10 panels of type (*b*) an error of only 2 per cent in the maximum stress in web members.

219. Other Multiple Systems.—Fig. 7 illustrates a case of a double

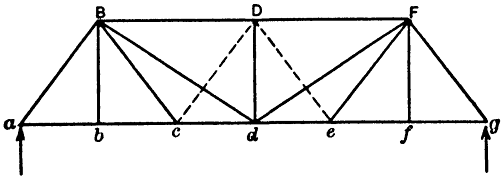


FIG. 7.

system of web members in which the two systems are very unlike in flexibility and the structure is not homogeneous. The dotted members may be considered as redundant. The loads at c and e can be carried by members $B\,c$ and $e\,F$ alone, and this path is much more rigid than the path over the dotted members and thence down *D d* and up the inclined members *d F* and *b B*. As a result, the dotted members will be found to take much less than half the loads at *c* and *e*, and the stresses can only be guessed at unless the theory of redundant members be applied. Such irregular systems are undesirable and uneconomical as they cannot be designed so that the unit stresses are of the proper values in all the members.

Again, if the double system is to be used in a cantilever arm, as in Fig. 8, an odd number of panels as shown in Fig. (*a*) is better than an even number as in Fig. (*b*). The heavy concentration *P* is carried by the member *a B* to the point *B* where it is divided between the two systems. In Fig. (*a*) the two paths over which it travels to the support on the right are *B b C d E f*, etc., and *B c D e F*, etc. These two paths contain an equal number of members and, for equal working stresses, have about the same rigidity. In Fig. (*b*) the number of panels from *b* to *i* is odd, and the dotted path from *B* to *i* contains nine members and the full path only seven. The two paths will therefore be quite unequal in rigidity and the full system will carry considerably more load than the other, assuming the same size of members. Being forced to deflect equally at *B* it follows that whatever the size of the members of the two systems, the unit stress in the dotted system will be considerably less than in the full system. Furthermore, this condition cannot be avoided no matter what the size of the several members.

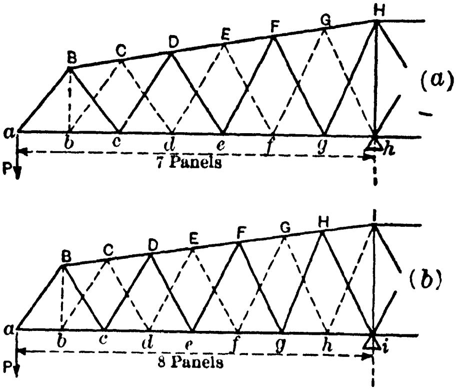


FIG. 8.

Other illustrations of a similar sort might be given, as, for example, the combination of an arch and a truss. The arch, being the more rigid under symmetrical loads, will carry nearly all the load in such cases and the truss will serve mainly as a stiffening truss under moving loads, as in the case of the suspension bridge.

220. General Conclusions Regarding Multiple Systems.—From the results of the preceding analysis, and also from a general view of the matter, it may be concluded that, where multiple systems are so arranged as to be very nearly alike in design so that their relative rigidities will be about the same, the loads will distribute themselves nearly equally among the several systems, a condition which permits them to be assumed to act independently in the analysis. The slight uncertainty of stress in such systems is of small moment as compared to the advantages which they may afford in the case of very large structures especially with reference to convenient panel length and size of joint details. In fact their use will, in some respects, give a more homogeneous design and one freer from secondary stresses than a system of sub-paneling such as employed in the Pettit truss. For large structures where the web members of a double system are large enough to be of satisfactory size such a system is well worth consideration.

In one respect a structure with redundant members is theoretically not, in general, the most economical form. As illustrated by the foregoing discussion a structure with redundant members cannot be so designed that all members shall be subjected to any prescribed working stress. If there is, for example, one redundant member, the distortion of this member is fixed by the distortion of all the others; that is to say, its unit stress is a definite function of the unit stresses in all the other members, no matter what its size. In the example of the swing bridge of Art. 72 it was shown that, whatever the size of the centre diagonals, their unit stresses would be very small. This defect in structures with redundant members is of importance only in cases where the several parts or systems are of unequal flexibility, as in the example cited and in such a truss as shown in Fig. 7. This condition shows, however, that the principle of equal flexibility or rigidity of each part of a redundant system is essential not only for convenient calculation but for economy of design.

Again, in the double system with an odd number of panels, it is seen that under full load, tension exists in both of the diagonals in the central panel, some shear being transmitted each way across this panel. This is, in general, an uneconomical arrangement and shows that an odd number of panels is not desirable. In a multiple intersection truss the principle of symmetrical arrangement and equal flexibility should therefore prevail.

221. The Double Triangular Truss with Verticals.—Not infrequently a case will arise where an analysis is required of a truss of the type shown in Fig. 9. This truss may be looked upon as a double triangular truss with the two systems connected at each panel point. Under this assumption the two diagonals of any panel are made of about equal size and the verticals are all alike (excepting the end posts), and of sufficient size to equalize the loads on the two systems, or to carry a half-joint load in tension, in case of a through bridge, or in compression, in case of a deck bridge.

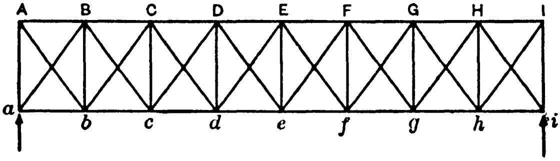


FIG. 9.

In some respects a truss of this kind is advantageous as compared to the double system without verticals. It provides convenient points of attachment for the floor beams in the case of a through bridge, it avoids excessive variations in web stress due to concentrated loads passing from one system to another, and it reduces the secondary stresses, as will be shown in Chapter VII.

Under a full uniform load the verticals perform little service, as the deformations of the two diagonal systems are nearly alike and hence the verticals can receive but little stress. In this respect they are in a somewhat similar condition to the vertical stiffeners of a plate girder. Under partial loads, however, they tend rapidly to equalize the stresses in the two diagonal systems. To just what extent this is accomplished and the errors involved in an approximate analysis based on such an assumption, together with the probably maximum stress in any one vertical, may be approximately determined by the analysis of a typical problem. Such an analysis will also serve to illustrate a case of several redundant members.

Assume dimensions as given in Fig. 10. The cross-sections of the members are also given in the figure. The two diagonals in each panel are assumed to be of equal section, and all verticals except the end posts are 12 sq. in. in area. The panels are numbered as shown.

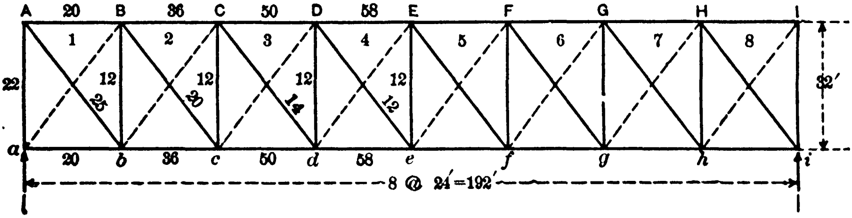


FIG. 10.

There are eight redundant members in this structure, each intermediate vertical being redundant, and, in addition, one such member due to the double system. It will be more convenient in the analysis, however, to assume one of the diagonals in each panel as redundant, which, if removed, gives a Pratt system of bracing with no redundant members. The dotted diagonals will be assumed as the redundant members and their stresses will be called S_1, S_2, S_3, etc., the subscript corresponding to the panel in question.

222. *General Equations.*—With the truss loaded in any manner a series of eight equations may be written out, similar to eq. (3)′, Art. 212, in which the stresses in the eight redundant members appear as the unknowns. These are then determined by a solution of the eight equations. The modulus E will be omitted as it is constant. In these equations u_1, u_2, u_3, etc., are the stresses in the members due to tensile stresses of one pound in the several redundant members, No. 1, No. 2, etc., all other redundant members being removed excepting the one in question; and S' is the stress in any member due to the given loads, all the redundant members being removed in this case. It will be noted that the first equation contains the several products $\Sigma \frac{u_1^2 l}{A}$, $\Sigma \frac{u_1 u_2 l}{A}$, $\Sigma \frac{u_1 u_3 l}{A}$, $\Sigma \frac{u_1 u_4 l}{A}$, etc., and the second equation the products $\Sigma \frac{u_2 u_1 l}{A}$, $\Sigma \frac{u_2^2 l}{A}$, $\Sigma \frac{u_2 u_3 l}{A}$, $\Sigma \frac{u_2 u_4 l}{A}$, etc. These products are independent of the loading on the structure and will be first determined.

223. *Values of* $\Sigma \frac{u^2 l}{A}$, *etc.*—Let Fig. 11 represent any panel, member No. 6 being the redundant member. Obviously a tension in this member will cause no stress in any member of the truss except in the quadrilateral *M m n N*. Represent the dimensions as shown, and number the members 1, 2, 3, 4, and 5. Then the stresses u and the products $\frac{u^2 l}{A}$ will be as given below (the term $\frac{l}{A}$ for the redundant member is included by considering that for this member $u = +1$). The sectional areas of the chords are denoted by A_c and A'_c, of the verticals by A_v and A'_v, and of the diagonals by A_d and A'_d.

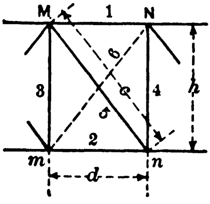


FIG. 11.

Member.	l	A	u	$\frac{u\,l}{A}$	$\frac{u^2 l}{A}$
1	d	A_c	$-\frac{d}{c}$	$-\frac{d^2}{c A_c}$	$+\frac{d^3}{c^2 A_c}$
2	d	A'_c	$-\frac{d}{c}$	$-\frac{d^2}{c A'_c}$	$+\frac{d^3}{c^2 A'_c}$
3	h	A_v	$-\frac{h}{c}$	$-\frac{h^2}{c A_v}$	$+\frac{h^3}{c^2 A_v}$
4	h	A'_v	$-\frac{h}{c}$	$-\frac{h^2}{c A'_v}$	$+\frac{h^3}{c^2 A'_v}$
5	c	A_d	$+1.0$	$+\frac{c}{A_d}$	$+\frac{c}{A_d}$
6	c	A'_d	$+1.0$	$+\frac{c}{A'_d}$	$+\frac{c}{A'_d}$

In this case the areas of the two chord members of any panel are equal, and also the areas of the two diagonals. The areas of the two verticals are different for panels 1 and 8. We have then, in general, for any panel

$$\Sigma \frac{u^2 l}{A} = \frac{2\,d^3}{c^2 A_c} + \frac{h^3}{c^2 A_v} + \frac{h^3}{c^2 A'_v} + \frac{2\,c}{A_d}. \quad . \quad . \quad . \quad (9)$$

For simplicity the values of d, h, and c will be taken at 3, 4, and 5 units. Substituting these values in (9), and the values of A_c, A_v, and A_d for the several panels, we get values of $\Sigma \frac{u^2 l}{A}$ (including l/A for redundant members) as follows:

Panel	$\Sigma \frac{u^2 l}{A}$
1	0.838
2	0.987
3	1.184
4	1.297
5	1.297
6	1.184
7	0.987
8	0.838

The values of $u_1 u_2$, $u_1 u_3$, etc., are very readily found in this case. The only members having a u_1 are those of the quadrilateral $A\ a\ b\ B$, and the only members having a u_2 are those of the quadrilateral $B\ b\ c\ C$, hence the member $B\ b$ is the only one having both a u_1 and a u_2. For this member the value of $u_1 = -\frac{h}{c}$ and the value of u_2 also equals $-\frac{h}{c}$, hence $\frac{u_1 u_2 l}{A} = + \frac{h^3}{c^2 A_v} = + 0.213$. The only members having a u_3 are those of panel 3, and hence no member has both a u_1 and a u_3, therefore $\Sigma \frac{u_1 u_3 l}{A} = 0$. Likewise $\Sigma \frac{u_1 u_4 l}{A} = 0$, etc. It follows, in the same manner, that in the second equation of (3), $\Sigma \frac{u_2 u_1 l}{A} = + 0.213$, and $\Sigma \frac{u_2 u_3 l}{A} = + 0.213$, and all other products of this kind are zero. Likewise in the third equation, $\Sigma \frac{u_3 u_2 l}{A} = + 0.213$ and $\Sigma \frac{u_3 u_4 l}{A} = + 0.213$, etc., etc.

224. *Values of* $\Sigma \frac{S' u l}{A}$.—The analysis will be made for a single load at each of the joints h, g, f, and e, so that the problem will be completely solved for any combination of loads. Inasmuch as the five members of a panel are the only ones that have a stress u due to the stress in any one redundant member, the values of $\Sigma \frac{S' u l}{A}$ are readily calculated. Let $M\ m\ n\ N$, Fig. 12, represent any panel of the truss and suppose the truss to be loaded with a single load at any joint, either on the left or right. Let V = resultant of all forces on the left of a section through the panel, assumed as acting at a distance a from m, and taken as positive upward. Then the values of S' for the members of panel $m\,n$ are as follows: $m\,n = + V a/h$, $M\,N = - V(a + d)/h$; $M\,m = N\,n = - V$, $M\,n = V c/h$. If the load is applied at n then the stress in $N\,n = P - V$ The values of $\frac{S' u l}{A}$ for any panel are then as follows:

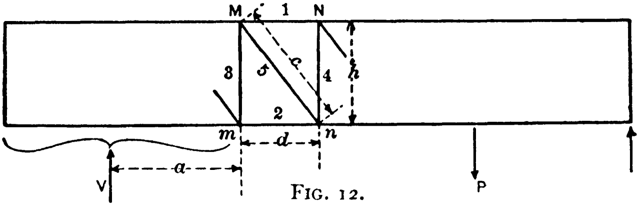


FIG. 12.

Member.	$\frac{S' u l}{A}$
1	$\frac{V(a + d)\,d^2}{h c A_c}$
2	$-\frac{V a d^2}{h c A'_c}$
3	$\frac{V h^2}{c A_v}$
4	$\frac{V h^2}{c A'_v}$
5	$\frac{V c^2}{h A_d}$

For $A_c = A'_c$ and $A_v = A'_v$ we have

$$\Sigma \frac{S' u l}{A} = V \left[\frac{d^3}{h c A_c} + \frac{2 h^2}{c A_v} + \frac{c^2}{h A_d} \right]. \quad . \quad . \quad . \quad . \qquad (10)$$

Where the load is located at n, then the value of $\Sigma \frac{S' u l}{A}$ is increased by $-\frac{P h^2}{c A_v}$. In the end panels the term $\frac{2 h^2}{c A_v}$ is $\frac{h^2}{c A_v} + \frac{h^2}{c A'_v}$.

The remainder of the calculations are given in the following table.

VALUES OF $\frac{S'\,u\,l}{A}$ FOR JOINT LOADS OF 8 UNITS

Panel.	$\frac{d^3}{h\,c\,A_c}$	$\frac{2\,h^2}{c\,A_v}$	$\frac{c^2}{h\,A_d}$	$\Sigma\frac{S'\,u\,l}{A}$ for $V = 1$.	$\Sigma\frac{S'\,u\,l}{A}$ for $P = 8$			
					Load at *h*.	Load at *g*.	Load at *f*.	Load at *e*.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	.067	.413	.250	.730	+ .730	+1.460	+2.190	+2.920
2	.037	.533	.312	.882	+ .882	+1.764	+2.646	+3.528
3	.027	.533	.447	1.007	+1.007	+2.014	+3.021	+4.028
4	.023	.533	.521	1.077	+1.077	−2.154	+3.231	+2.176
5	.023	.533	.521	1.077	+1.077	+2.154	+1.099	−4.308
6	.027	.533	.447	1.007	+1.007	− .118	−5.035	−4.028
7	.037	.533	.312	.882	−1.250	−5.292	−4.410	−3.528
8	.067	.267	.250	.584	−4.088	−3.504	−2.920	−2.336

In Col. (5) are given the sums of the values in Cols. (2), (3), and (4). The load P is assumed to be eight units for sake of convenience in the calculation of shears. The several values of $\Sigma\frac{S'\,u\,l}{A}$ are found by multiplying the values in Col. (5) by the respective shears. Thus for load of 8 at h, the shear in all panels from 1 to 7 is unity and positive, hence for all panels from 1 to 6 the value of $\Sigma\frac{S'\,u\,l}{A}$ is the same as given in Col. (5). For panel No. 7 the load is at h and hence, as shown on page 301, the quantity $-\frac{P\,h^2}{c\,A_v}$, or -2.132, is to be added to this sum, giving a total of -1.250. In panel No. 8 the value of $\frac{2\,h^2}{c\,A_v}$ is given as .267 only, as the member $I\,i$ has zero stress for S' in all cases. For load at h, the shear in panel 8 $= -7$, hence $\Sigma\frac{S'\,u\,l}{A} = -7 \times .584 = -4.088$. For a load at g, the shear on the left of $g = 2$, and on the right it is -6, hence the values of $\Sigma\frac{S'\,u\,l}{A}$ are found by multiplying the values in Col. (5) by these shears respectively, and for the panel No. 6 adding the quantity -2.132 as before. The values for loads at f and e are found in a similar manner. We are now prepared to write out the several equations for a load at any joint.

225. *Solution of Equations.*—Following the general eq. (3) the several equations between the eight redundant stresses for a load at *h* are as follows:

$$\begin{array}{lrrrrl}
(1) & .838\,S_1 + & .213\,S_2 & & & = -\ \ .730 \\
(2) & .213\,S_1 + & .987\,S_2 + & & .213\,S_3 & = -\ \ .882 \\
(3) & .213\,S_2 + & 1.184\,S_3 + & & .213\,S_4 & = -\ 1.007 \\
(4) & .213\,S_3 + & 1.297\,S_4 + & & .213\,S_5 & = -\ 1.077 \\
(5) & .213\,S_4 + & 1.297\,S_5 + & & .213\,S_6 & = -\ 1.077 \\
(6) & .213\,S_5 + & 1.184\,S_6 + & & .213\,S_7 & = -\ 1.007 \\
(7) & .213\,S_6 + & .987\,S_7 + & & .213\,S_8 & = +\ 1.250 \\
(8) & .213\,S_7 + & .838\,S_8 & & & = +\ 4.088
\end{array}$$

The solution of these eight equations is a comparatively simple matter on account of the manner in which the unknowns appear. The work is given below in detail, and the explanation follows:

$$\begin{array}{lrl}
(1') & S_1 + \ \ .254\,S_2 & = -\ \ .872 \\
(2') & S_1 + 4.63\ \ S_2 + S_3 & = -\ 4.14 \\ \hline
(a\ ) & 4.38\ \ S_2 + S_3 & = -\ 3.27 \\ \hline
(a') & S_2 + \ \ .228\,S_3 & = -\ \ .747 \\
(3') & S_2 + 5.56\ \ S_3 + S_4 & = -\ 4.73 \\ \hline
(b\ ) & 5.33\,S_3 + S_4 & = -\ 3.98 \\ \hline
(b') & S_3 + \ \ .188\,S_4 & = -\ \ .748 \\
(4') & S_3 + 6.09\ \ S_4 + S_5 & = -\ 5.06 \\ \hline
(c) & 5.90\ \ S_4 + S_5 & = -\ 4.31 \\ \hline
(c') & S_4 + \ \ .170\,S_5 & = -\ \ .730 \\
(5') & S_4 + 6.09\ \ S_5 + S_6 & = -\ 5.06 \\ \hline
(d) & 5.92\ \ S_5 + S_6 & = -\ 4.33 \\ \hline
(d') & S_5 + \ \ .169\,S_6 & = -\ \ .732 \\
(6') & S_5 + \ 5.56\ \ S_6 + S_7 & = -\ 4.73 \\ \hline
(e\ ) & 5.39\ \ S_6 + S_7 & = -\ 4.00 \\ \hline
(e') & S_6 + \ \ .186\,S_7 & = -\ \ .742 \\
(7) & S_6 + 4.63\ \ S_7 + S_8 & = +\ 5.87 \\ \hline
(f) & 4.44\ \ S_7 + S_8 & = +\ 6.61 \\ \hline
(f') & S_7 + \ \ .223\,S_8 & = +\ 1.490 \\
(8') & S_7 + 3.93\ \ S_8 & = +19.19 \\ \hline
(g\ ) & 3.71\ \ S_8 & = +17.70 \\
(g') & S_8 & = +4.770
\end{array}$$

Dividing (1) and (2) by the coefficients of S_1 gives (1′) and (2′), and subtracting we get (*a*); then from (*a*) and (3) in like manner we get (*a*′), (3′), and (*b*); then from (*b*) and (4) we get (*c*), etc. Finally, from (*g*) we get $S_8 = +4.77$. Substituting back in (*f*′) we get S_7; then in (*e*′) we get S_6, etc. The values of the stresses in the several redundant members are given in the table below.

In like manner the stresses for loads of 8 units at the other joints are found. The results are all given in the following table:

VALUES OF STRESSES IN REDUNDANT MEMBERS

Panel.	Stresses S_1, S_2, etc., for a Load of 8 Units at Each Joint.			
	Load at *h*.	Load at *g*.	Load at *f*.	Load at *e*.
1	− .71	−1.44	−2.15	−2.88
2	− .61	−1.21	−1.82	−2.38
3	− .63	−1.25	−1.84	−2.65
4	− .63	−1.23	−2.00	−1.76
5	− .59	−1.37	−1.16	+3.26
6	− .82	− .53	+3.94	+2.35
7	+ .43	+4.86	+3.03	+2.62
8	+4.77	+2.93	+2.73	+2.12

For loads on the other joints the stresses can readily be obtained from the above values by considering the total shear in the panel. Thus for load at *d*, symmetrical with *f*, the stress in the member *a B* is the same as that in *Hi* for load at *f*. But for load at *f* the stress in *h I* = + 2.73, and as the shear = − 5.0, the stress in *H i* = $5.00 \times \frac{5}{4} -$ 2.73 = 3.52 compression, which is the desired stress in *a B*.

226. *Stresses and Influence Lines for the Diagonals.*—For convenience in determining maximum values the stresses in all the diagonals, both the redundant members and others, have been calculated for all panels in the left half of the truss, and for a load of eight units at each joint. The results are tabulated in table on next page.

The values of the stresses in the two diagonals of a panel indicate to what extent the verticals cause the shears to be equally divided between the two members. For loads somewhat remote from the panel in question the stresses are nearly equal, as, for example, for a load at

STRESSES IN ALL DIAGONALS FOR JOINT LOADS OF 8 UNITS

Joint Loaded.	Panel 1.		Panel 2.		Panel 3.		Panel 4.	
	a B	*A b*	*b C*	*B c*	*c D*	*C d*	*d E*	*D e*
h	− .71	+ .54	− .61	+ .64	− .63	+ .62	− .63	+ .62
g	−1.44	+1.06	−1.21	+1.29	−1.25	+1.25	−1.23	+1.27
f	−2.15	+1.60	−1.82	+1.93	−1.84	+1.91	−2.00	+1.75
e	−2.88	+2.12	−2.38	+2.62	−2.65	+2.35	−1.76	+3.24
d	−3.52	+2.73	−3.22	+3.03	−2.31	+3.94	+2.59	−1.16
c	−4.57	+2.93	−2.64	+4.86	+1.97	− .53	+1.13	−1.37
b	−3.98	+4.77	+1.68	+ .43	+ .43	− .82	+ .66	− .59
All joints	−19.25	+15.75	−10.20	+14.80	−6.28	+8.72	−1.24	+3.76

h or *g*, the stresses in the diagonals of panels 2, 3, and 4 are nearly equal. In panel 1 the member *a B* invariably receives the greater stress on account of it being the more direct path to the support at *a*. For a load at *f* the stresses in panel 4 are 2.00 and 1.75, showing a greater variation than for load at *g;* and for load at *e*, they are 1.76 and 3.24, or nearly as 1 to 2. In panel 3 they are 2.35 and 2.65 respectively.

These results are graphically shown in Fig. 13, plotted as influence lines for diagonal stress in the four panels. In each case the straight line *a′ i* represents one-half of the shear. Fig. (*f*) shows the influence line for diagonal stress in panel No. 4 if no verticals are used. The equalizing effect of the verticals is very apparent. From these influence lines, the maximum stresses in the members *B c*, *C d*, and *D e*, are found to be about 10 per cent (11.4, 10.9, and 9.4 per cent) larger than if calculated on the basis of their carrying one-half of the shear. The maximum stresses in *b C*, *c D*, and *d E* are correspondingly smaller. In the end panel, the stress in *a B* is about 7.5 per cent greater and in *B a* 7.5 per cent less than one-half of the shear. Without the verticals, the stresses range from 15 per cent to 30 per cent greater than half the shear.

227. *Stresses in the Intermediate Verticals.*—The stress in any vertical is conveniently found from the stresses in the two diagonals meeting the vertical at the upper chord joint. A detailed analysis for loads of 8 units at the various joints gives the following results:

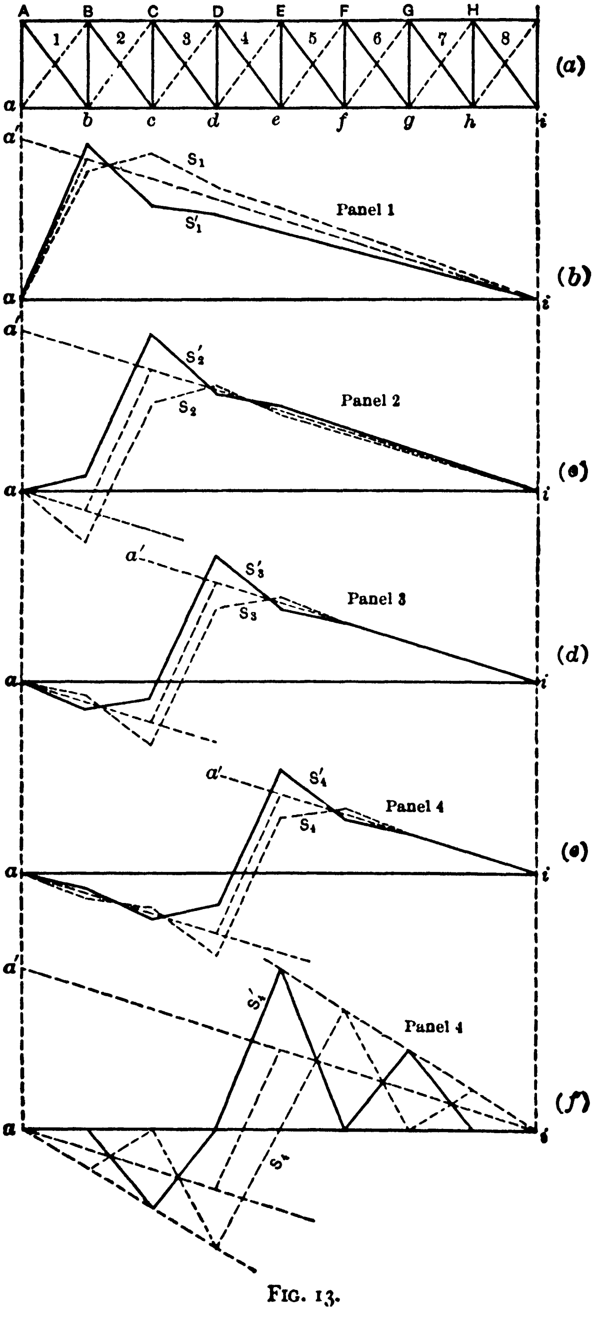

A
B
C
D
E
F
G
H
I
1
2
3
4
5
6
7
8
(a)
a
b
c
d
e
f
g
h
i
a'
S_1
S'_1
Panel 1
(b)
a
i
a'
S'_2
S_2
Panel 2
(c)
a
i
a'
S'_3
S_3
Panel 3
(d)
a
i
a'
S'_4
S_4
Panel 4
(e)
a
i
a'
S'_4
Panel 4
(f)
S_4
a
i

FIG. 13.

STRESSES IN THE VERTICALS FOR JOINT LOADS OF 8 UNITS

Joint Loaded.	*Bb*	*Cc*	*Dd*	*Ee*
h	+ .06	− .02	+ .01	− .02
g	+ .12	− .03	− .02	+ .08
f	+ .17	− .08	+ .07	− .47
e	+ .21	+ .02	− .47	+2.81
d	+ .39	− .58	+2.78	− .47
c	− .23	+2.53	− .48	+ .08
b	+2.84	− .69	+ .13	− .02
All joints	+3.56	+1.15	+2.02	+1.99

It is seen that the stresses due to loads at joints remote from the member in question are small. For a load at the adjacent joint the stresses are generally from 5 to 8 per cent of the joint load and compressive; and for a load at the joint, it ranges from 2.53 to 2.84 units, or from 32 per cent to 35 per cent of a joint load, and is tension. If the verticals were of larger section these stresses would be somewhat larger, approaching one-half of a joint load, and the diagonal stresses would be more nearly equalized; if smaller, their stresses would be less and the diagonal stresses more unequal. For loads at the upper joints, the stresses in the verticals would be compressive and about the same in amount as above given. For equal loads at upper and lower joints the stresses in the verticals would be practically zero.

228. *General Conclusions.*—For the double triangular truss with verticals it may be concluded from the above analysis that the stresses in the diagonals will be practically equalized except for loads at joints adjacent to the panel in question, and that the maximum stresses in the diagonals of any panel will not vary greatly from the half shear (about 10 per cent in the truss here analyzed). In the analysis of such a structure, therefore, closely approximate results will be reached on the basis of equal distribution of the shear. The errors in chord stress under this assumption will be very small near the centre and will increase toward the end, where they are the same as in the web members.

The maximum stresses in the verticals will be less than one-half of the excess of the joint load of the loaded chord above that of the unloaded chord. In a through bridge this stress will be tension and in a deck bridge, compression. The presence of verticals avoids excessive web stresses and local deformations, which occur in the ordinary double intersection truss, due to concentrated loads that may happen to be spaced about two panels apart. This condition is favorable to low secondary stresses as demonstrated more fully in Chapter VII.

229. General Case of Double Diagonals in a Pratt System.—In Art. 224, Part I, an example of equal double diagonals was analyzed, assuming the chord members to be indefinitely large and the verticals six times as large as the diagonals (as in an upper lateral system). In that case it was shown that the stresses in the two diagonals were nearly equal but of opposite sign. A more general solution will here be given so that the relative stresses for any such case can be determined by substitution. It is here assumed that both diagonals are in action. If both members are slender bars, this condition will require an initial tension equal to the calculated maximum compression.

Fig. 14 represents any such panel. V is the shear in the panel, or resultant of forces on the left, and is applied a distance a from m. Let the sectional areas be represented by A_c, A'_c, etc., as given in the table following. If member 6 be taken as the redundant member the values of u, $u\,l/A$, S', etc., are as follows:

Member.	l	A	u	$\frac{u\,l}{A}$	$\frac{u^2\,l}{A}$	S'	$\frac{S'\,u\,l}{A}$
1	d	A_c	$-\frac{d}{c}$	$-\frac{d^2}{cA_c}$	$\frac{d^3}{c^2A_c}$	$-V\frac{a+d}{h}$	$+V\frac{(a+d)d^2}{h\,c\,A_c}$
2	d	A'_c	$-\frac{d}{c}$	$-\frac{d^2}{cA'_c}$	$\frac{d^3}{c^2A'_c}$	$+V\frac{a}{h}$	$-V\frac{a\,d^2}{h\,cA'_c}$
3	h	A_v	$-\frac{h}{c}$	$-\frac{h^2}{cA_v}$	$\frac{h^3}{c^2A_v}$	$-V$	$+\frac{V\,h^2}{c\,A_v}$
4	h	A'_v	$-\frac{h}{c}$	$-\frac{h^2}{cA'_v}$	$\frac{h^3}{c^2A'_v}$	$-V$	$+\frac{V\,h^2}{c\,A'_v}$
5	c	A_d	$+1$	$-\frac{c}{A_d}$	$\frac{c}{A_d}$	$+V\frac{c}{h}$	$+V\frac{c^2}{h\,A_d}$
6	c	A'_d	$+1$	$-\frac{c}{A'_d}$	$\frac{c}{A'_d}$	0	

The value of the stress in the redundant member is then written out from the general formula $S_r = -\dfrac{\Sigma \dfrac{S' u l}{A}}{\Sigma \dfrac{u^2 l}{A}}$. It is, therefore,

$$S_6 = -V \frac{d^2 c\left(\dfrac{a+d}{A_c} - \dfrac{a}{A'_c}\right) + h^3 c\left(\dfrac{1}{A_v} + \dfrac{1}{A'_v}\right) + \dfrac{c^4}{A_d}}{d^3 h\left(\dfrac{1}{A_c} + \dfrac{1}{A'_c}\right) + h^4\left(\dfrac{1}{A_v} + \dfrac{1}{A'_v}\right) + c^3 h\left(\dfrac{1}{A_d} + \dfrac{1}{A'_d}\right)}. \quad (11)$$

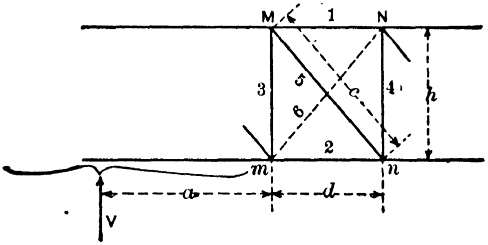


FIG. 14.

If the chords are nearly equal and the verticals also, as in the case of a lateral system, then

$$S_6 = -\frac{V}{2} \cdot \frac{\dfrac{d^3 c}{A_c} + 2\dfrac{h^3 c}{A_v} + \dfrac{c^4}{A_d}}{\dfrac{d^3 h}{A_c} + \dfrac{h^4}{A_v} + \dfrac{c^3 h}{2}\left(\dfrac{1}{A_d} + \dfrac{1}{A'_d}\right)}. \quad . \quad . \quad . \quad (12)$$

Again, if the chords and verticals are very large as compared to the diagonals, the terms containing these areas disappear and we have, approximately,

$$S_6 = -V\frac{c}{h} \cdot \frac{A'_d}{A_d + A'_d}, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

that is, the stress in each diagonal will be proportional to its area, and the unit stresses in the two will be the same.

The above analysis excludes from consideration the effect of double diagonals in adjacent panels. The exact solution of such a case may be made in the same manner as the problem of the previous article.

230. The Counters of a Baltimore or Pettit Truss.—The true distribution of stress in the diagonals of a Baltimore or Pettit truss may readily be determined in a manner similar to that used in the preceding article. Fig. 15 represents a panel in a Baltimore truss in which a counter *N O* or *m O* is required. It is assumed that there is sufficient initial tension in the members so that none will become slack. Assume *O N* (No. 8) to be the redundant member, V = shear on section cutting panel *o n*, and P = load applied at *O*. The required calculations are given below. The areas of sections are designated as in Art. 229, the areas of the diagonals being called A_{d1} and A_{d2} for *M n*, and A'_{d1} and A'_{d2} for *m N*. The values of u are the same as in Art. 260.

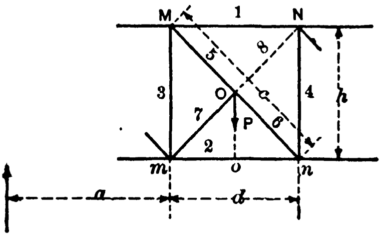


FIG. 15.

Member.	l	A	u	$\frac{u\,l}{A}$	$\frac{u^2 l}{A}$	S'	$\frac{S'\,u\,l}{A}$
1	d	A_c	$-\frac{d}{c}$	$-\frac{d^2}{c\,A_c}$	$\frac{d^3}{c^2 A_c}$	$-V\frac{a+d}{h}$	$+V\frac{(a+d)\,d^2}{h\,c\,A_c}$
2	d	A'_c	$-\frac{d}{c}$	$-\frac{d^2}{c\,A'_c}$	$\frac{d^3}{c^2 A'_c}$	$+V\frac{a}{h}+\frac{P\,d}{2\,h}$	$-\frac{V\,a\,d^2}{h\,c\,A'_c}-\frac{P\,d^3}{2\,h\,c\,A'_c}$
3	h	A_v	$-\frac{h}{c}$	$-\frac{h^2}{c\,A_v}$	$\frac{h^3}{c^2\,A_v}$	$-\left(V+\frac{P}{2}\right)$	$+\left(V+\frac{P}{2}\right)\frac{h^2}{c\,A_v}$
4	h	A'_v	$-\frac{h}{c}$	$-\frac{h^2}{c\,A'_v}$	$\frac{h^3}{c^2 A'_v}$	$-V$	$+V\frac{h^2}{c\,A'_v}$
5	$\frac{c}{2}$	A_{d1}	$+1$	$+\frac{c}{2\,A_{d1}}$	$\frac{c}{2\,A_{d1}}$	$+\left(V+\frac{P}{2}\right)\frac{c}{h}$	$+\left(V+\frac{P}{2}\right)\frac{c^2}{2\,h\,A_{d1}}$
6	$\frac{c}{2}$	A_{d2}	$+1$	$+\frac{c}{2\,A_{d2}}$	$\frac{c}{2\,A_{d2}}$	$+V\frac{c}{h}$	$+\quad V\frac{c^2}{2\,h\,A_{d2}}$
7	$\frac{c}{2}$	A'_{d1}	$+1$	$+\frac{c}{2\,A'_{d1}}$	$\frac{c}{2\,A'_{d1}}$	$-\frac{P\,c}{2\,h}$	$-\frac{P\,c^2}{4\,h\,A'_{d1}}$
8	$\frac{c}{2}$	A'_{d2}	$+1$	$+\frac{c}{2\,A'_{d2}}$	$\frac{c}{2\,A'_{d2}}$	0	

Then, as before, $S_8 = -\dfrac{\dfrac{S'ul}{A}}{\Sigma \dfrac{u^2 l'}{A}}$, from which we derive

$$S_8 = -\frac{V\left[d^2c\left(\frac{a+d}{A_c} - \frac{a}{A'_c}\right) + h^3c\left(\frac{1}{A_v} + \frac{1}{A'_v}\right) + \frac{c^4}{2}\left(\frac{1}{A_{d_1}} + \frac{1}{A_{d_2}}\right)\right] - \frac{P}{2}\left[\frac{d^3c}{A'c} - \frac{c^4}{2}\left(\frac{1}{A_{d_1}} - \frac{1}{A'_{d_1}}\right) - \frac{h^3c}{A_v}\right]}{d^3h\left(\frac{1}{A_c} + \frac{1}{A'_c}\right) + h^4\left(\frac{1}{A_v} + \frac{1}{A'_v}\right) + \frac{c^3h}{2}\left(\frac{1}{A_d} + \frac{1}{A_{d_2}} + \frac{1}{A'_{d_1}} + \frac{1}{A'_{d_2}}\right)}. \quad . \quad (14)$$

If the chords and posts are large and A_{d1} is placed $= A_{d2} = A_d$ we have

$$S_8 = -\frac{V\frac{c^4}{A_d} + \frac{Pc^4}{4}\left(\frac{1}{A_d} - \frac{1}{A'_{d1}}\right)}{\frac{c^3h}{2}\left(\frac{2}{A_d} + \frac{1}{A'_{d1}} + \frac{1}{A'_{d2}}\right)}. \quad . \quad . \quad . \quad (15)$$

and if $A'_{d1} = A'_{d2} = A'_d$, we have

$$S_8 = -\frac{c}{h}\left[V\frac{A'_d}{A_d + A'_d} + \frac{P}{4} \cdot \frac{A'_d - A_d}{A_d + A'_d}\right]. \quad . \quad . \quad (16)$$

Comparing with eq. (13) it is seen that the stress carried by the counter ON is the same as given by eq. (13), modified by the term $\frac{P}{4} \cdot \frac{A'_d - A_d}{A_d + A'_d}$, which is of little consequence. Hence we may say as in Art. 229. that the shear is distributed approximately as the sections of the diagonals, provided neither diagonal in question becomes slack through lack of initial tension or rigidity. In panel mo the same principle will hold, and the stresses in MO and mO will be approximately as their sections, the one taking tension and the other compression.

In case some or all of the diagonal members are thin bars, incapable of carrying compression excepting by reason of initial stress, it is generally assumed that no compression will be carried by such members. In this case the member or members of the two panels which will first become slack are the ones that otherwise would receive the greatest compression. Suppose, for example, that all diagonals are eye-bars and that the shear in panel on is positive. The shear in panel mo will then be a greater positive shear. Members 7 and 3 will tend to take compression and 7 more than 8. Hence 7 will slacken and the load at *O* will be carried by *O N* acting as a sub-tie. If the shear in *o n* is negative and that in *m o* also negative, but smaller, *O n* will tend to slacken more than *M O* and hence may be considered as not acting. The member *M O* will then act as a sub-tie. If the shear in *o n* is negative and in *m o* positive, then both *O n* and *m O* tend to slacken and the one may be omitted in the panel of the greater shear. Either *M O* or *O N* acts then as a sub-tie.

If *m O* is made a compression member, then according to the same principles, when the shear in *o n* is positive *m O* will act and *O N* become idle; and when the shear in *o n* is negative, *O n* will drop out whether the shear in *m o* is positive or negative.

These conclusions accord with the assumptions made in the analysis of Part I. Similar assumptions may be made in the case of the Pettit truss.

SECTION II.—LATERAL TRUSS SYSTEMS

231. Forms of Lateral Trusses and General Methods of Analysis.—Lateral truss systems, whether for trestles, towers, buildings, or bridge trusses, may be of the single Warren type, as in Fig. 16 (*a*), the double Warren type, Fig. (*b*), or the type shown in Fig. (*c*), which may be considered a double Warren type with verticals (in case the diagonals

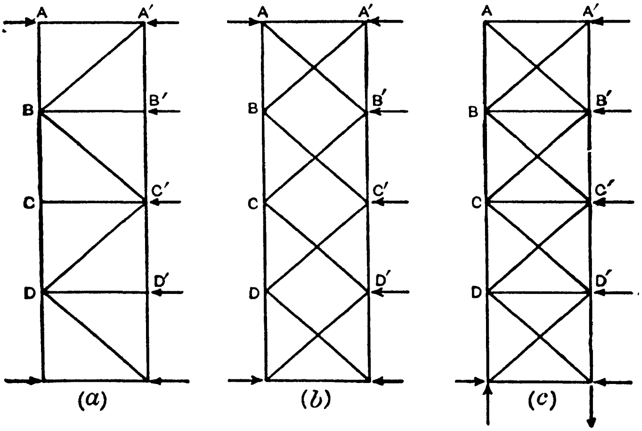


FIG. 16.

are stiff members), or a Pratt type with two diagonals in each panel (in case they are slender rods or shapes incapable of taking compression). By reason of the fact that lateral struts are generally required for other purposes, the most common type of lateral system is the third.

In Figs. (*a*) and (*b*) the stresses due to the lateral forces are calculated without difficulty, the two systems in Fig. (*b*) being treated as independent. In Fig. (*c*) the stresses may be closely determined by a consideration of the principles illustrated in the problem of Art. 229, The chords being large and the lateral struts generally of good size, the shear may be assumed as equally divided between the two diagonals, if they are capable of carrying compression. In that case the stresses in the struts are small—not more than a half-panel load. If the diagonals are slender rods, then the Pratt system is to be assumed. If both diagonals are assumed to act equally, then the chord stresses are determined by moments taken at a transverse section through the point of intersection of the web members. The moments of the stresses in the two web members will cancel.

232. Stresses Due to Distortions of the Main Members.—In forms (*a*) and (*b*) the direct stress in the main members, either tension or compression, causes no stresses in the laterals, as these members can resist stress only to the extent that they are prevented moving laterally at *B* and *B′* or *D* and *D′*, which is only through the bending resistance of the longitudinal members. In form (*c*), however, the presence of the members *B B′*, *C C′*, etc., prevents such lateral movement, and any direct stress in the main members will give rise also to some stress in the diagonals, with stresses of opposite sign in the lateral struts *B B′*, etc. In form (*a*) a compressive stress in the main members will cause them to be bent slightly to the left at *B* and *B′* and at *D* and *D′*, while *C* and *C′* stand fast. In form (*b*) they will be bent outward at *B* and *B′* and at *D* and *D′*. In form (*c*) they will be held more closely in line throughout. The secondary stresses due to this bending are greater in forms (*a*) and (*b*) than in form (*c*) and is a matter discussed more fully in Chapter VII. If the main members are in tension, as in the lower lateral system of a bridge. all conditions are reversed as to sign.

It will be desirable to estimate the stresses in form (*c*) due to stresses in the main members. Let Fig. 17 represent any panel in a lower lateral system of a bridge truss, and let member *m N* be taken as the redundant member. Consider equal longitudinal forces S_1 applied at *M*, *N*, *m*, and *n*, and let S_2 = the diagonal stress, assumed to be the same in all members. The total longitudinal force acting at each joint is $S_1 + S_2\frac{d}{c}$, and is generally a known quantity. Call this force P. Then the calculations are as follows:

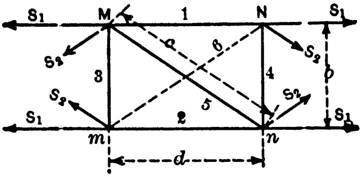


FIG. 17.

Member.	l	A	u	$\frac{ul}{A}$	$\frac{u^2 l}{A}$	S'	$\frac{S' u l}{A}$
1	d	A_c	$-\frac{d}{c}$	$-\frac{d^2}{c A_c}$	$\frac{2 d^3}{c^2 A_c}$	$+P$	$-\frac{2 P d^3}{c A_c}$
2	d	A_c	$-\frac{d}{c}$	$-\frac{d^2}{c A_c}$		$+P$	
3	b	A_v	$-\frac{b}{c}$	$-\frac{b^2}{c A_v}$	$\frac{2 b^3}{c^2 A_v}$	$-S_2 \frac{b}{c}$	$\frac{2 S_2 b^3}{c^2 A_v}$
4	b	A_v	$-\frac{b}{c}$	$-\frac{b^2}{c A_v}$		$-S_2 \frac{b}{c}$	
5	c	A_d	$+1$	$+\frac{c}{A_d}$	$\frac{2 c}{A_d}$	0	
6	c	A_d	$+1$	$+\frac{c}{A_d}$		0	

The value of S_6 is then equal to

$$S_6 = \frac{P\frac{d^2}{c A_c} - S_2 \frac{b^3}{c^2 A_v}}{\frac{d^3}{c^2 A_c} + \frac{b^3}{c^2 A_v} + \frac{c}{A_d}}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Noting that $S_6 = S_2$ we derive

$$S_6 = P\frac{\dfrac{d^2 c}{A_c}}{\dfrac{2\,b^3}{A_v} + \dfrac{d^3}{A_c} + \dfrac{c^3}{A_d}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the case where the vertical members are large, as the floor beams of a through bridge, we have, very closely

$$S_6 = P\frac{d^2 c\,A_d}{d^3 A_d + c^3 A_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Suppose, for example, that $A_d = \frac{1}{4}A_c$ and $c = 1.3\,d$. Then $S_6 = 0.13\,P$. That is, the diagonal stress is 13 per cent of the total longitudinal force P. If the verticals in this case had an area equal to the diagonals, the value of S_6 would be $0.091\,P$.

It will be seen that the narrower the truss the greater the proportion of stress carried by the diagonals. In the end panels of a bridge the laterals are large and the chords small, so that a very considerable percentage may be carried by the former. The unit stress in the diagonals resulting from these indirect stresses can never be as great as that in the chords themselves, as they are stressed less for equal longitudinal distortion. From eq. (3) we have approximately (A_d being small as compared to A_c), $\frac{S_6}{A_d} = \frac{P}{A_c} \cdot \frac{d^2}{c^2}$. That is, the unit stress in the diagonal will be approximately equal to that in the chord multiplied by $\frac{d^2}{c^2}$ or by $\frac{d^2}{b^2 + d^2}$.

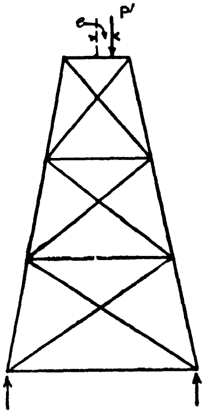


FIG. 18.

In the case of trestle towers of the form shown in Fig. 18, sufficiently approximate results may still be obtained by assuming all the stresses in the diagonals, due to vertical loads, to be equal. An unsymmetrical load P' is equivalent to a central load P' and a moment $P' \times e$. This moment, being uniform from top to bottom, causes little or no stress in the web members, while the central load P' is divided equally between the two posts and causes the same stresses in the web members as above calculated. Hence we may say in general that the same stresses are caused in the lateral members whether the load is central or eccentric.

The braced arch with chords parallel, or nearly so (Fig. 19), is another form of truss in which the chord members generally are all

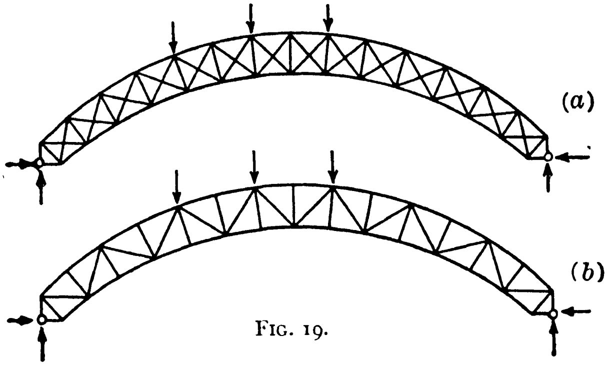


FIG. 19.

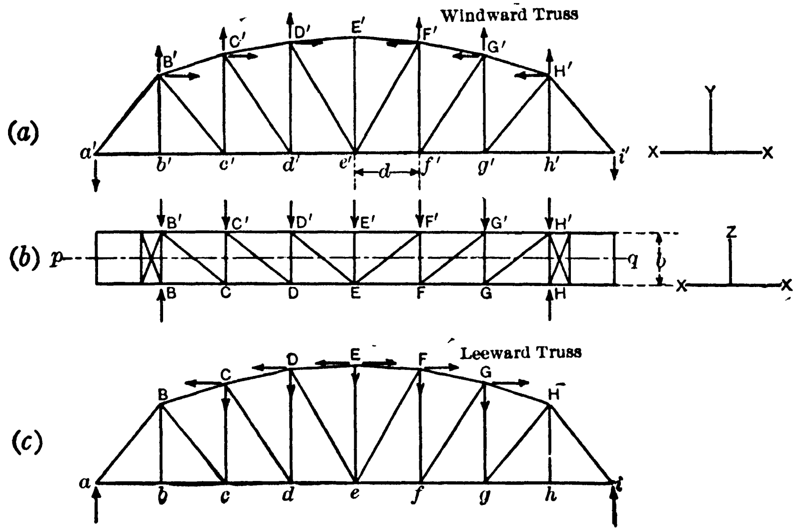


FIG. 20.

in compression. Such longitudinal deformation will cause considerable stress in a web system of the form shown in Fig. (*a*). And for the reasons already given, a system such as shown in (*b*), while comparatively free from lateral stress, is likely to be subjected to large secondary stresses. (See Art. 323 for further consideration of this matter.)

233. Lateral Stresses in Curved-Chord Trusses.—Fig. 20 (*b*) represents the upper lateral system of a curved-chord truss. Under the lateral forces the shears in the panels of the lateral truss will be the same as if the truss were a plane figure. In analyzing such a structure these shears should first be found. Suppose then the stresses in the lateral diagonals be resolved into X, Y, and Z components, as indicated in the figure. Then the Z-component of any diagonal is equal to the lateral shear. The stress in any diagonal member = shear $\times \frac{\text{length}}{b}$. The stresses in the struts are equal to the shears. The same result will be reached if the upper lateral truss be flattened out or developed into a plane figure, and the stresses found as for a plane truss, using the actual joint loads.

The stresses in the diagonals having been found, the stresses in the members of the main truss are found as follows: Pass the vertical section, $p\,q$. At each joint of the main truss, as E, resolve the diagonal stress into three components. The Z-component is balanced by the joint load and the stress in the strut. The X and Y components act in the plane of the main truss as external loads. This resolution being carried out for each joint we have the main truss on the windward side acted upon by forces as shown in Fig. (*a*), and that on the leeward side as shown in Fig. (*c*). (The portal effect is confined to the lower chord.) These external forces produce certain stresses in the main members which are readily determined in the usual way.

234. Lateral Bracing Necessary for Rigidity of Trusses.—Let Fig. 21 represent an ordinary framed structure in space, consisting of two main trusses, $a\,A\;B\,b$, and $a'\,A'\,B'\,b'$, and two lateral trusses, $A\,A'\,B'\,B$ and $a\,a'\,b'\,b$. The structure is made rigid in cross-section by the diagonals $A\,a'$ and $B'\,b$ at the ends. Evidently no further bracing is necessary for rigidity. Such a structure may theoretically be supported at any three joints and still retain its form. A less amount of bracing will give a structure not rigid in itself and which needs support at more than three points.

The number of necessary members for rigidity of a structure in space can readily be determined as follows: Starting with the simplest plane figure, the triangle, three members are required to fix three points, *a*, *b*, and *c*, Fig. 22. Having these points fixed, three additional members are required to fix a fourth point, *d*, not in the same plane. Then for another point, *e*, three more members will be required; and so on. Hence if m = total number of joints in the figure and n = number of necessary members, we have the relation

$$n - 3 = 3\,(m - 3), \quad \text{or} \quad n = 3\,m - 6. \quad . \quad . \quad . \quad (4)$$

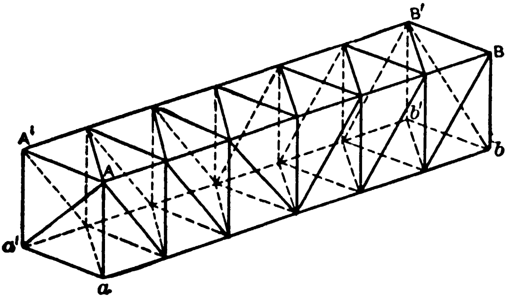


FIG. 21.

This relation will sometimes assist in determining the question of redundancy in space. In Fig. 21, for example, there are 28 points, which will require 3 × 28 − 6, or 78 members for rigidity. A count of the members shows 25 in each main truss, 13 additional members in each lateral system, and two more in the end bracing, or 78 in all, which is just sufficient to satisfy the requirements. Transverse bracing in intermediate panels is redundant.

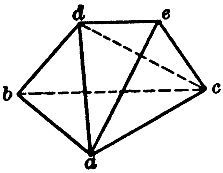


FIG. 22.

Rigidity may also be secured in Fig. 21, by omitting all but one of the lateral diagonals in one of the lateral systems and employing transverse bracing in each vertical panel. This gives the same total number of members as before. By the use of four supports a rigid system will result with transverse bracing in each panel and with all diagonals omitted in one lateral system. The use of two lateral trusses together with transverse bracing introduces redundancy, and while often desirable from a practical standpoint, this arrangement complicates the analysis somewhat, although in most cases the stresses are not large and this difficulty is of little practical consequence.

235. Redundant Reactions.—Theoretically a rigid frame needs support at three joints only, and if we resolve all reactions into components in three directions, parallel to rectangular axes, the number of components need be only six to provide for loads acting in all directions. Supports at more than three joints, or supports so arranged as to involve more than six possible component forces, give rise to redundant reactions, the exact determination of which requires the application of the theory of redundant members, as for the swing bridge and the arch of less than three hinges. Stated in another way, where reactions are redundant, they depend upon the dimensions of the members, and any errors in construction, or a settlement of supports, will cause stresses in the structure. Changes of temperature may also give rise to stresses in such structures. In the ordinary type of bridge structure redundant supports and redundant bracing are usually employed and the danger of undesirable stresses arising from such redundancy is generally of little consequence. However, as unusual conditions may require them to be considered, they will be briefly discussed.

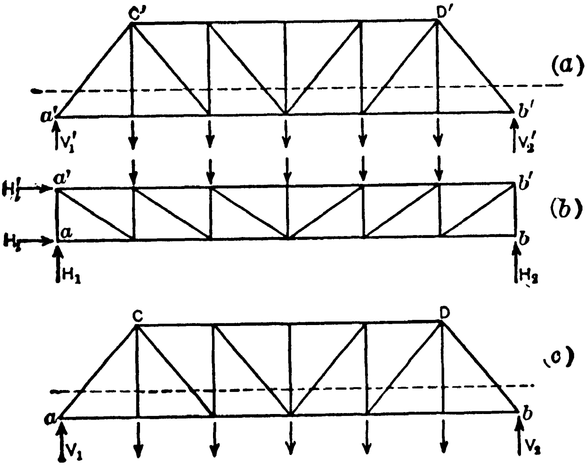


FIG. 23.

Fig. 23, (*a*), (*b*), and (*c*), shows the usual provision for vertical and horizontal reactions for a bridge truss. Supports are provided at he four points, *a*, *a'*, *b*, and *b'*. The reaction components are eight in number; four vertical reactions for the two main trusses, two horizontal lateral reactions, H_1 and H_2, to resist wind and other lateral forces and two horizontal reactions, H_l and H'_l, to resist longitudinal forces. The latter are supplied at the fixed end of the structure.

Theoretically, therefore, there are two redundant reactions. One of the vertical reactions may be considered as redundant and one of the horizontal reactions. Any three of the horizontal reactions would be sufficient for stability in a horizontal plane, but, obviously, such a combination as H_l, H'_l, and H_1 would be impracticable, as the structure would then act as a cantilever under wind forces, resulting in excessive lateral movements of the free end. The omission of one of the reactions H_l or H'_l would be admissible, as the longitudinal forces are small and reactions at *a* and *b*, as shown, would be sufficient. A fixed point at *a*, therefore, with the truss free to move in any direction at *a'* and *b'*, and longitudinally at *b*, would provide fully for horizontal forces without redundancy.

Usually both *a* and *a'* are fixed, in which case a further redundant component is introduced at *a'*, inasmuch as the reaction H_1 then becomes divided between the points *a* and *a'* in an indeterminate manner. Temperature changes will then also develop lateral reactions at *a* and *a'*. Practically, the ordinary truss is relatively so narrow that a rigid fastening at both *a* and *a'* are not objectionable. For wide structures, however, expansion and contraction need to be provided for by allowing free lateral movement of either *a* or *a'*. At the same time both points may be fixed as regards longitudinal movement.

The lateral forces being relatively small, and the lateral trusses flexible, the indeterminate character of the stresses due to redundancy of lateral supports in the ordinary bridge truss is of little importance and will not be further considered. The redundancy of vertical supports is of more importance, especially with reference to stresses caused by unequal settlement or other variation of level. This question will be considered more fully.

236. Stresses Due to Unequal Settlement of Supports.—In the construction of a large bridge, the ordinary procedure is such that the structure is fitted to its supports in the riveting of the lateral systems, so that slight irregularities in level of supports are taken care of.

Any subsequent settlement of one of the four supports, with respect to the plane of the other three, will cause stresses throughout the entire structure which will be now considered.

A deck structure will be considered with full end bracing, Fig. 24. The problem is solved by calculating the deflection of one point, as a, with respect to the plane of the other three, due to a force of unity acting upward at a. The corresponding reactions at the other supports will also be unity as indicated. The stresses u in all the members, due to these forces, are determined and then the desired deflection of a is found from the usual equation $\Delta = \Sigma \frac{u^2 l}{EA}$.

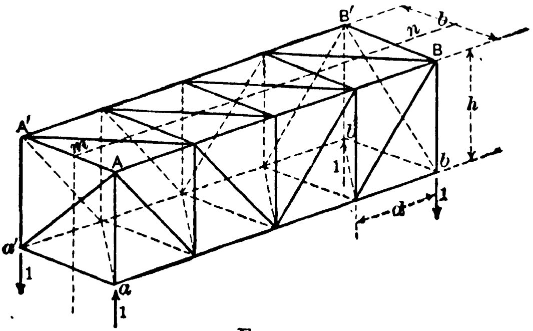


FIG. 24.

In finding the stresses u it is convenient to first pass a plane $m\,n$, cutting all the lateral members. Since there is no external load acting at any joint excepting at the four points, a, a', b', and b, the lateral or Z-components of all the lateral members are equal to each other, as shown by considering the lateral forces acting at any joint. Let U be this lateral component. Then let the stresses in all the upper and lower diagonals be resolved into Z- and X-components, and the stresses in the end diagonals into Z- and Y-components. The X-components of the upper and lower lateral diagonals will be equal to $U \times \frac{d}{b}$, and the Y-components of the end diagonals will be $U\frac{h}{b}$.

Consider now the forces acting on one of the vertical trusses (Fig. 25). At each joint where a lateral diagonal is connected there will be a horizontal force equal to $U\frac{d}{b}$, and at the end joints A and b there will be vertical forces of $U\dfrac{h}{b}$. The Z-components of the lateral diagonals and struts act at right angles to the plane of the main truss and balance each other. The various forces shown in Fig. 25 are therefore in equilibrium; hence we have, from moments about b, if n = number of panels,

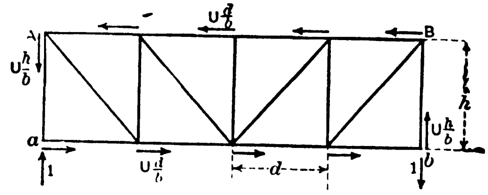


FIG. 25.

$$1 \times n\,d = U\frac{d}{b}\,.\,n\,h + U\frac{h}{b}\,.\,n\,d,$$

whence

$$U = \frac{b}{2\,h}. \qquad \qquad (5)$$

The value of U being found from (5), the remaining stresses are readily found, and thence the value of $\Sigma\,\dfrac{u^2\,l}{E\,A}$.

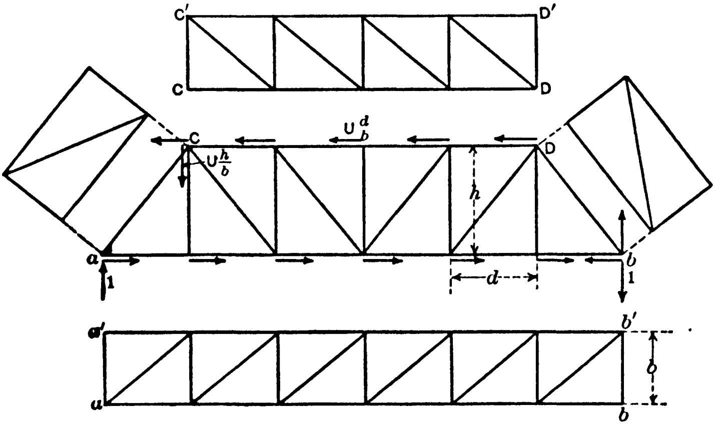


FIG. 26.

In the case of inclined end posts and full diagonal bracing, Fig. 26, the equation of moments gives

$$U = \frac{n}{n - 1} \cdot \frac{b}{2h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In the case of curved-chord trusses the lateral or Z-components will still be equal, but the diagonals of the laterals of the curved chord will have both vertical and horizontal components, as in Art. 233.

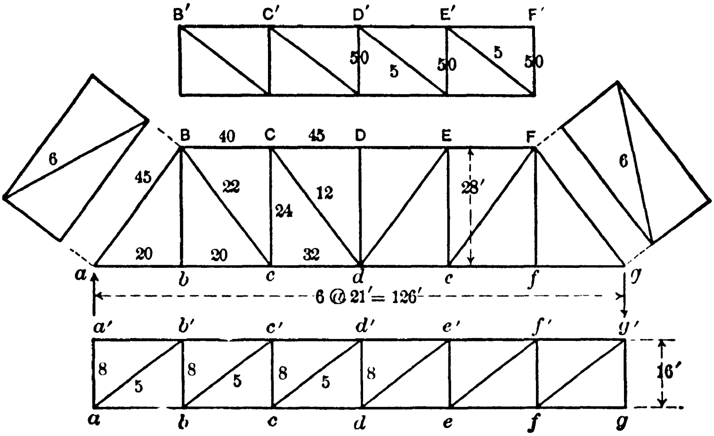


FIG. 27.

EXAMPLE.—Take a Pratt truss of six panels, with dimensions and cross-sections as given in Fig. 27. The lateral diagonals which are shown are those that will be in tension, with reactions at a and g as indicated.

The value of U is, from eq. (6), equal to $\frac{6}{5} \times \frac{16}{56} = 0.343$. Then $U\frac{d}{b} = 0.450$ and $U\frac{h}{b} = 0.600$. The horizontal force acting at each join of the main truss is therefore 0.450, and the vertical forces at B and g ar 0.600. These being known, the stresses in all the members are readily found. The value of $\Sigma\frac{u^2 l}{EA} = 0.0000156$, which is the deflection in inches for a one-pound reaction. A settlement of one-tenth inch therefore causes a change in reaction of 0.1/0.0000156 or 6,400 lbs. Such a reaction produces stresses in all members of 6,400 × u. For all laterals the lateral component of such stress will be .343 × 6,400 or 2,200 lbs. This example indicates the relative importance of stresses due to settlement of supports.

In the case of a through bridge the end bracing consists of portals instead of full diagonal bracing. The stresses in all members of the structure, except the members of the portals, due to the one-pound reaction will be the same as in the case already considered. In the portal there will be bending moments as well as direct stress. The quantity $\frac{u^2 l}{EA}$ will then be calculated for all direct stresses and $\int \frac{m^2 dx}{EI}$ for all members acting as beams. In the latter expression m is the bending moment at any section due to the one-pound reaction. The sum of all terms $\frac{u^2 l}{EA}$ and $\int \frac{m^2 dx}{EI}$ will give the desired deflection.

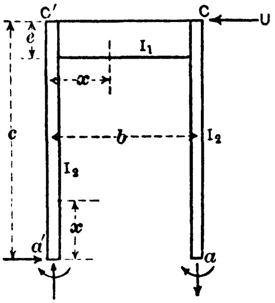


FIG. 28.

In the example given suppose the end bracing consists of a portal of the dimensions shown in Fig. 28; $c = 420$ in., $b = 192$ in., $I_1 = 4{,}000$, $I_2 = 2{,}000$. Assuming fixed ends, the value of m for the posts is approximately $\frac{U}{2}\left(\frac{c}{2} - x\right)$, and for the beam CC' it is $\frac{U}{4}c - U\frac{c}{2b}x = \frac{Uc}{2} \times \left(\frac{1}{2} - \frac{x}{b}\right)$. Integrating the value of $\frac{m^2 dx}{EI}$ for the post from 0 to c, and for the beam from 0 to b, we have the following:

	m	m^2	$\int \frac{m^2 dx}{EI}$
Post.	$\frac{U}{2}\left(\frac{c}{2} - x\right)$	$\frac{U^2}{4}\left(\frac{c^2}{4} - cx + x^2\right)$	$\frac{U^2 c^3}{48 E I_2}$
Beam.	$\frac{Uc}{2}\left(\frac{1}{2} - \frac{x}{b}\right)$	$\frac{U^2 c^2}{4}\left(\frac{1}{4} - \frac{x}{b} + \frac{x^2}{b^2}\right)$	$\frac{U^2 c^2 b}{48 E I_1}$

The total value of $\Sigma \int \frac{m^2 dx}{EI}$ is therefore $U^2 \left(\frac{c^3}{24 E I_2} + \frac{c^2 b}{48 E I_1}\right)$. Substituting numerical values, this is found to be equal to $U^2 \times \frac{1{,}720}{E} = \frac{202.4}{E}$. In the case of the full diagonal bracing, instead of this term we had the value of $\frac{u^2 l}{EA}$ for the diagonal member. This amounted to $\frac{52.4}{E}$. Thus the portal frame lends to the distortion or deflection about four times as much as the full diagonal bracing. The total deflection is in this case 0.000026 in. in place of 0.0000156 for the fully braced ends, or almost double. This illustrates very well the flexibility of portals as compared to full diagonal bracing.

237. Stresses in Lateral Trusses Due to Vertical Loads.—The usual form of lateral bracing, arranged symmetrically with respect to the centre of the structure, receives no stress from vertical loads excepting that due to the distortions of the chords of the main trusses as explained in Art. 232. In the lateral system shown in Fig. 27, the members are not arranged symmetrically and it will be found that vertical loads on the main trusses will cause slight stresses in the laterals, for the summation $\Sigma \frac{S' u l}{EA}$ will not be zero.

In the solution of such a problem one of the laterals may be taken as the redundant member. The stresses S' are then simply the stresses in the main trusses due to the vertical loads. The stresses u will be the stresses in all the members due to a one-pound tension in the lateral. These are readily found. The forces acting on the main truss will be similar to those shown in Fig. 26, but here the value of U is unity, and the end reactions are to be determined.

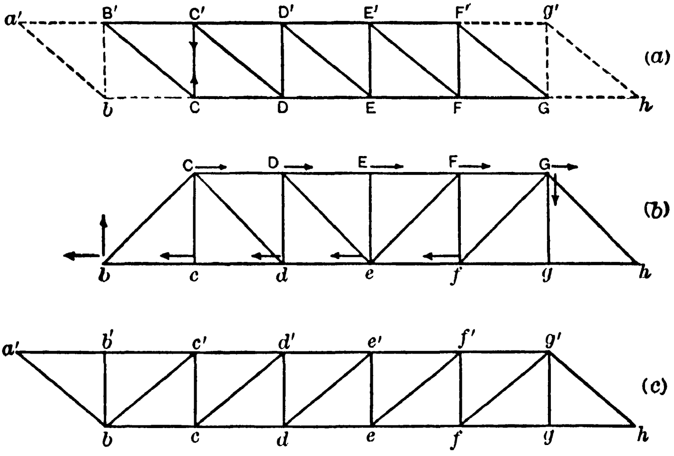


FIG. 29.

In the case of a skew bridge the lateral stresses due to vertical loads are considerable, especially in a double-track structure where the skew is large. As an example of such a case assume a truss of the general dimensions and sectional areas given in Fig. 27, but built in the form shown in Fig. 29, the skew being one panel. Suppose one of the trusses ($bCGh$) be loaded with a unit load at each upper joint. Let it be required to determine the resulting stresses in the laterals.

Assume any one of the laterals, as CC', to be the redundant member. The stresses u will be the stresses throughout the structure due to a one-pound tension in this member. They are found as explained in Art. 236. The lateral component of all lateral members will be unity, tension in some and compression in others. The stresses S' will be those in truss $b\,C\,G\,h$, due to the vertical loads only. Calculating the summations $\Sigma \frac{S'ul}{A}$ and $\Sigma \frac{u^2 l}{A}$, we find the stress in CC', from the formula $S_r = -\Sigma \frac{S'ul}{A} \Big/ \Sigma \frac{u^2 l}{A}$, to be $-$ 0.129 lbs.

The relation of the skew to the lateral arrangement being the same, when viewed from either side, it is evident that unit joint loads on the truss $a'\,B'\,F'\,g'$ will cause a stress in FF', and the lateral struts in general, of 0.129 lbs. compression. If the load per lineal foot is 5,000 lbs., the total floor-beam load is 5,000 × 21 = 105,000 lbs., and the stress in each lateral strut, due to such load, will be 105,000 × 0.129 = 13,500 lbs. compression, and in each diagonal it will be 13,500 × $\frac{317}{192}$ = 22,300 lbs. tension. The unit stresses for the areas assumed will be 270, 1,700, and 4,460 lbs. per sq. in. respectively.

The use of smaller laterals will result in a greater flexibility and a smaller stress in each member, but the unit stress will be slightly larger.

238. Transverse Bracing.—As shown in Art. 234, a bridge may be considered as fully braced when provided with an upper and a lower lateral system and end bracing. For sake of additional rigidity against lateral vibration under moving loads, transverse bracing is generally inserted at each panel point. In through bridges this is made of portal form and as deep as the head room will permit; in deck bridges it consists of full diagonal bracing. This intermediate transverse bracing may be considered as redundant, and an exact analysis of its stresses can be made only on that theory. However, it is not important that an exact analysis be made, as the stresses are small under any reasonable assumption, and minimum sections are generally sufficient.

The stresses in the transverse bracing will be considered in two parts: (a), the stresses due to lateral forces such as wind pressure, and (*b*), the stresses due to vertical loads applied unequally upon the main trusses.

239. (*a*) *Stresses Due to Lateral Forces.*—Consider any panel, *D d*, of a through bridge, Fig. 30. Lateral forces applied at the panel are *P* and *P′*. Were the transverse bracing not present, the load *P* would be carried wholly by the upper laterals to the portal and thence to the supports; and the load *P′* would be carried wholly by the lower laterals to these same points. The transverse bracing, however, connects the two lateral systems at each panel point and modifies more or less this distribution of the load.

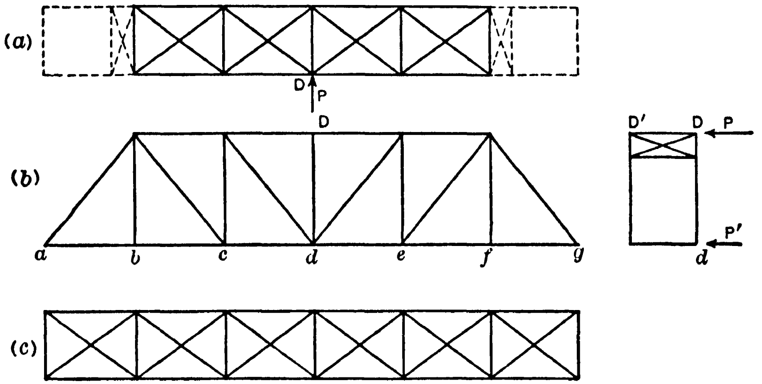


FIG. 30.

The relative amount of stress in the transverse frame may be estimated by considering the deflections of the lateral systems at *D* and *d*. The joint load *P* is much smaller than *P′*, and, likewise, the sections of the upper lateral members are smaller than those of the lower system. On the whole, the sections are in proportion to the loads, and hence the unit stresses and deflections in the two lateral systems will be approximately the same under full load. If the upper lateral system were of the same length and was supported at the ends in the same manner as the lower system, the deflections would be equal and therefore the transverse bracing would remain practically unstressed. Instead of this being the case, however, the upper laterals transmit their load down the portals, which are much more flexible than full diagonal bracing. Hence the upper laterals, with the portals, constitute a more flexible path than the lower laterals. The point *D* will, therefore, tend to deflect more than d, and to throw some stress on the transverse bracing, with a corresponding additional load upon the lower laterals. Inasmuch, however, as the intermediate transverse bracing is more flexible than the end portals, the proportion of the load P which will be transferred to the lower laterals by means of the transverse bracing will be small. An assumption commonly made is that the transverse bracing may transmit as much as one-half the upper load, P. The upper laterals are, at the same time, designed to carry all the upper loads and the lower laterals the lower load. These assumptions are sufficient for all ordinary cases of simple truss spans.

If the bridge is a deck structure the end bracing will be full diagonal bracing, and each lateral system will be more nearly equal in flexibility. In this case the transverse bracing will be but slightly stressed.

In general, the distribution of loads in the case of redundant lateral systems can be estimated with sufficient accuracy by considering the relative rigidities of the several systems. If one complete system (preferably the most rigid one) is made sufficient to carry the entire load, the auxiliary systems may, with entire safety, be designed arbitrarily. Their object is usually to increase the general stiffness of the structure, and minimum sections consistent with this end are suitable for the purpose.

240. (*b*) *Stresses Due to Unequally Loaded Main Trusses.*—When the vertical trusses are unequally loaded, as in the case of a double-track bridge, their deflections will be unequal and the transverse frame will tend to throw the trusses out of a vertical position. If it were not for the lateral bracing there would be very little resistance to such warping of the main trusses, and the stresses in the transverse bracing would be very slight. Where, however, an upper and a lower lateral system exist, such warping is resisted, and to that extent throws stress upon the transverse frame. Under unsymmetrical loads, therefore, the transverse bracing receives some stress, the amount depending upon its rigidity and the rigidity of the lateral trusses. The maximum effect will occur in the case of a deck bridge where the transverse frames and end bracing are composed of diagonals under direct stress. Such a structure will be considered in detail.

Let Fig. 31 represent the transverse section of a deck bridge, and P an eccentric load. The load transferred to the truss $D\,d$ will be

$P\left(\frac{1}{2}+\frac{e}{b}\right)$, and to the truss $D'\,d'$ will be $P\left(\frac{1}{2}-\frac{e}{b}\right)$. The excess on $D\,d$ over $D'\,d'$ will be $\frac{2\,P\,e}{b}$, which is the only portion of the load which need be here considered. A similar condition will exist at all the other panel points.

An exact solution of this problem would require the simultaneous consideration of all the transverse frames, each one furnishing one redundant member. However, for the purposes of this discussion, it will be sufficient to consider a single panel, such as $D\,d$, with its transverse bracing, and determine the stresses in the diagonal of such bracing due to a load of one pound applied at D.

We will assume a truss of the dimensions shown in Fig. 27. Transverse diagonal bracing will be assumed in panel $D\,d$, consisting of two bars of 2 sq. in. each. If a considerable initial tension is applied these may be considered equivalent to a single member of 4 sq. in. in area, arranged as shown in Fig. 31. A load of 1 pound is applied at D and the problem is to determine the tensile stress in the transverse diagonal due to this one-pound load. It is solved in the usual way by the theory of redundant members. The diagonal $D'\,d$ is taken as redundant, and the stresses u determined throughout the entire structure due to a one-pound tension in this member (including the value $u=1$ for this member itself); also the stresses S' in the truss $a\,B\,F\,g$, due to the vertical load of one pound at D. Then the stress in $D'\,d = \Sigma\frac{S'\,u\,l}{A}\Big/\Sigma\frac{u^2\,l}{A}$. The result of this calculation is a tension in $D'\,d$ of 0.20 lb. The vertical component of this is 0.175 lb. That is, the transverse bracing transfers 17 per cent of the load at D to the opposite truss.

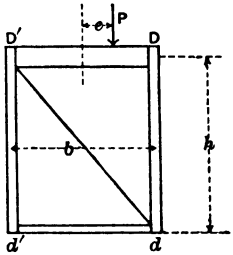


FIG. 31.

In this calculation it is found that the deformation of the member $D'd$, itself, has a large influence on its stress. If, for example, it is made 2 sq. in. in area instead of 4 sq. in., its resulting stress will be 0.166 lb, with a vertical component of 0.144 lb., a decrease of 17 per cent.

Again, if the bridge were a through bridge instead of a deck structure, with the portal form of end and transverse bracing, the flexibility of the lateral systems would be more than doubled, reducing the possible load transferred to less than half of the above values. In such a case it is a very liberal estimate to assume that 10 per cent of the excess vertical load may be transferred to the opposite truss.

In double-track bridges the excess load on the near truss is approximately one-half the total load on one track. Ten per cent of this will equal 250 to 300 lbs. per foot, as the amount transferred. The lateral component of this will be only about one-half, or 125 to 150 lbs. per foot as the load on the lateral trusses due to eccentric loads.

We may therefore conclude, in general, that the stresses in the transverse frame from eccentric loads will not be greater than that caused by assuming one-half the lateral load on the unloaded chord to be carried by the transverse frame.

241. Proportions of Lateral Bracing for Maximum Ridigity.— In many cases the question arises as to the most suitable panel length for the lateral system. Thus, for example, where the panel length of the main truss is long, either one or two panels of the lateral system may be made to correspond with one panel of the main truss; or, on the other hand, in double track bridges of short panel, the panel length of the lateral truss may be made equal to either one or two panels of the main truss. In general, the panel length selected should be such as to give the maximum ridigity to the lateral system for the given working stresses. An approximate theoretical analysis of this problem will be of some assistance in selecting the best proportions.

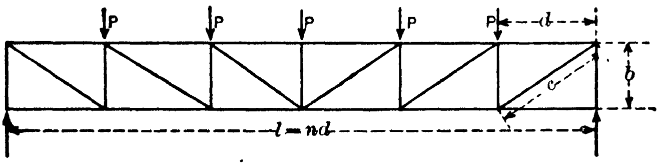


FIG. 32.

Let Fig. 32 represent any lateral truss of n panels. Under a load P the truss will deflect a distance determined by the formula $\Sigma \frac{Sul}{EA}$. It will be assumed that the chord members are large as compared to the web members so that their influence upon the deflection may be neglected.

Let f_t = working stress in the diagonal members and f_c = working stress in the struts. Under full load the stress S in any member will be approximately equal to $f_t A$, or $f_c A$, as the case may be. The stress u, for deflection at the centre, will be ½ for each strut and $\frac{c}{2b}$ for each diagonal. Hence, summing the products $\frac{Sul}{EA}$ we get, for the struts, $\Sigma \frac{Sul}{EA} = \frac{nbf_c}{2E}$, and for the diagonals, $\Sigma \frac{Sul}{EA} = \frac{nc^2f_t}{2bE}$. The deflection is therefore

$$\Delta = \frac{n}{2bE}(b^2 f_c + c^2 f_t). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For a variable panel length d, we have $n = l/d$. Substituting in (6), and placing $c^2 = b^2 + d^2$, we get, in terms of d, etc.,

$$\Delta = \frac{l}{2Edb}[b^2(f_c + f_t) + d^2 f_t]. \quad . \quad . \quad . \quad . \quad (7)$$

Differentiating this with respect to d, equating to zero, etc., we find that for a minimum value of Δ, or for maximum rigidity, the length of panel is

$$d = b\sqrt{\frac{f_c + f_t}{f_t}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If, for example, f_c is approximately equal to $\frac{1}{2}f_t$, we have $d = 1.22\, b$. Where the struts are very large, as in the case of floor beams used as struts, then f_c may be placed equal to zero and for maximum rigidity, $d = b$.

SECTION III.—THE QUADRANGULAR OR PORTAL FRAME*

242. The Quadrangular Frame.—Many problems arise in which an analysis is required of the stresses in a framework of four sides without diagonal bracing, such framework depending for its stability upon the bending resistance of its various members. An example of such framework is the ordinary bridge portal, or the viaduct or elevated railroad bent in which diagonal bracing is omitted. Wind bracing in tall buildings is frequently made of the portal or quadrangular form and occasionally high trestles are made of this type. Each transverse panel of a through bridge also forms such a framework, in which a correct determination of the stresses due to the load on the floor beam as well as lateral forces requires a consideration of the deformations of all the members of the frame.

---

* See Ch. VIII for application of slope-deflection method to the problems of this section.

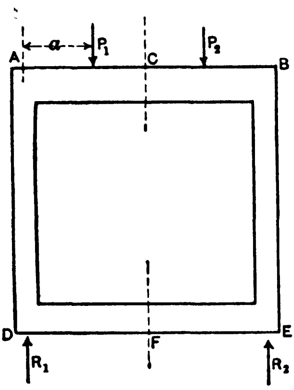


Fig. 33.

The stresses in a quadrangular frame are not statically determinate as they are dependent upon the deformations of the several members. A general solution of this problem will first be given, after which the analysis will be applied to several special cases.

243. General Solution.—Let $A\,B\,E\,D$, Fig. 33, be a complete rectangular frame supporting any loads, P_1, P_2, etc. The reactions, R_1 and R_2, are supposed to be determined by considering the structure as a whole, and the conditions of its supports. The problem is to determine the stresses in the frame.

The general method of solution used in the case of the arch with fixed ends can be advantageously applied here. The frame is cut at a convenient point, as at the centre, C, and the internal forces, M_0, V_0, and H_0, acting at this section treated as the unknown quantities. They are indicated in Fig. 34 in the direction which will be considered as positive. Then, selecting any other convenient section, as F, as the section of reference, assume this section to stand fast and treat the right and left half of the frame exactly as the two halves of the arch in Art. 154. In brief, express the deflections and changes of angle at C, of each half, with respect to the fixed section

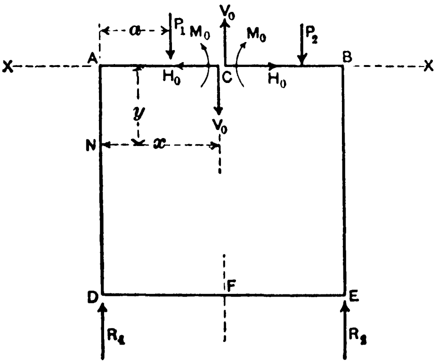


Fig. 34.

at F, by means of the general equations of Arts. 2 and 3. Then equate these functions for the two halves, thus deriving three equations from which the three unknowns, M_0, H_0, and V_0 are found.

As in Art. 154, let M'_r and M'_l represent the bending moment at any section in the right and left portions respectively, due to the given external forces, M_r and M_l, the total bending moments, and x and y the co-ordinates of any section, measured from C, as origin, x to be considered as positive toward the right for the right half and toward the left for the left half. Let I = moment of inertia of the section in general. Then, since the horizontal and vertical components of the deflection at C, for the two halves must be equal, and likewise the angular change, we have, as in Art. 154, with due regard to sign, and considering E as constant,

$$\left.\begin{aligned} \int \frac{M_l x\,ds}{I} &= \int \frac{M_r x\,ds}{I} \\ \int \frac{M_l y\,ds}{I} &= -\int \frac{M_r y\,ds}{I} \\ \int \frac{M_l\,ds}{I} &= -\int \frac{M_r\,ds}{I} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the left side,

$$M_l = M'_l + M_0 + H_0 y - V_0 x.$$

For the right side,

$$M_r = M'_r + M_0 + H_0 y + V_0 x.$$

Substituting in (1), we have for the general case, the following three equations between the unknowns M_0, H_0, and V_0,

$$\left(\int_l \frac{x ds}{I} - \int_r \frac{x ds}{I}\right) M_0 + \left(\int_l \frac{x y ds}{I} - \int_r \frac{x y ds}{I}\right) H_0 - V_0 \int \frac{x^2 ds}{I} = \int \frac{M'_r x ds}{I} - \int \frac{M'_l x ds}{I} \quad . \ (2)$$

$$M_0 \int \frac{y\,ds}{I} + H_0 \int \frac{y^2\,ds}{I} + \left(\int_r \frac{x\,y\,ds}{I} - \int_l \frac{x\,y\,ds}{I}\right) V_0 = -\int \frac{M'\,y\,ds}{I} \quad (3)$$

$$M_0 \int \frac{ds}{I} + H_0 \int \frac{y\,ds}{I} + \left(\int_r \frac{x\,ds}{I} - \int_l \frac{x\,ds}{I}\right) V_0 = -\int \frac{M'\,ds}{I}. \quad . \quad . \quad . \ (4)$$

In any problem the numerical values of the various coefficients of M_0, H_0, and V_0 should be first calculated, after which these equations are readily solved.

244. Symmetrical Frames.—For a frame symmetrical about a vertical axis, equations (2) to (4) become much simplified, the various functions of x and y for the left side being equal to similar functions for the right side. The several equations reduce to the following:

$$\left.\begin{aligned} V_0 \int_{\frac{1}{2}} \frac{x^2\,ds}{I} &= \tfrac{1}{2}\int \frac{M'_l\,x\,ds}{I} - \tfrac{1}{2}\int \frac{M'_r\,x\,ds}{I} \\ M_0 \int_{\frac{1}{2}} \frac{y\,ds}{I} + H_0 \int_{\frac{1}{2}} \frac{y^2\,ds}{I} &= -\tfrac{1}{2}\int \frac{M'\,y\,ds}{I} \\ M_0 \int_{\frac{1}{2}} \frac{ds}{I} + H_0 \int_{\frac{1}{2}} \frac{y\,ds}{I} &= -\tfrac{1}{2}\int \frac{M'\,ds}{I} \end{aligned}\right\} \quad . \quad . \quad (5)$$

In this expression, $\int_{\frac{1}{2}}$ signifies the integration for one-half only. Solving for H_0, V_0, and M_0 we derive as in Art. 154,

$$H_0 = \frac{\int_{\frac{1}{2}} \frac{ds}{I} \int \frac{M'\,y\,ds}{I} - \int_{\frac{1}{2}} \frac{y\,ds}{I} \int \frac{M'\,ds}{I}}{2\left[\left(\int_{\frac{1}{2}} \frac{y\,ds}{I}\right)^2 - \int_{\frac{1}{2}} \frac{ds}{I} \int_{\frac{1}{2}} \frac{y^2\,ds}{I}\right]} \quad . \quad . \quad (6)$$

$$V_0 = \frac{\int \frac{M'_l\,x\,ds}{I} - \int \frac{M'_r\,x\,ds}{I}}{2\int_{\frac{1}{2}} \frac{x^2\,ds}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$M_0 = -\frac{\int \frac{M'\,ds}{I} + 2H_0 \int_{\frac{1}{2}} \frac{y\,ds}{I}}{2\int_{\frac{1}{2}} \frac{ds}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Inasmuch as each term contains a value of I, the result will be the same if, instead of using actual values, only relative values are employed. This is a more convenient method in the investigation of general problems in which the relative proportions of the various members may be assumed.

If the dimensions of a symmetrical frame be as shown in Fig. 35, the values of the several integrals of the above equations, which d not involve M', are as follows (for one-half only):

Integral.	C to A	A to D	D to F
$\int\frac{ds}{I}$	$\frac{b}{2I_1}$	$\frac{h}{I_2}$	$\frac{b}{2I_3}$
$\int\frac{y\,ds}{I}$	0	$\frac{h^2}{2I_2}$	$\frac{bh}{2I_3}$
$\int\frac{y^2\,ds}{I}$	0	$\frac{h^3}{3I_2}$	$\frac{bh^2}{2I_3}$
$\int\frac{x^2\,ds}{I}$	$\frac{b^3}{24I_1}$	$\frac{b^2h}{4I_2}$	$\frac{b^3}{24I_3}$

The integrals involving M' depend upon the external loads. For a single load P, applied a distance a, less than $\frac{b}{2}$, from A, there will be

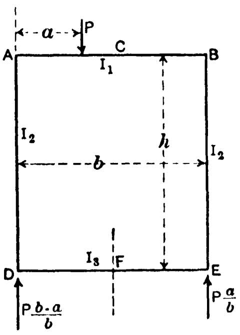


Fig. 35.

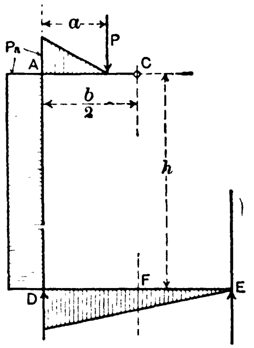


Fig. 36.

moments M', as shown in Fig. 36, and the various integrals are as follows:

Integral.	C to A	A to D	D to F	F to E
$\int\frac{M'\,ds}{I}$	$-\frac{Pa^2}{2I_1}$	$-\frac{Pah}{I_2}$	$-\frac{Pab}{2I_3}$	
$\int\frac{M'\,y\,ds}{I}$	0	$-\frac{Pah^2}{2I_2}$	$-\frac{Pabh}{2I_3}$	
$\int\frac{M'\,x\,ds}{I}$	$-\frac{Pa^2}{2I_1}\left(\frac{b}{2}-\frac{a}{3}\right)$	$-\frac{Pabh}{2I_2}$	$-\frac{5Pab^2}{48I_3}$	$-\frac{Pab^2}{48I_3}$

EXAMPLE.—Assume the following data (Fig. 35): $h = 150$ in.; $b = 120$ in.; $I_1 = 8{,}000$; $I_2 = 2{,}000$; $I_3 = 4{,}000$; $a = b/4 = 30$ in. It will be convenient, and lead to the same results, to call $I_2 = 1$; $I_3 = 2$; $I_1 = 4$.

We then have, for one-half of the frame,

$$\int \frac{ds}{I} = 15 + 150 + 30 = 195.$$

$$\int \frac{y\,ds}{I} = 11{,}250 + 4{,}500 = 15{,}750.$$

$$\int \frac{y^2\,ds}{I} = 1{,}125{,}000 + 675{,}000 = 1{,}800{,}000.$$

$$\int \frac{x^2\,ds}{I} = 18{,}000 + 540{,}000 + 36{,}000 = 594{,}000.$$

And for the entire frame,

$$\int \frac{M'\,ds}{I} = -P\,(112 + 4{,}500 + 900) = -\,5{,}512\,P.$$

$$\int \frac{M'\,y\,ds}{I} = -P\,(337{,}500 + 135{,}000) = -\,472{,}500\,P.$$

$$\int_l \frac{M'\,x\,ds}{I} - \int_r \frac{M'\,x\,ds}{I} = -P\,(5{,}625 + 270{,}000 + 22{,}500 - 4{,}500)$$
$$= -\,293{,}625\,P.$$

Substituting in Eqs. (6), (7), and (8), we have,

$$H_0 = \frac{195\,(-\,472{,}500) - 15{,}750\,(-\,5{,}512)}{2\,[\,(15{,}750)^2 - 195 \times 1{,}800{,}000]}\,P = +\,.0258\,P.$$

$$V_0 = \frac{-\,293{,}625}{2 \times 594{,}000}\,P = -\,.247\,P.$$

$$M_0 = -\,\frac{-5{,}512 + 2 \times .0258 \times 15{,}750}{2 \times 195}\,P = +\,12.05\,P.$$

245. Quadrangular Frames with Brackets.—Figs. 37 and 38 illustrate other arrangements with respect to the manner of applying the load. In Fig. 38 the moments M' in $A\,C$ and $A\,G$ are zero, and in $G\,D$ it is $+\,P\,a$. The general formulas given in Art. 243 apply directly to these cases, the only feature requiring notice being the values of the moment M'.

These forms are often used in highway bridges where sidewalks are supported on brackets.

246. Frame with Posts Fixed at Ends.—If the posts *A D* and *B E* are fixed at the lower end, Fig. 39, then the formulas above developed will apply by making $I_3 = \infty$. For a single load P the integrals given in Art. 244 may be substituted directly in eqs. (6), (7), and (8), omitting values for member *D E*. By such substitution and reduction, there result the following values:

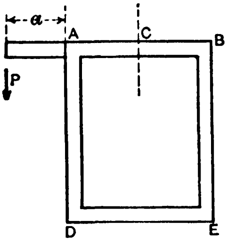


FIG. 37.

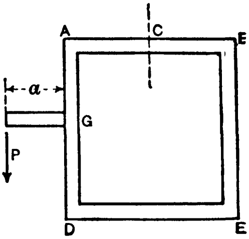


FIG. 38.

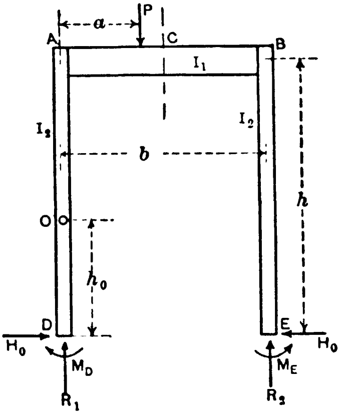


FIG. 39.

$$H_0 = \frac{3\,Pa\,(b-a)\,I_2}{2\,h^2 I_1 + 4\,b\,h\,I_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$V_0 = -P\frac{a}{b}\left(1 - \frac{(b-a)\;(b-2\,a)\;I_2}{b^2 I_2 + 6\,b\,h I_1}\right) \quad . \quad . \quad . \quad (10)$$

$$M_0 = \frac{Pa}{2}\left(1 - \frac{2\;(b-a)\;I_2}{hI_1 + 2\,bI_2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The value of a is taken less than $b/2$.

Having the values of H_0, V_0, and M_0 for section C, the values of the moments and stresses at any section are readily found. Thus at A we have $M_A = M_0 - V_0\frac{b}{2} - P\,a$; at D, $M_D = M_0 - V_0\frac{b}{2} - P\,a + H_0\,h$; etc.

For two loads P, symmetrically placed, $V_0 = 0$, and H_0 and M_0 are twice the values given by (9) and (11). Moments at A and D are then

$$M_A = -\frac{2\,P\,a\,(b-a)\,I_2}{h\,I_1 + 2\,b\,I_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$M_D = +\frac{P\,a\,(b-a)\,I_2}{h\,I_1 + 2\,b\,I_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The point of inflection, O, is located a distance above the base equal to

$$h_0 = \frac{h}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

247. Frame with Posts Hinged at One End.—This case is solved by placing the horizontal deflection of E with respect to D equal to zero, or, what is the same thing, treating the forces H as redundant forces. In either case we have $\int \frac{M'\,m\,d\,s}{E\,I} + H\int \frac{m^2\,d\,s}{E\,I} = 0$, in which M' is the moment due to the load P, and m is the moment due to a

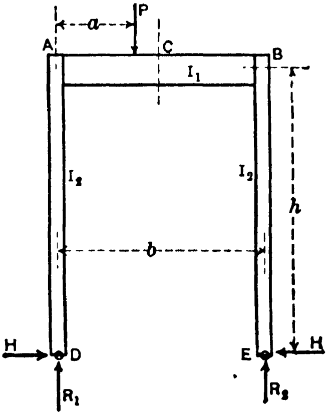


FIG. 40.

value of H equal to one unit. This reduces to the same general expression developed for the two-hinged arch:

$$H = \frac{\int \frac{M' y\, ds}{I}}{\int \frac{y^2\, ds}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

The integrals are as follows:

Integral.	D to A	A to B	B to E
$\int \frac{y^2\, ds}{I}$	$\frac{h^3}{3 I_2}$	$\frac{b h^2}{I_1}$	$\frac{h^3}{3 I_2}$
$\int \frac{M' y\, ds}{I}$	0	$\frac{P a (b-a) h}{2 I_1}$	0

Substituting in (15), we have

$$H = \frac{3 P a (b-a) I_2}{4 h^2 I_1 + 6 b h I_2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

The value of H being known, the bending moment at any section is easily found.

EXAMPLE.—Let Fig. 41 represent an elevated railway bent with the dimensions as shown. Let $I_1 = 50{,}000$; $I_2 = 640$; area of column $= A_2 =$ 20 sq. in.; width of column = 15 in. Calculate the bending moments and fibre stresses at A and D, due to four loads of 60,000 lbs. applied as shown.

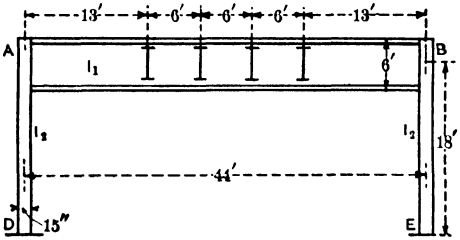


FIG. 41.

(*a*) *Fixed Ends.*—Eq. (12) may be used, calculating the term $a\ (b - a)$ for the two symmetrical pairs of loads. There results:

$M_A = -$ 846,600 in. lbs.; f_a = 9,960 lbs. per sq. in.

$M_D = +$ 423,300 in. lbs.; f_d = 4,980 lbs. per sq. in.

The direct stress in the columns is equal to 120,000/20 = 6,000 lbs. per sq. in. In this case therefore the bending stress is greater than the direct stress.

(*b*) *Hinged Ends.*—Use eq. (16), giving $H = 3{,}000$ lbs. Then $M_A = 3{,}000 \times 18 \times 12 = 648{,}000$ in. lbs.; and $f_a = 7{,}600$ lbs. per sq. in.

It is to be noted that in both cases the bending moments are nearly proportional to the value of I_2, and that having the bending moments the value of f is proportional to the width of the member. Hence it may be said that the fibre stress in such a case is approximately proportional to $I_2\, c$, or the moment of inertia times the width. For low stresses, therefore, a relatively narrow column is very advantageous.

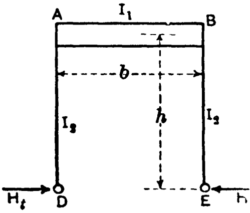


FIG. 42.

248. Temperature Stresses.—Where one side of the frame is fixed in length, as in the preceding two cases, changes of temperature will cause stresses in the structure. These are found by placing the deflection of E with respect to D equal to the change of length of $A\,B$ due to change of temperature. For hinged ends, Fig. 42, the deflection is $\Delta_t = \int \frac{M\,y\,d\,s}{E\,I} = H \int \frac{y^2\,d\,s}{E\,I}$. This reduces to

$$\Delta_t = \frac{H\,h^2}{E}\left(\frac{2\,h}{3\,I_2} + \frac{b}{I_1}\right). \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Let ω = coefficient of expansion. Then $\Delta_t = \omega\,t\,b$ and we have for temperature changes,

$$H_t = \frac{E\,\omega\,t\,b}{h^2\left(\frac{2\,h}{3\,I_2} + \frac{b}{I_1}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

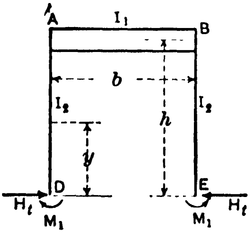


FIG. 43.

For fixed ends at D and E, Fig. 43, we may determine the total angular change $\Delta\,\varphi$ from D to E, and the total horizontal movement $\Delta\,x$, and write $\Delta\,\varphi = 0$ and $\Delta\,x = -\Delta_t = -\,\omega\,t\,h$. As in Art. 154, we have

$$\Delta\,\varphi = \int \frac{M\,d\,s}{E\,I} \text{ and } \Delta\,x = \int \frac{M\,y\,d\,s}{E\,I}.$$

Let M_1 = moment at D and E. Then $M = M_1 - H_t y$. Substituting this value of M in the above integrals, we have

$$M_1 \int \frac{d\,s}{E\,I} - H_t \int \frac{y\,d\,s}{E\,I} = 0.$$

$$M_1 \int \frac{y\,d\,s}{E\,I} - H_t \int \frac{y^2\,d\,s}{E\,I} = -\Delta_t.$$

Solving for H_t and M_1, we derive

$$H_t = \frac{\Delta_t \int \frac{d\,s}{E\,I}}{\int \frac{d\,s}{E\,I} \int \frac{y^2\,d\,s}{E\,I} - \left(\int \frac{y\,d\,s}{E\,I}\right)^2}. \quad . \quad . \quad . \quad (19)$$

$$M_1 = \frac{H_t \int \frac{y\,d\,s}{I}}{\int \frac{d\,s}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Expressing the several integrals in terms of the dimensions of the frame as given in Fig. 43, we have

Integral.	D to A	A to B	B to E
$\int \frac{d\,s}{I}$	$\frac{h}{I_2}$	$\frac{b}{I_1}$	$\frac{h}{I_2}$
$\int \frac{y\,d\,s}{I}$	$\frac{h^2}{2\,I_2}$	$\frac{b\,h}{I_1}$	$\frac{h^2}{2\,I_2}$
$\int \frac{y^2\,d\,s}{I}$	$\frac{h^3}{3\,I_2}$	$\frac{b\,h^2}{I_1}$	$\frac{h^3}{3\,I_2}$

Substituting and reducing, we have,

$$H_t = \frac{3\,E\,I_2\,(b\,I_2 + 2\,h\,I_1)\,\omega\,t\,b}{h^3\,(2\,b\,I_2 + h\,I_1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$M_1 = \frac{3\,E\,I_2\,(b\,I_2 + h\,I_1)\,\omega\,t\,b}{h^2\,(2\,b\,I_2 + h\,I_1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$M_A = M_1 - H_t\,h = -\frac{3\,E\,I_1\,I_2\,\omega\,t\,b}{h\,(2\,b\,I_2 + h\,I_1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The distance of the point of inflection above the base is

$$h_0 = \frac{M_1}{H_t} = h\frac{bI_2 + hI_1}{bI_2 + 2hI_1}. \quad . \quad . \quad . \quad . \quad (24)$$

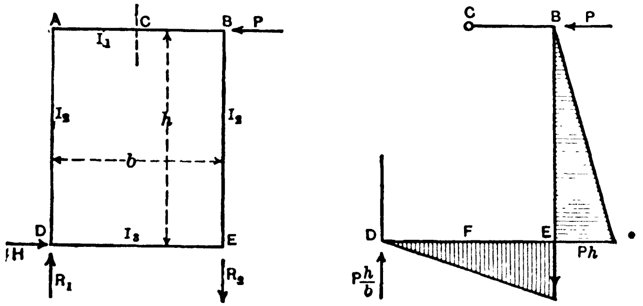
EXAMPLE.—Let it be required to calculate the temperature stresses in the bent of Art. 247 for an increase of temperature of 50 degrees, the coefficient of expansion being .0000065.

a. Fixed Ends.—The value of $\omega t b$ = .0000065 × 50 × 44 × 12 = 0.1716 in. Then by Eq. (22) M_d = 206,000 in. lbs.; and f_d = 2,410 lbs. per sq. in. From (23) M_a = − 199,000 in. lbs., and f_a = 2,330 lbs. per sq. in.

b. Hinged Ends.—Eq. (18) gives H_t = 467 lbs., and hence M_a = 467 × 18 × 12 = 101,400 in. lbs., and f_a = 1,182 lbs. per sq. in.

Temperature stresses are about double for fixed ends as for hinged ends. The values would be exactly double if I_1 were indefinitely large.

249. Effect of Lateral Forces.—Fig. 44 represents a quadrangular frame, symmetrical about a vertical axis and acted upon by the horizontal force P, applied at B. The reactions are assumed to be applied as shown. It is required to determine the stresses in the frame. The general solution of the problem is the same as given in Art 274. A section is taken at C and the values of H_0, V_0, and M_0 determined by the general formulas of Art. 243. The moment diagram for the moments M' are shown in Fig. 44 (b). The values of the several integrals involving M' are as follows:



(a) (b)

FIG. 44.

Integrals.	C to B	B to E	E to F	F to D
$\int \frac{M' ds}{I}$	0	$-\frac{P h^2}{2 I_2}$	$\overbrace{-\frac{P b h}{2 I_3}}$	
$\int \frac{M' y ds}{I}$	0	$-\frac{P h^3}{3 I_2}$	$\overbrace{-\frac{P h^2 b}{2 I_3}}$	
$\int \frac{M' x ds}{I}$	0	$-\frac{P h^2 b}{4 I_2}$	$-\frac{5 P b^2 h}{48 I_3}$	$-\frac{P b^2 h}{48 I_3}$

Substituting in eqs. (6), (7), and (8), we derive the following:

$$M_0 = 0. \qquad (25)$$

$$V_0 = -\frac{P\left(\frac{b h^2}{4 I_2} + \frac{b^2 h}{12 I_3}\right)}{2\left[\frac{b^3}{24 I_1} + \frac{b^2 h}{4 I_2} + \frac{b^3}{24 I_3}\right]}. \qquad (26)$$

$$H_0 = \frac{P}{2}. \qquad (27)$$

The important results are, that, where the members $A D$ and $B E$ are alike, $M_0 = 0$ and $H_0 = \frac{P}{2}$, that is, the point C is a point of inflection and the value of H_0 is $\frac{P}{2}$, which signifies that the lateral pressure, P, is carried equally by the two sides of the frame. It also follows that the point of inflection in $D E$ is at the centre, F, and the value of the shear at this point is found by interchanging I_3 and I_1 in eq. (26). It is

$$V_F = -\frac{P\left(\frac{b h^2}{4 I_2} + \frac{b^2 h}{12 I_1}\right)}{2\left[\frac{b^3}{24 I_1} + \frac{b^2 h}{4 I_2} + \frac{b^3}{24 I_3}\right]}. \qquad (28)$$

The sum of $V_0 + V_F$ must equal $R_1 = \frac{P h}{b}$, as will be found by adding (27) and (28).

If $I_1 = I_3$, then $V_0 = V_F = \frac{Ph}{2b}$.

The forces acting on one-half the frame are clearly shown in Fig. 45.

If the posts are fixed at the base, then $I_3 = \propto$, and we have

$$V_0 = -P\frac{3h^2 I_1}{b^2 I_2 + 6bhI_1}. \quad \cdot \quad (29)$$

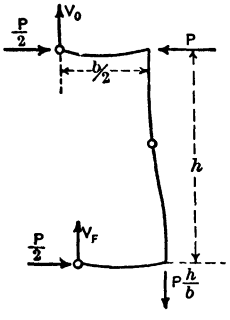


FIG. 45.

The point of inflection in the vertical post is found by placing $\frac{V_0 b}{2} + \frac{P(h - h_0)}{2} = 0$, where h_0 is the distance of this point above the base.

250. Partially Trussed Portal Frames.—Frames of the type shown in Fig. 46 are partly trussed and partly a framework of beams. Members FG and HK are hinged at their ends, and hinges exist at A and B, and may or may not exist at D and E. As in Art. 249, there will be a point of inflection at the centre C, and the compression at C will be $\frac{P}{2}$. The horizontal reactions at D and E will each be $\frac{P}{2}$. If the posts are fixed at D and E, there remain to be determined the moments M_1 at these points, or the distance h_0 of the point of inflection above D. If the posts are hinged, all external forces become known and the problem is solved as explained in Part I. The influence of the deformations due to direct stresses will be neglected as it is very small in any case.

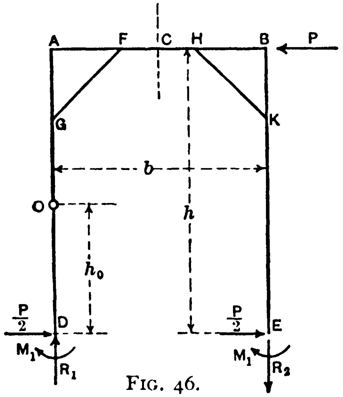


FIG. 46.

In the solution of this problem it is sufficient to consider one-half only, as shown in Fig. 47. The problem is solved by the application of the theory of redundant members. A convenient form of solution is to place the deflection of the point C equal to zero, as is evident from the symmetry of conditions. The general expression for this deflection, neglecting the effect of direct stress, is $\Delta = \Sigma \int \frac{M\,m\,d\,s}{E\,I}$, in which M is the actual moment at any section and m is the moment due to unit load acting at C. Fig. 48 shows the moment diagrams for moments M and m. The values of M and m, and of the several integrals, are as follows, origins taken at C, A, and D:

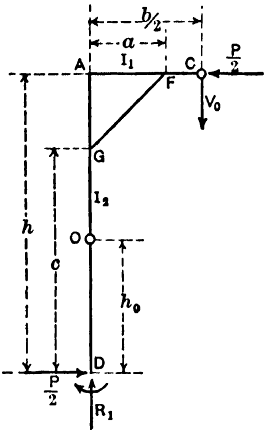
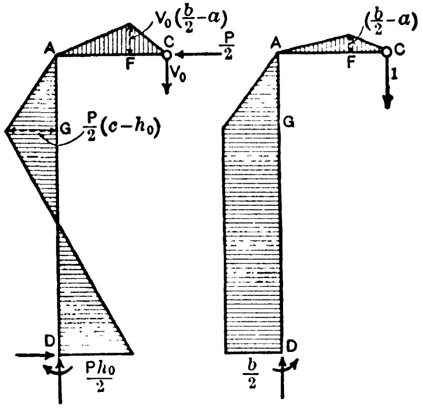


FIG. 47.



Moments M Moments m.

FIG. 48.

Quantity.	C to F	A to F	A to G	G to D
M	$-V_0 x$	$-V_0 \frac{\left(\frac{b}{2}-a\right)}{a} x$	$-\frac{P(c-h_0)\,x}{2\,(h-c)}$	$\frac{P(h_0-x)}{2}$
m	$-x$	$-\frac{\frac{b}{2}-a}{a} x$	$-\frac{b\,x}{2\,(h-c)}$	$-\frac{b}{2}$
$M\,m$	$V_0 x^2$	$V_0 \left(\frac{\frac{b}{2}-a}{a}\right)^2 x^2$	$\frac{Pb\,(c-h_0)\,x^2}{4\,(h-c)^2}$	$-\frac{Pb\,(h_0-x)}{4}$
$\int \frac{M\,m\,d\,s}{I}$	$\frac{V_0 \left(\frac{b}{2}-a\right)^3}{3\,I_1}$	$\frac{V_0 \left(\frac{b}{2}-a\right)^2 a}{3\,I_1}$	$\frac{P\,b(c-h_0)\,(h-c)}{12\,I_2}$	$\frac{Pb\,c\,(c-2\,h_0)}{8\,I_2}$

Adding the several integrals, substituting $V_0 = \frac{H(h - h_0)}{b}$ and solving for h_0, we derive

$$h_0 = \frac{h(b - 2a)^2 I_2 + 2bc(c + 2h) I_1}{(b - 2a)^2 I_2 + 4b(2c + h) I_1}. \quad . \quad . \quad (30)$$

The value of h_0 being known, the stresses in the portal are found as for the case of hinged ends, the point of inflection O being taken as the hinge. For such cases as shown in Fig. 49, the solution will be

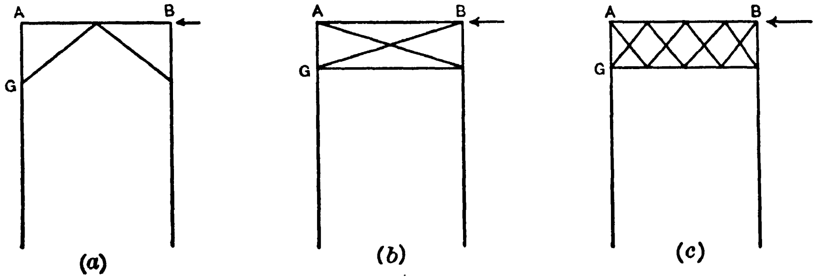


FIG. 49.

reached from eq. (30), by placing $I_1 = \infty$; or, what is the same thing, by placing the lateral deflection of the two points A and G equal. Dividing both numerator and denominator of (30) by I_1, and then placing $I_1 = \infty$, we derive

$$h_0 = \frac{c(c + 2h)}{2(2c + h)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

which is the same value as deduced in Art. 199, Part I, eq. (29).

251. Frames with Inclined Posts.—The general methods of Art. 243 are obviously applicable to such frames as illustrated in Fig. 50. If the frame is symmetrical about a vertical axis the effect of a horizontal force P is similar to that in a rectangular frame. The centre points of members $A\ B$ and $D\ E$ are points of inflection, and the direct stress in $A\ B$ is $\frac{P}{2}$.

Placing the deflection of C with respect to F equal to zero (Fig. 51), enables the values of V_1 and V_2 to be determined in the same manner as in Art. 249. This deflection is most readily found by the general expression $\Delta = \int \frac{M\,x\,d\,s}{E\,I}$, where x is the abscissa measured from the origin C. For member $C\,A$, $M = V_1\,x$ and $\int \frac{M\,x\,d\,s}{I} = \frac{V_1\,b_1^3}{24\,I_1}$; likewise for $F\,D$, $\int \frac{M\,x\,d\,s}{I} = -\frac{V_2\,b_2^3}{24\,I_3}$, writing this deflection as

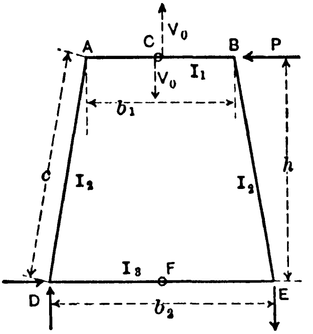


FIG. 50.

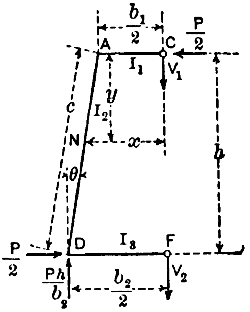


FIG. 51.

minus, since it tends to diminish the deflection of C relative to F. For the member $A\,D$ the integral $\int \frac{M\,x\,d\,s}{I}$ is found as follows: For any point N, the moment $M = V_1\,x - \frac{P\,y}{2}$. Also we have $d\,s = \frac{d\,x}{\sin\theta} = \frac{2\,c}{b_2 - b_1} \cdot d\,x$, and $y = \left(x - \frac{b_1}{2}\right) \cot\theta = \frac{h}{b_2 - b_1}(2x - b_1)$

We then have for $A\,D$,

$$\int \frac{M\,x\,d\,s}{I} = \frac{2\,V_1\,c}{(b_2 - b_1)\,I_1} \int_{\frac{b_1}{2}}^{\frac{b_2}{2}} x^2 d\,x - \frac{P\,h\,c}{I_2\,(b_2 - b_1)^2} \int_{\frac{b_1}{2}}^{\frac{b_2}{2}} (2x^2 - b_1 x)\,d\,x$$

$$= \frac{V_1\,c\,(b_2^2 + b_1\,b_2 + b_1^2)}{12\,I_2} - \frac{P\,h\,c\,(2\,b_2 + b_1)}{24\,I_2}.$$

We therefore have, after multiplying through by 12,

$$\frac{V_1\,c\,(b_2^2 + b_1\,b_2 + b_1^2)}{I_2} + \frac{V_1\,b_1^3}{2\,I_1} - \frac{V_2\,b_2^3}{2\,I_3} = \frac{P\,h\,c\,(2\,b_2 + b_1)}{2\,I_2}. \quad (32)$$

Also

$$V_1 + V_2 = \frac{P\,h}{b_2} \quad . \; . \; . \; . \; . \; . \; . \quad (33)$$

from which V_1 and V_2 are readily found.

252. Portal Frames of Multiple Stories.—In building construction, and occasionally in other structures, quadrangular lateral bracing is used consisting of several such frames placed one above the other or side by side (Fig. 52). The analysis of such frames may be made in the same manner as for a single quadrangle. A section $p\,q$ is taken through the centre, and at each point C, there are three unknowns, M, V, and H. The necessary equations are obtained by writing out for each story the same relations as expressed in Art. 243, thus giving three equations for each section, corresponding to the number of unknowns.

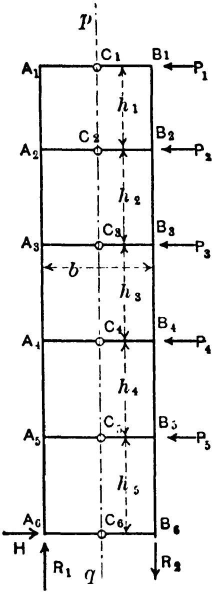


FIG. 52.

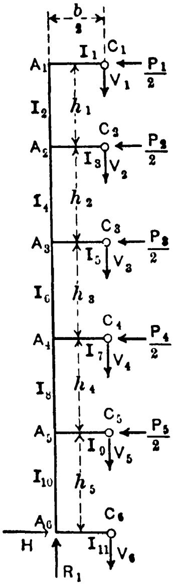


FIG. 53.

If the framework is symmetrical about the section $p\,q$, as is usually the case, then the points of inflection of the several transverse beams, $A\,B$, will be at the centre point C, and the several values of H will be one-half the lateral force, P, applied at the respective level. There remain to be determined then only the shears, V, at the sections C. The necessary equations are obtained by placing equal to zero the deflection of C_1 with respect to C_2, also that of C_2 with respect to C_3, etc. This gives five equations in the structure shown. The sixth is found by taking moments of all the forces about A_6, giving

$$\Sigma V \frac{b}{2} = \tfrac{1}{2}\,[P_1\,(h_1+h_2+h_3+h_4+h_5) + P_2\,(h_2+h_3+h_4+h_5) + \text{etc.}]$$

The deflections are very easily expressed in terms of V and P by use of the general expression $\Delta_y = \int \frac{M\,x\,d\,x}{E\,I}$. The necessary equations will be given here for a five-story frame, using the notation shown in Fig. 53.

For the deflection of C_1 relative to C_2, we need consider only the members $C_1\,A_1\,A_2\,C_2$. Using C_1, A_1, and C_2 as origins, the moment in $C_1\;A_1 = V_1\,x$; that in $A_1\,A_2 = V_1\frac{b}{2} - \frac{P_1}{2}\,y$, and in $A_2\,C_2$ it is $-\;V_2\,x$, considering as plus a moment which contributes to the downward deflection of C_1 relative to C_2. The value of the integrals $\int \frac{M\,x\,d\,x}{I}$ will be as follows:

for $C_1\,A_1$, $\frac{V_1\,b^3}{24\,I_1}$;

for $A_1\,A_2$, $\frac{V_1\,b^2\,h_1}{4\,I_2} - \frac{P_1\,h_1^2\,b}{8\,I_2}$;

for $A_2\,C_2$, $-\,\frac{V_2\,b^3}{24\,I_3}$;

whence we have

$$\frac{V_1\,b\,h_1}{I_2} + \frac{V_1\,b^2}{6\,I_1} - \frac{V_2\,b^2}{6\,I_3} = \frac{P_1\,h_1^2}{2\,I_2}. \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

For the deflection of C_2 with respect to C_3, consider only the frame $C_2\,A_2\,A_3\,C_3$. The moment in $C_2\,A_2 = V_2\,x$; in $A_2\,A_3$ it is $(V_1 + V_2)\,\frac{b}{2} - \frac{P_1}{2}\,h_1 - \frac{P_1 + P_2}{2}\,y$; and in $A_3\,C_3$ it is $-\;V_3\,x$. Except for the member $A_2\,A_3$ the values are similar to those in the first story. The value of $\int \frac{M\,x\,d\,x}{I}$ for $A_2\,A_3$ is

$$(V_1 + V_2)\,\frac{b^2\,h_2}{4\,I_4} - \frac{P_1\,h_1\,h_2\,b}{4\,I_4} - \frac{(P_1 + P_2)\,h_2^2\,b}{8\,I_4}.$$

We then derive the expression, corresponding to (34),

$$(V_1 + V_2) \times \frac{b\,h_2}{I_4} + \frac{V_2 b^2}{6\,I_3} - \frac{V_3 b^2}{6\,I_5} = \frac{(P_1 + P_2)\,h_2^2 + 2\,P_1\,h_1\,h_2}{2\,I_4}. \qquad (35)$$

For the next section, with origin at C_3, the moment in $A_3 A_4$ is equal to $(V_1 + V_2 + V_3)\frac{b}{2} - \frac{P_1(h_1 + h_2) + P_2 h_2}{2} - \frac{(P_1 + P_2 + P_3)\,y}{2}$. We derive then, as before,

$$(V_1 + V_2 + V_3)\frac{bh_3}{I_6} + \frac{V_3 b^2}{6\,I_5} - \frac{V_4 b^2}{6\,I_7} = \frac{(P_1 + P_2 + P_3)\,h_3^2 + 2\,[P_1(h_1 + h_2) + P_2 h_2]\,h_3}{2\,I_8}. \qquad (36)$$

And for the lower story we derive in a similar manner:

$$(V_1 + V_2 + V_3 + V_4 + V_5)\frac{bh_5}{I_{10}} + \frac{V_5 b^2}{6I_9} - \frac{V_6 b^2}{6I_{11}} =$$

$$\frac{(P_1 + P_2 + P_3 + P_4 + P_5)h_5^2 + 2[P_1(h_1 + \ldots h_4) + P_2(h_2 + \ldots h_4) + P_3(h_3 + h_4) + P_4 h_4]h_5}{2\,I_{10}}. \quad . \; (37)$$

Then taking moments about A_6, we have

$$(V_1 + .. + V_6)\frac{b}{2} = \frac{P_1(h_1 + .. + h_5) + P_2(h_2 + .. + h_5) + P_3(h_3 + h_4 + h_5) + P_4(h_4 + h_5) + P_5 h_5}{2} \; (38)$$

These six linear equations between the values of V are readily solved after substituting numerical coefficients. The values of V being known, the moments are readily found at any section. For more numerous stories additional equations are easily written out following the form of eq. (37). If the lower ends of the posts are fixed, then the value of I for the lower strut is ∞. If hinged, then the value of V_6 for the lower strut is zero in eq. (37), and equations of deflection are written out for all stories above the lowest.

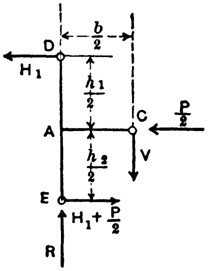


FIG. 54.

For a large number of stories an exact analysis becomes quite laborious. Approximate results may be reached by assuming that the point of inflection of the columns is midway between the lateral struts. For any particular joint the forces acting are then as shown in Fig. 54. H_1 represents half the sum of all the lateral forces above this story, and $\frac{P}{2}$ the lateral force applied at this section. Taking moments about E, we have $V\frac{b}{2} = \frac{H_1(h_1 + h_2)}{2} + \frac{P h_2}{4}$, whence

$$V = \frac{2 H_1 (h_1 + h_2) + P h_2}{2 b}. \quad . \quad . \quad . \quad . \quad (39)$$

253. Portal-Braced Towers with Inclined Posts.—The Kinzua Viaduct, constructed in 1900, is of the type shown in Fig. 55. The lateral rigidity is secured by relatively deep struts riveted to the main columns of the towers. The analysis of such a structure may be made in the manner explained in the preceding article. Points C_1, C_2, etc., are points of inflection, and their deflection with respect to each other will be zero. Fig. 56 represents one-half the structure. The equations will vary from those of the preceding article on account of the inclined posts, in the same manner as those of Art. 251 differ from those of Art. 244. For the struts, the integrals are of the same form as in Art. 244. For the posts they are different.

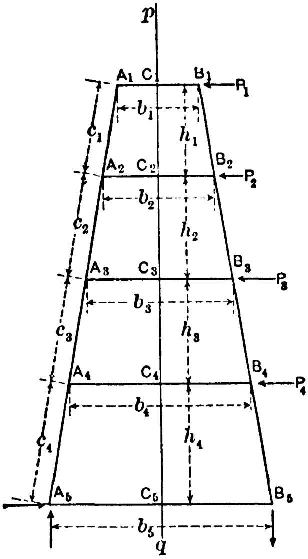


FIG. 55.

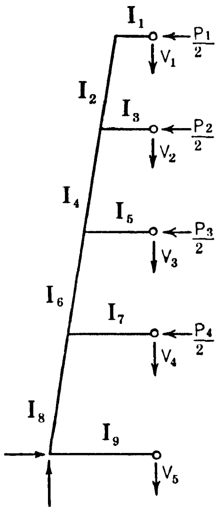


FIG. 56.

As deduced in Art. 251 the integral $\int \frac{M x\, d x}{I}$ for $A_1 A_2$ is

$$\frac{V_1 c_1 (b_1^2 + b_1 b_2 + b_2^2)}{12 I_2} - \frac{P_1 h_1 c_1 (2 b_2 + b_1)}{24 I_2},$$

and for the top story the equation is the same as eq. (32) of Art. 251. For the second story the integrals for $C_2 A_2$ and $C_3 A_3$ are equal to $\frac{V_2 b_2^3}{24 I_3}$ and $-\frac{V_3 b_3^3}{24 I_5}$ respectively. For $A_2 A_3$ the effect of V_1 and V_2 is the same as that of a force $V_1 + V_2$ acting at C_2, and is found by substituting $V_1 + V_2$ for V_1 in the expression above, and changing subscripts. It is $\frac{(V_1 + V_2)\, c_2\, (b_2^2 + b_2 b_3 + b_3^2)}{12\, I_4}$. For the horizontal forces $\frac{P_1}{2}$ and $\frac{P_2}{2}$, their effect on $A_2 A_3$ is the same as a single force $\frac{P_1 + P_2}{2}$ applied at C_2, together with a constant moment equal to $\frac{P_1 h_1}{2}$. The former is equal to $\frac{(P_1 + P_2)\, h_2 c_2\, (2\, b_3 + b_2)}{24\, I_4}$. The constant moment $\frac{P_1 h}{2}$ applied in $A_2 A_3$ is at an average distance x from C_2 equal to $\frac{b_2 + b_3}{4}$. hence for this part $\int \frac{M\, x\, d\, x}{I} = \frac{P_1\, h_1\, c_2\, (b_2 + b_3)}{8\, I_4}$. After multiplying through by 12 the equation for the second story becomes then

$$\frac{(V_1 + V_2)\, c_2\, (b_2^2 + b_2 b_3 + b_3^2)}{I_4} + \frac{V_2 b_2^3}{2\, I_3} - \frac{V_3 b_3^3}{2\, I_5}$$
$$= \frac{(P_1 + P_2)\, h_2 c_2\, (2\, b_3 + b_2) + 3\, P_1\, h_1\, c_2\, (b_2 + b_3)}{2\, I_4} \quad . \quad (40)$$

For the third story the equation is

$$\frac{(V_1 + V_2 + V_3)\, c_3\, (b_3^2 + b_3 b_4 + b_4^2)}{I_6} + \frac{V_3 b_3^3}{2\, I_5} - \frac{V_4 b_4^3}{2\, I_7}$$
$$= \frac{(P_1 + P_2 + P_3) h_3 c_3 (2 b_4 + b_3) + 3[P_1 (h_1 + h_2) + P_2 h_2] c_3 (b_3 + b_4)}{2\, I_6} \quad . \quad (41)$$

etc. etc. etc.

Then from moments about A_5, we have

$$V_1 + V_2 + V_3 + V_4 + V_5 = \frac{P_1 (h_1 + \ldots h_4) + P_2 (h_2 + \ldots h_4) + P_3 (h_3 + h_4) + P_4 h_4}{b_5} \quad . \quad . \quad (42)$$

EXAMPLE.—The Kinzua Viaduct. (Trans. Am. Soc. C. E., Vol. 46, 1901.) Fig. 57 represents a section of the Kinzua Viaduct, at a point of maximum height, where five stories are employed. The dimensions *c.* to *c.* are as shown and also the values of the several moments of inertia.

The posts are assumed as fixed at the base, which is equivalent to assuming $I_{11} = \infty$

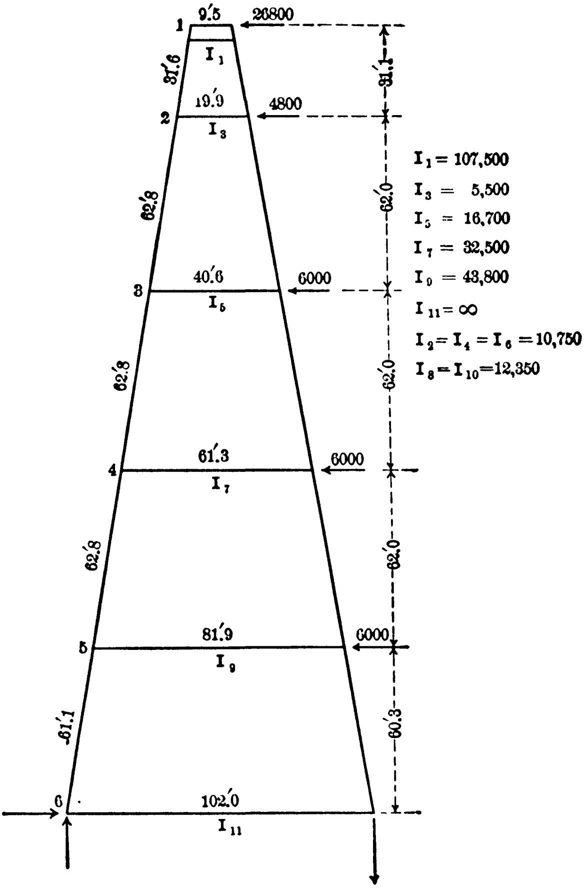


FIG. 57.

For the top story we have, by substituting in (32),

$$\frac{V_1 \times 31.6\,(9.5^2 + 9.5 \times 19.9 + 19.9^2)}{10,750} + \frac{V_1 \times 9.5^3}{2 \times 107,500} - \frac{V_2 \times 19.9^3}{2 \times 5,500}$$

$$= \frac{26,800 \times 31.1 \times 31.6\,(2 \times 19.9 + 9.5)}{2 \times 10,750}$$

or

$$1.989\, V_1 - 0.716\, V_2 = 60,500. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Then from (40) we have

$$16.68\ V_1 + 17.40\ V_2 - 2.005\ V_3 = 1{,}021{,}000. \quad . \quad . \quad . \quad (b)$$

In a similar manner, for the 3d, 4th, and 5th stories we have

$$\begin{aligned} 46.1\ V_1 + 46.1\ V_2 + 48.11\ V_3 - 3.54\ V_4 &= 3{,}602{,}000. \quad (c)\\ 78.65\ V_1 + 78.65\ V_2 + 78.65\ V_3 + 82.19\ V_4 - 6.26\ V_5 &= 7{,}163{,}000. \quad (d)\\ 126.0\ V_1 + 126.0\ V_2 + 126.0\ V_3 + 126.0\ V_4 + 132.26\ V_5 &= 12{,}785{,}000. \quad (e) \end{aligned}$$

(V_6 does not appear here as $I_{11} = \infty$).

In eliminating, note that V_5 may be eliminated from the last two equations, then V_4 from the resulting equation combined with (c), then V_3 and V_2, getting finally the value of V_1. Then by substitution the other values are readily found. The results are,

$V_1 = 38{,}900$ lbs., $V_4 = 12{,}400$ lbs.,
$V_2 = 23{,}400$ lbs., $V_5 = 9{,}900$ lbs.
$V_3 = 16{,}300$ lbs.,

These agree very closely with those given by Grimm in Trans. Am. Soc. C. E., Vol. 46, p. 32, calculated from the theory of least work.

If the posts were hinged at the base, then eq. (e) would no longer hold good. In its place we would make use of eq. (42), taking moments about the foot of the post.

254. Deflection of Quadrangular Frames.—The relative rigidity of quadrangular frames, or portals, as compared with full diagonal bracing is a matter of some importance. In any given case the deflection of such a frame is readily found from the general formula, $\Delta = \Sigma \int \frac{M\,m\,d\,s}{E\,I}$, where m is the moment due to a one-pound load acting at the point whose deflection is desired. General formulas are unwieldy except in simple cases. Such a case is that of the symmetrical frame subjected to a lateral force as discussed in Art. 249. A comparison of its deflection with that of a fully braced frame will be instructive.

The lateral deflection of the frame of Art. 249 can be calculated by considering the lateral deflection of point C with respect to F, of the half-frame as shown in Fig. 58. Let M_1 = moment at A due to the lateral force, and M_2 = moment at D. The moment in $A\,D$, at any point distant x from A, will be $M = M_1 + (M_2 - M_1)\,\frac{x}{h}$

$= M_1\left(1 - \frac{x}{h}\right) + M_2\frac{x}{h}$. The moment in DF at any point distant x from $F = M_2\frac{2x}{b}$. For a one-pound load applied horizontally at C,

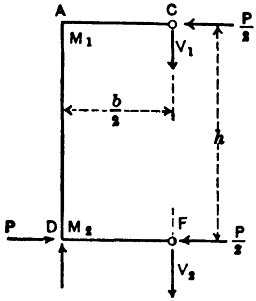


FIG. 58.

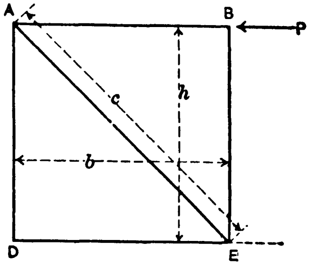


FIG. 59.

the moment in $AC = 0$, that in $AD = m = x$; and that in $DF = m = \frac{2hx}{b}$. The integrals $\int\frac{Mm\,dx}{I}$ are then as follows:

For AD, $M_1\frac{h^2}{6I_2} + M_2\frac{h^2}{3I_2}$.

For DF, $M_2\frac{bh}{6I_3}$.

The deflection is, therefore,

$$\Delta_1 = \frac{h}{6E}\left[(M_1 + 2M_2)\frac{h}{I_2} + M_2\frac{b}{I_3}\right]. \quad . \quad . \quad . \qquad (43)$$

For a fully braced rectangular frame (Fig. 59), the deflection is found from the formula $\Delta = \Sigma\frac{Sul}{EA}$. Assuming approximately that the section of the diagonal is small as compared to the other members of the frame, we derive the value

$$\Delta_2 = \frac{S(h^2 + b^2)}{EAb} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

in which S is the total stress in the diagonal due to the load P.

To compare the relative values of Δ in (43) and (44), suppose that $b = h$, and $I_2 = I_3$. Then $M_1 = M_2$. Also let $S/A = s =$

working stress in the diagonal of Fig. 59, and let f = maximum fibre stress in the members of Fig. 58, due to the moment M_1, and let c = the half width of members of Fig. 58 = distance from neutral axis to extreme fibre. Then in eq. (43), $\frac{M}{I} = f/c$, and hence $\Delta_1 = \frac{2\,h^2 f}{3\,E c}$. And in (44) $\Delta_2 = \frac{2\,h\,s}{E}$. The ratio of the two values is

$$\frac{\Delta_1}{\Delta_2} = \frac{h\,f}{3\,c\,s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (45)$$

Generally the adopted working stress f, for lateral forces, will be much less than s. Suppose $f/s = 1/4$, then

$$\frac{\Delta_1}{\Delta_2} = \frac{h}{12\,c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

In this case, therefore, the deflections of the two forms will be equal if the half width, c, of the members, is equal to 1/12 the height of the panel. Generally the members are of much more slender proportions.

The preceding analysis shows that the quadrangular frame will be relatively flexible as compared to a diagonal bracing unless the working stresses for the lateral forces are small and the width of members large. The analysis also shows that for given working stresses the flexibility is inversed proportional to the widths of the members (not to the moments of inertia).

SECTION IV.—TRUSSED BEAMS

255. The King Post Truss.—A combination form of beam and truss often employed is that shown in Fig. 60. The member $A\,B$ is a relatively large beam, supported by the trussing at C. Loads may be applied at any point. The beam $A\,C\,B$, therefore, acts as a strut in the truss $A\,D\,B$ as well as a beam carrying the applied loads directly. The stresses may be solved by the general method of redundant members, assuming the stress in $C\,D$ as redundant. This member is cut at C and the deflections of the beam above, and the truss below, are placed equal.

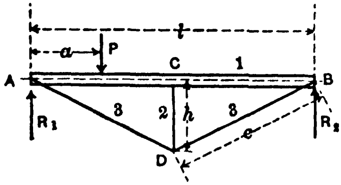


FIG. 60.

Let S_r = stress in CD. The deflection of the beam is that due to the load P and the force S_r. By Table 1, Art. 7, this is equal to $\frac{Pa}{12\,EI}\left(\frac{3\,l^2}{4} - a^2\right) - \frac{S_r l^3}{48\,EI}$. The deflection of the truss is found by calculating the value of $\Sigma \frac{S\,u\,l}{E\,A}$, for a load of S_r at C. The calculations are as follows:

Member.	S	u	$\frac{u\,l}{A}$	$\frac{S\,u\,l}{A}$
1	$-\frac{S_r l}{4\,h}$	$-\frac{l}{4\,h}$	$-\frac{l^2}{8\,h\,A_1}$	$\frac{S_r l^3}{32\,h^2 A_1}$
2	$-S_r$	-1	$-\frac{h}{A_2}$	$\frac{S_r h}{A_2}$
3	$+\frac{S_r c}{2\,h}$	$+\frac{c}{2\,h}$	$+\frac{c^2}{2\,h\,A_3}$	$\frac{S_r c^3}{4\,h^2 A_3}$

Adding, and taking twice the given values for members 1 and 3, we have for the truss =

$$\Delta = S_r\left[\frac{l^3}{16\,h^2 E_1 A_1} + \frac{h}{E_2 A_2} + \frac{c^3}{2\,h^2 E_3 A_3}\right] \quad .$$

whence, writing the deflections equal and solving for S_r, we have

$$S_r = \frac{\frac{Pa}{12\,E_1 I_1}\left(\frac{3\,l^2}{4} - a^2\right)}{\frac{l^3}{16\,h^2 E_1 A_1} + \frac{h}{E_2 A_2} + \frac{c^3}{2\,h^2 E_3 A_3} + \frac{l^3}{48\,E_1 I_1}} \quad . \quad . \quad (47)$$

Frequently the beam is of wood and the trussing iron or steel, in which case the values of the various moduli must be used. Generally the terms in the denominator of (47) involving direct stress are small as compared to those involving moment, and may be neglected, giving

$$S_r = \frac{4\,Pa}{l^3}\left(\frac{3\,l^2}{4} - a^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

which is the centre reaction for a beam continuous over three supports.

For a uniform load w, extending from a distance $a = x_1$ to a distance $a = x_2$, the numerator of (47) is changed by placing $P =$

$w\,d\,x$ and $a = x$, and integrating $P\,a\,(\frac{3}{4}\,l^2 - a^2)$ from x_1 to x_2. For a full uniform load the numerator is the central deflection of a beam uniformly loaded $= \dfrac{5\,w\,l^4}{384\,E\,I}$.

256. The Queen Post Truss Without Diagonals.—Trussed beams are sometimes made of the form shown in Fig. 61, the member $A\,D$ being a continuous beam. No diagonals being used in the middle panel, the rigidity of the construction depends upon the strength of the beam, the trussing furnishing equal reactions at B and C.

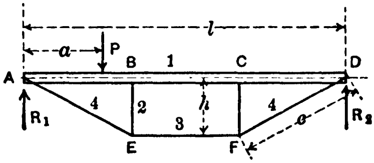


FIG. 61.

Let S_r = stress in $B\,E$ and $C\,F$, assumed as equal. This is very nearly true, as the small deflections of E and F do not appreciably change the inclinations of $A\,E$, $E\,F$, and $F\,D$. The horizontal components of $A\,E$ and $F\,D$ being equal (equal to the stress in $E\,F$), their vertical components are equal and hence the stresses in $B\,E$ and $C\,F$.

The problem is solved by determining the *sum* of the deflections of points B and C as regards the beam, and placing this equal to the same sum as calculated from the truss. Since the truss alone is unstable, the deflection of one of the points depends upon the other, but the sum of the two is a definite quantity.

For a load P, placed between A and B, the deflections of B and C are given by the table of Art. 7. They are

$$\Delta_b = \frac{P\,a}{9\,E\,I}\left(\frac{5\,l^2}{9} - a^2\right)$$

$$\Delta_c = \frac{P\,a}{18\,E\,I}\left(\frac{8\,l^2}{9} - a^2\right)$$

Then

$$\Delta_b + \Delta_c = \frac{P\,a}{18\,E\,I}\,(2\,l^2 - 3\,a^2).$$

For two upward forces S_r, applied at B and C, the deflection of each point is equal to

$$\Delta'_b = \frac{5\,S_r l^3}{162\,EI}$$

and the sum of the deflections for the two points is $\frac{5\,S_r l^3}{81\,EI}$. The downward deflection of the truss at B and C, due to two forces S_r. is calculated as before. It is

$$\Delta'' = S_r\left[\frac{l^3}{9\,h^2 E_1 A_1} + \frac{2\,h}{E_2 A_2} + \frac{l^3}{27\,h^2 E_3 A_3} + \frac{2\,c^3}{h^2 E_4 A_4}\right].$$

Placing $\Delta_b + \Delta_c - 2\,\Delta'_b = \Delta''$, and solving for S_r, we have,

$$S_r = \frac{\dfrac{P\,a}{18\,E_1 I_1}(2\,l^2 - 3\,a^2)}{\dfrac{l^3}{9\,h^2 E_1 A_1} + \dfrac{2\,h}{E_2 A_2} + \dfrac{l^3}{27\,h^2 E_3 A_3} + \dfrac{2\,c^3}{h^2 E_4 A_4} + \dfrac{5\,l^3}{81\,E_1 I_1}} \quad . \quad . \quad (49)$$

For a load between B and C

$$S_r = \frac{\dfrac{P\,l}{162\,E_1 I_1}[27\,(a\,l - a^2) - l^2]}{D_{49}} \quad . \quad . \quad . \quad . \quad (50)$$

in which D_{49} is the denominator in eq. (49). For a full uniform load of w per unit length, the value of S_r is

$$S_r = \frac{\dfrac{11\,w\,l^4}{486\,EI}}{D_{49}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

If the direct stresses are neglected in (49), the value of the denominator becomes equal to $\frac{5\,l^3}{81\,EI}$. For a load at B or C, $a = l/3$, and we have $S_r = P/2$, that is, the truss carries one-half the load and the beam one-half. The upward pressure at C is also $P/2$, thus giving a point of inflection in the beam at the centre.

For trussed beams of three or more panels, with full diagonal bracing, the case becomes similar to the calculation of secondary stresses in trusses with rigid joints. This case is fully treated in the following chapter.

257. Truss of Several Panels with Diagonals Omitted in One Panel.—It is sometimes desired to estimate the supporting power of a truss from which one of the web members has been removed, the chords being continuous and the joints riveted. The strength of such a truss is dependent upon the bending resistance of the chord members $C D$ and $E F$, but the exact stresses therein, due to a given loading, can be calculated only by considering all the joints of the structure. Generally a sufficiently exact result can be obtained by assuming the points of inflection of $C D$ and $E F$ to be at their centres, G and H. Let V = shear in panel $E F$, caused by any load, Fig. 62. The sum of the moments at C and E, in $C D$ and $E F$, is then $V \times \frac{d}{2}$, and the relative moments carried by the two members will be in proportion to their moments of inertia (their deflections being equal). Hence we have $M_1 + M_2 = V \frac{d}{2}$, and $\frac{M_1}{M_2} = \frac{I_1}{I_2}$, whence

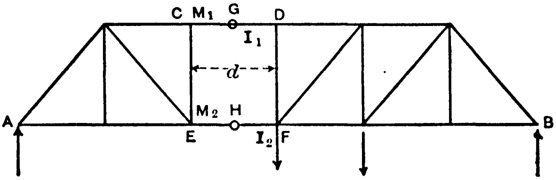


FIG. 62.

$$M_1 = \frac{V d I_1}{2 (I_1 + I_2)} \text{ and } M_2 = \frac{V d I_2}{2 (I_1 + I_2)}. \quad . \quad . \quad (52)$$

For a more exact solution recourse must be had to the methods used in calculating secondary stresses (Chapter VII).

SECTION V.—BEAMS ON MULTIPLE ELASTIC SUPPORTS

258. The General Problem.—The problem of determining the distribution of loads upon elastic supports, transferred thereto by means of an intermediate beam, is one which frequently arises in practice, particularly in connection with bridge floors of various types. The maximum load supported by a single cross-tie, or one element of a steel floor system, consisting of closely spaced transverse members, such as *I*-beams, or trough-shaped sections, are problems of this kind. Usually a solution is reached by making an approximate estimate which will be certain to be on the safe side. It is not difficult, however, to arrive at a reasonably exact solution by methods used in solving other statically indeterminate combinations. In the following articles a general solution will be given, and a diagram from which numerical results may readily be obtained for a variety of conditions.

259. General Method of Solution.—Fig. 63 represents an arrangement such as here considered. *A B* is a longitudinal beam supporting a concentrated load, *P*. The beam, *A B*, rests upon numerous transverse beams, 1, 2, 3, 4, etc., which in turn are supported at their ends by relatively deep and rigid girders or other rigid supports. Fig. (*b*) shows one of the transverse beams *M N*, supporting two longitudinal beams, *A B*, symmetrically placed, as in the case of the usual single-track railway bridge-floor. The beams *M N* may be ties resting upon stringers, or units of a steel floor system riveted into longitudinal girders; and the beam, *A B*, may be merely the rail, or a rail together with channel, *I*-beam, or other supplementary support. In any case the beam, *A B*, as here considered, includes all of the elements acting together to support the load *P*. In determining the moment of inertia of this combination the moment of inertia of each independent part is to be taken about its own gravity axis and the results added. If one element is of wood and another of steel, then, as will later appear, the product *E I* of each element is to be taken and the results added.

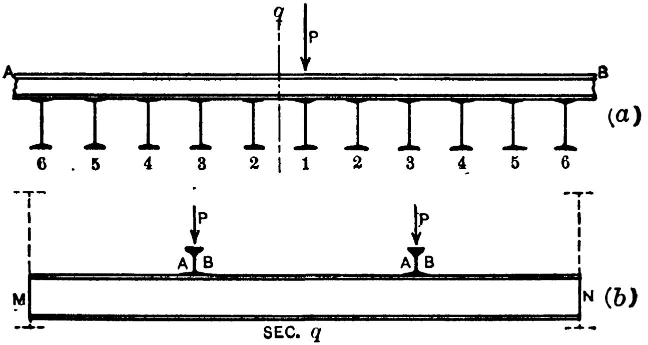


FIG. 63.

In the solution of this problem it is assumed that the supports of the transverse beam *M N* are so rigid, compared to the beam *A B*, that they may be assumed as absolutely so. The various transverse beams, 1, 2, 3, etc., will, therefore, deflect in proportion to the load brought upon them, or in proportion to the reaction which they supply to the beam *A B*. Generally the beam *A B* is continuous over a large number of transverse supports, and, in turn, it supports several loads *P*, of various weights and spacing.

In arriving at a solution it will be sufficiently exact to consider not more than nine supports, symmetrically spaced with respect to the load *P*, and to assume that the beam *A B* ends at the last support at either end. The principal problem is to determine the load carried by beam No. 1, immediately beneath the load *P*, but at the same time the loads carried by the other beams should be found. The error involved by neglecting the supports beyond the nine here considered is very small, and in most cases the results are accurate enough if only seven supports are considered.

If additional loads are applied to the beam *A B*, within the length considered, their effect upon the load carried by No. 1, can be found from the results obtained for the load *P*. Thus if a load *P′* is placed over beam No. 3, the proportion of *P′* transferred to No. 1 may be taken as the same as the proportion of *P* carried by No. 3, etc. Loads beyond the last support considered do not appreciably affect the load on No. 1.

260. Deflection of the Transverse Beam.—Fig. 64 represents a transverse beam of length *b*, loaded with two equal loads, *Q*, symmetrically spaced a distance *z* from the ends. The deflection of points *O*, where the loads are applied, is given by the formula of Table 1, Art. 7. In terms of the load *Q*, it is

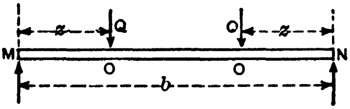


FIG. 64.

$$\Delta = \frac{Q z^2}{6 E I}(3 l - 4 z). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It will also be desirable to express this deflection in terms of the maximum fibre stress due to the load Q. The bending moment $= Q z$ and the fibre stress $= f = \frac{M h}{2 I} = \frac{Q z h}{2 I}$; whence $Q z = \frac{2 I f}{h}$. Substituting in (1), we then have also

$$\Delta = \frac{f z}{3 E h}(3 b - 4 z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which h is the depth of the beam, assumed as symmetrical about its neutral axis.

In the subsequent analysis it is necessary to have a measure of the flexibility of the transverse beams. A convenient measure or coefficient of this flexibility is the amount of deflection at the points O, due to a load of one pound placed at each point. If k represents this coefficient of flexibility, then

$$k = \frac{\Delta}{Q} = \frac{z^2}{6 E I}(3 b - 4 z), \quad . \quad . \quad . \quad . \quad . \quad (3)$$

also

$$k = \frac{f z}{3 E h Q}(3 b - 4 z). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Generally the problem relates to a specific design already determined by approximate methods. In this case k is determined from (3), as the value of I is already known. In making a new design,

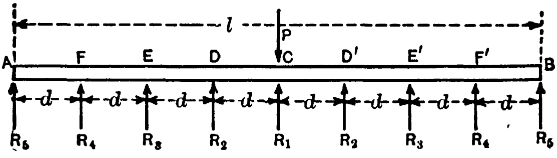


FIG. 65.

however, it is considerably more expeditious to fix only upon the depth h of the floor or beam. Then, using an estimated value of Q, the value of k is at once found from (4), without fully working out the cross-section of the floor element. Having k, the true value of Q is found as described later, and then if desired a corrected value of k. A considerable variation in k will affect the value of Q but little.

261. The Longitudinal Beam.—Fig. 65 represents the longitudinal beam, $A B$, and the several reactions, $R_1 \ldots\ldots\ldots R_5$, spaced a distance,

d, apart. The total moment of inertia of the beam $A\,B$ is I, and its modulus of elasticity is E. These enter into the calculation only as the product $E\,I$, and if two kinds of material are used, the total value of $E\,I$ is to be obtained by adding these products for the several elements.

The forces being vertical, only two reactions can be determined by statics. There are, therefore, seven redundant reactions. These may be taken as reactions $R_1 \ldots\ldots\ldots R_4$, the end reactions, R_5, being considered as the two "necessary" reactions. By reason of symmetry of arrangement the reactions on one side are equal to those on the other side, symmetrically located; hence there are practically only four unknowns, $R_1 \ldots\ldots\ldots R_4$, which requires for solution four condition equations. These condition equations may be obtained by the method of least work, expressing the total work performed in the deflection of the beam and its supports, then differentiating with respect to each support and placing such derivative equal to zero; or they may be obtained by the method of redundant members, placing the deflection of the beam at each support equal to the settlement of that support. The latter method is the simpler, and in its application the deflection formula given in the Table in Art. 7 will be made use of directly, instead of using the fundamental expressions of Art. 212.

The four condition equations will then be obtained by placing the deflection of the beam at points C, D, E, and F, equal to the deflection of the respective support, which in general is equal to $R\,k$, where R is the reaction at any point. If Δ_c, Δ_d, etc., are the deflections at the several points, then the equations are, in general terms:

$$\left.\begin{aligned} \Delta_c &= R_1\,k \\ \Delta_d &= R_2\,k \\ \Delta_e &= R_3\,k \\ \Delta_f &= R_4\,k \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

In calculating the deflections Δ, the effect of the load P, and of the several reactions, $R_1 \ldots\ldots\ldots R_4$, may be expressed separately, and the several results added.

262. *Deflections Due to Load P.*—In calculating the deflection due to the load P, it is assumed that the forces, $R_1 \ldots\ldots\ldots R_4$, are removed. The deflection of any point will then be that due to: (1) the bending of the beam under load P, with reactions at A and B, and (2) the settlement of the supports at A and B, which will be $\frac{P}{2}k$. The deflection of any point of a beam, supporting a single load P, at the centre, is given by the general formula (Table 1, p. 10) $y = \frac{Px}{48\,EI}(3\,l^2 - 4\,x^2)$, in which x is the distance of the point from the near end of the beam. Applying this formula to the several points, C, D, E, and F, the value of x will be equal to $4\,d$, $3\,d$, $2\,d$, and d. We have also, $l = 8\,d$. Adding in each case the quantity $\frac{P}{2}k$, we obtain the following values for the deflections due to P:

$$\left.\begin{aligned}\delta_c &= P\left(\frac{32\,d^3}{3\,EI} + \frac{k}{2}\right)\\ \delta_d &= P\left(\frac{39\,d^3}{4\,EI} + \frac{k}{2}\right)\\ \delta_e &= P\left(\frac{22\,d^3}{3\,EI} + \frac{k}{2}\right)\\ \delta_f &= P\left(\frac{47\,d^3}{12\,EI} + \frac{k}{2}\right)\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6)$$

263. *Deflections Due to* R_1.—These deflections will be given by the same equations as for P, the deflections being upward.

264. *Deflections Due to* R_2.—The deflections due to the two symmetrical loads, R_2, are obtained by the general formulas from Table 1. They are:

$$\left.\begin{aligned}&\text{for } x < z, \quad y = \frac{Px}{6\,EI}(3\,lz - 3\,z^2 - x^2)\\ &\text{for } x > z, \quad y = \frac{Pz}{6\,EI}(3\,lx - 3\,x^2 - z^2)\end{aligned}\right\} \quad \ldots \quad (7)$$

In this case, since there are two equal forces, R_2, the end reactions due to these loads are each equal to R_2, and the movement of end supports due to their elasticity, will be $R_2\,k$. In applying eq. (7), $l = 8\,d$, and $z = 3\,d$. For point C, $x = 4\,d$, and the second equation (7) is used.

The several deflections are:

$$\left.\begin{aligned} \delta_c &= R_2\left(\frac{39}{2}\frac{d^3}{EI} + k\right) \\ \delta_d &= R_2\left(\frac{18\,d^3}{EI} + k\right) \\ \delta_e &= R_2\left(\frac{41}{3}\frac{d^3}{EI} + k\right) \\ \delta_f &= R_2\left(\frac{22}{3}\frac{d^3}{EI} + k\right) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (8)$$

By the principle of reciprocal deflections explained in Art. 222, Part I, it is observed that the deflection at C, due to a single force applied at D, is equal to the deflection at D due to a like force applied at C. Hence the value of δ_c of eq. (8) is double the value of δ_d of eq. (6).

265. *Deflections Due to R_3.*—In this case $z = 2\,d$ in eq. (7). The only values requiring calculation are δ_e and δ_f, as the value of δ_c is twice δ_e of eq. (6), and $\delta_d = \delta_e$ of eq. (8). The results are:

$$\left.\begin{aligned} \delta_c &= R_3\left(\frac{44}{3}\frac{d^3}{EI} + k\right) \\ \delta_d &= R_3\left(\frac{41}{3}\frac{d^3}{EI} + k\right) \\ \delta_e &= R_3\left(\frac{32}{3}\frac{d^3}{EI} + k\right) \\ \delta_f &= R_3\left(\frac{35}{6}\frac{d^3}{EI} + k\right) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (9)$$

266. *Deflections Due to R_4.*—The values of δ_c, δ_d, and δ_e are obtained from the values of δ_f of eqs. (6), (8), and (9). The value of δ_f is calculated from eq. (7). The results are:

$$\left.\begin{aligned} \delta_c &= R_4\left(\frac{47}{6}\frac{d^3}{EI} + k\right) \\ \delta_d &= R_4\left(\frac{22}{3}\frac{d^3}{EI} + k\right) \\ \delta_e &= R_4\left(\frac{35}{6}\frac{d^3}{EI} + k\right) \\ \delta_f &= R_4\left(\frac{10}{3}\frac{d^3}{EI} + k\right) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (10)$$

267. *Resulting Equations.*—Placing the total deflection at C, for all loads and forces, equal to $R_1\,k$, we have the following equation:

$$P\left(\frac{32\,d^3}{3EI}+\frac{k}{2}\right)-R_1\left(\frac{32\,d^3}{3EI}+\frac{k}{2}\right)-R_2\left(\frac{39\,d^3}{2\,EI}+k\right)-R_3\left(\frac{44\,d^3}{3\,EI}+k\right)-R_4\left(\frac{47\,d^3}{6\,EI}+k\right)=R_1\,k.$$

Similar equations are derived by placing the total deflection at D equal to $R_2\,k$, that at E equal to $R_3\,k$, and that at F equal to $R_4\,k$.

Simplifying these equations by dividing by $\frac{d^3}{EI}$ and placing $\frac{k\,EI}{d^3}$ $= a$, we derive the four simple equations:

$$\left.\begin{aligned}R_1(\ 64+9a)+R_2(117+\ 6a)+R_3(\ 88+\ 6a)+R_4(47+\ 6a)&=P(\ 64+3a)\\R_1(117+6a)+R_2(216+24a)+R_3(164+12a)+R_4(88+12a)&=P(117+6a)\\R_1(\ 44+3a)+R_2(\ 82+\ 6a)+R_3(\ 64+12a)+R_4(35+\ 6a)&=P(\ 44+3a)\\R_1(\ 47+6a)+R_2(\ 88+12a)+R_3(\ 70+12a)+R_4(40+24a)&=P(\ 47+6a)\end{aligned}\right\}\quad .\ .\ (11)$$

The algebraic solution need not be carried further. In any problem the numerical value of $a = \frac{k\,E\,I}{d^3}$ should be calculated, then substituted in eq. (11), and the values of R_1, R_2, R_3, and R_4 determined by elimination.

268. *Equations for Seven and Five Supports.*—As it may be desired to analyze a case where a less number of supports are involved, there are given here the necessary equations for the case of seven supports and five supports, calculated in the same manner as above described. For fairly rigid supports the equations for seven or even five supports will give sufficiently accurate results for R_1.

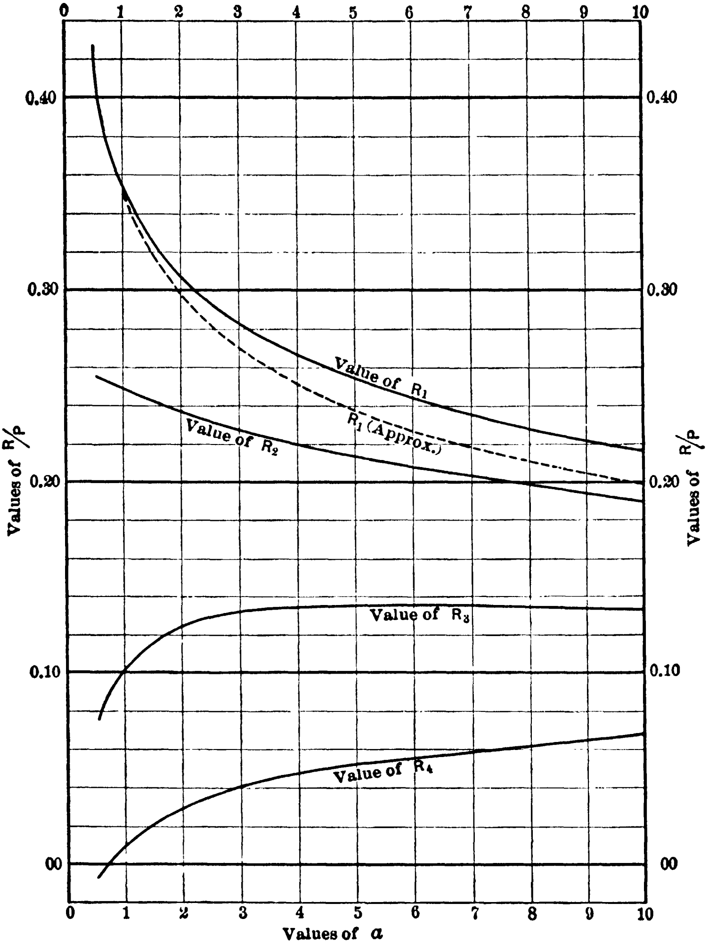
For seven supports:

$$\left.\begin{aligned}R_1\,(27+9\,a)+R_2\,(46+\ 6\,a)+R_3\,(26+\ 6\,a)&=P\,(27+3\,a)\\R_1\,(23+3\,a)+R_2\,(40+12\,a)+R_3\,(23+\ 6\,a)&=P\,(23+3\,a)\\R_1\,(13+3\,a)+R_2\,(23+\ 6\,a)+R_3\,(14+12\,a)&=P\,(13+3\,a)\end{aligned}\right\}\quad .\ .\ (12)$$

For five supports:

$$\left.\begin{aligned}R_1\,(\ 8+9\,a)+R_2\,(11+\ 6\,a)&=P\,(\ 8+3\,a)\\R_1\,(11+6\,a)+R_2\,(16+24\,a)&=P\,(11+6\,a)\end{aligned}\right\}\quad .\ .\ (13)$$

269. Diagram for Values of Reactions.—In the usual problem of bridge-floor design the value of a will rarely be less than 1 nor more



$$a = \frac{k E I}{d^3},$$

k = deflection in inches of the transverse beam for a load of one pound,

d = spacing in inches of transverse beams,

E and I refer to longitudinal beam A B.

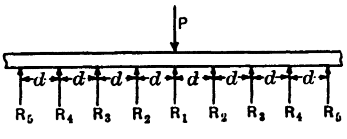


FIG. 66.—DIAGRAM OF REACTIONS FOR A BEAM ON MULTIPLE SUPPORTS.

than 10. The practical application of this method can therefore be greatly facilitated by constructing a diagram of the values of the ratio of maximum reaction R_1, to the concentrated load P. Such a diagram is given in Fig. 66, calculated from eq. (11), Art. 267. In addition, the values of R_2, R_3, and R_4 are also given for use where two concentrations are spaced closer together than 4 d.

270. The foregoing is a good illustration of a method of solution applicable to quite a variety of cases. For unequally spaced supporting beams, the number of unknowns would be increased and the equations would be modified accordingly.

Where the beams or individual supports are not too widely spaced, sufficiently accurate results can be obtained by considering the support to be continuous, an assumption which leads to a simpler solution. This is arrived at as follows:

Let p = pressure (upward load) per unit length of support, and k' = coefficient of flexibility of support = deflection for a load of unity per unit length. Then the upward pressure on the longitudinal beam per unit length will be $p = y/k'$, and hence, from Eq. (10), Art. 1,

$$E I \frac{d^4 y}{d x^4} = - \frac{y}{k'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Integrating this, we have

$$y = e^{cx}(A \sin c x + B \cos c x) + e^{-cx} (C \sin c x + D \cos c x),$$

in which $c = \sqrt[4]{\frac{1}{4 E I k'}}$. The constants of integration are determined by the conditions that $y = 0$ for $x = \infty$; $\frac{d y}{d x} = 0$ for $x = 0$ and $x = \infty$; and shear $\left(= E I \frac{d^3 y}{d x^3}\right) = - \frac{P}{2}$ for $x = 0$. Applying these conditions, we get finally

$$y = - \frac{P}{8 E I c^3} e^{-cx} (\sin cx + \cos c x). \quad . \quad . \quad . \quad (15)$$

For $x = 0$, $y = - \frac{P}{8 E I c^3}$ and the pressure per unit length at this point is

$$p_0 = - \frac{y}{k'} = \frac{P}{8 E I c^3 k'} = \frac{P}{\sqrt[4]{64 E I k'}}$$

Applying this to beams spaced d units apart, we have $p_0\, d = R_1$ and $k' = k\, d$; hence

$$R_1 = \frac{P\, d}{\sqrt[4]{64\, E\, I\, k\, d}}$$

or

$$R_1 = \frac{P}{2.83\, \sqrt[4]{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

The dotted curve in Fig. 66 is calculated from this equation.

The reactions at other beams can be obtained from Eq. (15) by using the proper value of x. A general expression is

$$R_n = R_1\, e^{-n\sqrt[4]{\frac{1}{a}}}\left(\sin n \sqrt[4]{\frac{1}{a}} + \cos n \sqrt[4]{\frac{1}{a}}\right) \quad . \quad . \quad . \qquad (17)$$

where n is the number of spaces from R_1 to R_n.*

271. Examples.—(1) What will be the maximum load on one tie of a bridge floor, the ties being of oak, 8″ x 8″ in section, spaced 12 in. apart and supported by stringers spaced 7 ft. apart. The rail is a 90-lb. rail with a moment of inertia of 35.2.

The modulus of elasticity of the tie may be taken at 2,000,000. Its moment of inertia is 341. Then from eq. (3) the value of the coefficient k is equal to $\frac{12^2}{6 \times 2{,}000{,}000 \times 341}(21 - 4) \times 12 = .0000072$. The value of a of eq. (11) is then

$$a = \frac{k\, E\, I}{d^3} = \frac{.0000072 \times 30{,}000{,}000 \times 35.2}{12^3} = 4.40.$$

From the diagram, Fig. 66, we find, for $a = 4.4$, $R_1/P = 0.26$, that is, the maximum load on one tie is 26 per cent of the wheel concentration. The load on the next tie is about 22 per cent, on the third tie is about 14 per cent, and on the fourth it amounts to 5 per cent. A second wheel spaced three ties (three feet) from the first will add about 5 per cent of its weight to the load under the wheel in question.

(2) Suppose a single-track steel floor system is to consist of elements spaced 16 ins. apart and of a depth of 18 in. Suppose, further, that the working stress to be employed is 15,000 lbs. per sq. in., and that the wheel concentration is 30,000 lbs., plus 100 per cent for impact. Let it be required to determine the maximum load on one of the floor elements. The width of the floor between supports is 12 ft., and the rails are 100 lb. rails having a moment of inertia of 44.4. They are supported in 10-in. channels having a

---

* For detailed application of this method to railroad track, see Transactions of the American Society of Civil Engineers, Vol. 82, 1918, p. 1203.

moment of inertia of 4.0. The total value of I for the combination is therefore 48.4.

The value of k will be first estimated from eq. (4). Assuming approximately that $Q = 40$ per cent of the wheel load, $= 24{,}000$ lbs., we have, for $l = 12$ ft., $z = 3.5$ ft., $h = 18$ in.,

$$k = \frac{15{,}000 \times 3.5 \times 12 \times (36 - 14) \times 12}{3 \times 18 \times 24{,}000 \times E} = \frac{128}{E}$$

$$a = \frac{k E I}{d^3} = \frac{128 \times 48.4}{16^3} = 1.5.$$

With $a = 1.5$ we find from Fig. 66, $R_1 = 32$ per cent of P or 19,200 lbs.

We can now correct the value of k in eq. (4), finding, for $Q = 19{,}200$, $k = \dfrac{160}{E}$. Then $a = 1.9$ and the diagram gives $R_1 = 31$ per cent of P. If the transverse beam is now designed for a load of $.31 \times 60{,}000 = 18{,}600$ lbs., at a working stress of 15,000 lbs. per sq. in., it will meet the conditions of the problem.

272. Stringer and Slab Floors for Highway Bridges.—Floor systems of the type shown in Fig. 67 are in common use in highway bridge floors. In this system, a concrete slab acts as a transverse beam to transfer loads to longitudinal beams called stringers, which in turn transfer their load to the floorbeams. The determination of the load carried by the stringers due to concentrated loads applied to the floor slab forms a special case of the problem considered in the preceding articles.

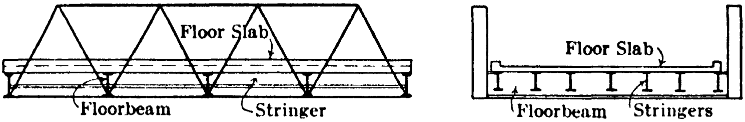


FIG. 67.

It may be assumed that the floor beams are very rigid, compared to the rest of the system, for their design load is generally much greater than any loading resulting from concentrated loads in adjacent panels. The deflection of the floorbeams will therefore be neglected. Only a portion of the floor slab acts effectively as a transverse beam, the width of this beam depending upon the slab thickness, the proportions of the panel, and the spacing of the stringers.

The problem under discussion may be illustrated by the conditions shown in Fig. 68 for a transverse beam, or slab, supported by five equally spaced longitudinal beams, or stringers. As shown in Fig. 68, the concentrated load is not centrally placed, as in the preceding articles. The analysis there given must therefore be modified to account for the changed conditions.

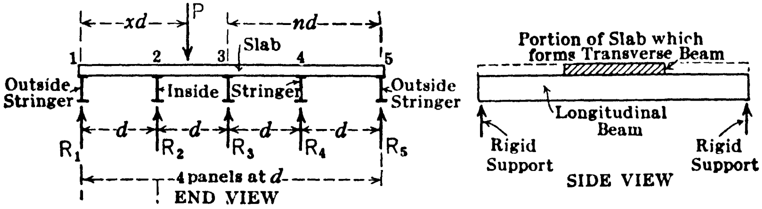


FIG. 68.

It is generally the case that the outside stringers differ in design somewhat from the inside stringers and therefore have a different coefficient of flexibility.

Let

k_i = coefficient of flexibility of the inner stringers;

k_o = coefficient of flexibility of outer stringers;

$N = \dfrac{k_o}{k_i}$;

E_T and I_T = respectively the modulus of elasticity and moment of inertia of the transverse slab;

$$a = \frac{k_i E_T I_T}{d^3}.$$

As in the previous problem, the reactions R_2, R_3, and R_4 are taken as redundant and the solution obtained by placing the deflections at points 2, 3, and 4 equal to the reactions at these points multiplied by k_i. Three condition equations are thus established from which the three redundant reactions can be calculated. A complete solution requires the load P to be placed at points 1, 2, and 3 in turn. To illustrate the method, the load will be assumed applied at point 2 and the condition equation for deflection at point 2 will be formulated. The deflection will be calculated in four parts, namely, that due to

P, R_2, R_3, and R_4, respectively, each of these forces considered as acting separately, with beam supported at 1 and 5.

Deflection Due to P.—From the general formula given in Table I, Art. 7, with $x = d$, $l = 4\,d$, $z_1 = d$, $z_2 = 3\,d$, the deflection of point 2 due to bending in the transverse beam $= \frac{3}{4}\,P\,\frac{d^3}{E_T\,I_T} = \frac{3}{4}\,P\,\frac{k_o}{a\,N}$. The reaction at 1 is $\frac{3}{4}\,P$ and at 5 is $\frac{1}{4}\,P$. The settlement at 1 due to bending of the longitudinal beam is $\frac{3}{4}\,P\,k_o$ and at 5 is $\frac{1}{4}\,P\,k_o$. At point 2 it will be in proportion or $[\frac{1}{4} + \frac{3}{4}\,(\frac{3}{4} - \frac{1}{4})]\,P\,k_o = \frac{5}{8}\,P\,k_o$. Total deflection $= P\,(\frac{3}{4}\,\frac{k_o}{a\,N} + \frac{5}{8}\,k_o)$.

Deflection Due to R_2.—This deflection is given by the same expression as that for P, but is upward. It is $R_2\,(\frac{3}{4}\,\frac{k_o}{a\,N} + \frac{5}{8}\,k_o)$.

Deflection Due to R_3.—Here z_1 and $z_2 = 2\,d$. The deflection due to bending $= \frac{11}{12}\,R_3\,\frac{d_3}{E_T\,I_T} = \frac{11}{12}\,R_3\,\frac{k_o}{a\,N}$. Deflection due to movement of end supports $= \frac{R_3}{2}\,k_o$. Total deflection $= R_3\left(\frac{11}{12}\,\frac{k_o}{a\,N} + \frac{k_o}{2}\right)$, upward.

Deflection Due to R_4.—Here $z_1 = 3\,d$, $z_2 = d$. The deflection due to bending $= \frac{7}{12}\,R_4\,\frac{k_o}{a\,N}$, and due to movement of supports $= \frac{3}{8}\,R_4\,k_o$. Total deflection $= R_4\left(\frac{7}{12}\,\frac{k_o}{a\,N} + \frac{3}{8}\,k_o\right)$, upward. Adding the several values and placing the sum equal to $R_2\,k_i = R_2\,\frac{k_o}{N}$, we have

$$P\left(\frac{3}{4}\,\frac{k_o}{a\,N} + \frac{5}{8}\,k_o\right) - R_2\left(\frac{3}{4}\,\frac{k_o}{a\,N} + \frac{5}{8}\,k_o\right) - R_3\left(\frac{11}{12}\,\frac{k_o}{a\,N} + \frac{k_o}{2}\right)$$
$$- R_4\left(\frac{7}{12}\,\frac{k_o}{a\,N} + \frac{3}{8}\,k_o\right) = R_2\,\frac{k_o}{N}.$$

This reduces to

$$\left.\begin{aligned} R_2\,(18 + 24a + 15\,a\,N) + R_3\,(22 + 12\,a\,N) \\ + R_4\,(14 + 9\,a\,N) = P\,(18 + 15\,a\,N) \end{aligned}\right\} \quad . \; . \; . \quad (18)$$

In a similar manner, condition equations for supports 3 and 4 are readily derived.

If it is desired to draw influence lines for load effect, similar condition equations must be formulated for loads at supports 1 and 3. Reactions for loads at supports 4 and 5 are readily derived from those at supports 2 and 1 by means of reciprocal relations between loads and reactions. Thus it is evident that the reaction at support 2, Fig. 68, due to a load at 4, is equal to the reaction at 4 due to a load at 2. In this manner all reactions may be determined.

In the following summary, Table A, all condition equations are given for the system shown in Fig. 68. It will be noted from Table A that the left-hand members of all equations are the same, but that the right-hand members differ for the several loading conditions.

The condition equations for a load at support 3, as given in Table A, may be simplified by noting that for a load at the center point, reactions R_2 and R_4 are equal. Therefore only two condition equations are required. These equations may be obtained by placing $R_2 = R_4$ in eqs. (1) and (2) of Table A. We then have

$$[16 + 12\,(1 + N)\,a]\,R_2 + (11 + 6\,N\,a)\,R_3 = (11 + 6\,N\,a)\,P$$

$$(11 + 6\,N\,a)\,R_2 + [8 + 3\,(2 + N)\,a]\,R_3 = (8 + 3\,N\,a)\,P.$$

These equations may be used in place of those given in Table A for the determination of the reactions due to a load at point 3.

Condition equations for a five-panel floor system are given in Table B.

In determining the reactions in any particular case, numerical values of N and a should be calculated and substituted in the condition equations of Table A or B. The resulting equations are then readily solved for values of the reactions for the inside beams. Reactions for the outside beams can then be determined from the equations given below each table.

Values of N are readily determined as soon as the sizes of the inner and outer longitudinal beams are known. The value of a is a function of the rigidity of both the longitudinal and transverse beams. In the case considered in this article, the transverse beam is formed by a portion of the floor slab. The width of slab to be considered as effective in forming the transverse beam will depend upon the slab thickness, the proportions of the panel, the spacing of the longitudinal

TABLE A

CONDITION EQUATIONS FOR A FOUR PANEL FLOOR

Eq. No.	Coefficients of R_2, R_3, and R_4			Absolute Terms.		
	R_2	R_3	R_4	Load at 1 $x = 0$	Load at 2 $x = 1$	Load at 3 $x = 2$
1	$18 + (24+15N)a$	$22 + 12Na$	$14 + 9Na$	$18NaP$	$(18 + 15Na)P$	$(22 + 12Na)P$
2	$11 + 6Na$	$16 + (12 + 6N)a$	$11 + 6Na$	$6NaP$	$(11 + 6Na)P$	$(16 + 6Na)P$
3	$14 + 9Na$	$22 + 12Na$	$18 + (24 + 15N)a$	$6NaP$	$(14 + 9Na)P$	$(22 + 12Na)P$

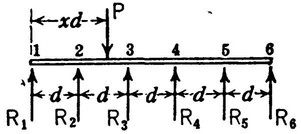
$$a = \frac{k_i E_T I_T}{d_3} \qquad N = \frac{k_o}{k_i}$$

$$R_1 = \tfrac{1}{4}[P(4 - x) - 3R_2 - 2R_3 - R_4]$$

$$R_5 = \tfrac{1}{4}[Px - R_2 - 2R_3 - 3R_4].$$

TABLE B

CONDITION EQUATIONS FOR A FIVE-PANEL FLOOR



Eq. No.	COEFFICIENTS OF R_2, R_3, R_4, AND R_5.				ABSOLUTE TERMS.		
	R_2	R_3	R_4	R_5	Load at 1 $x = 0$	Load at 2 $x = 1$	Load at 3 $x = 2$
1	$160+(150+102N)a$	$225+84\,N\,a$	$200+66\,N\,a$	$115+48\,N\,a$	$120\,N\,a\,P$	$(160+102N\,a)P$	$(225+84\,N\,a)P$
2	$225+84\,N\,a$	$360+(150+78N)a$	$340+72\,N\,a$	$200+66\,N\,a$	$90\,N\,a\,P$	$(225+84\,N\,a)\,P$	$(360+78\,N\,a)P$
3	$200+66\,N\,a$	$340+72\,N\,a$	$360+(150+78N)a$	$225+84\,N\,a$	$60\,N\,a\,P$	$(200+66\,N\,a)\,P$	$(340+72\,N\,a)P$
4	$115+48\,N\,a$	$200+66\,N\,a$	$225+84\,N\,a$	$160+(150+102N)a$	$30\,N\,a\,P$	$(115+48\,N\,a)\,P$	$(200+66\,N\,a)P$

$$a = \frac{k_i E_T I_T}{d_3} \qquad N = \frac{k_o}{k_i}$$

$$R_1 = \tfrac{1}{5}\left[P(5 - x) - 4R_2 - 3R_3 - 2R_4 - R_5\right]$$

$$R_6 = \tfrac{1}{5}\left[Px - R_2 - 2R_3 - 3R_4 - 4R_5\right].$$

beams, or stringers, and to some extent on the method of attachment between the slab and stringers. Slabs 5 or 6 inches thick, the depth generally used in bridge floors, supported by steel stringers whose top flanges are bedded in the slab, probably have an effective width of about two-thirds of the span of the stringers. If the slab is merely supported on the top flanges of the stringers, the effective width of slab is probably about 0.4 to 0.5 of the stringer span. Direct attachment between stringer and slab increases the rigidity of the slab in the direction of the stringers, with the result that the effective width of the slab is increased. For thin slabs, about 2½ or 3 inches thick, the effective width will probably not exceed the spacing of the stringers.

Values of k, the coefficient of flexibility of the longitudinal beam, which, as stated above, is equal to the deflection in inches of the longitudinal beam due to a one-pound load, will depend upon the manner in which the one-pound load is applied to the longitudinal beam. If the transverse beam is a steel beam, the one-pound load may be considered as applied directly to the longitudinal beam and $k = \frac{1}{48} \frac{l^3}{EI}$. If the transverse beam covers the entire longitudinal beam, the one-pound load is to be considered as uniformly distributed and $k = \frac{5}{384} \frac{l^3}{EI}$. If $c\,l$ = assumed width of transverse beam = length along which the one-pound load is assumed to act on the longitudinal beam, then a general expression for k is

$$k = \frac{8 - 4c^2 + c^3}{384} \times \frac{l^3}{EI}. \qquad (19)$$

In determining the moment of inertia of the transverse beam, it is sufficiently accurate to neglect the reinforcement and consider only the concrete area. The moment of inertia is then $\frac{1}{12} b h^3$, where b is the effective width, determined as stated above, and h = total thickness of slab. This approximation is permissible since the slab is not heavily stressed except in the vicinity of the applied load. Therefore the deformations in the greater part of the slab are due to stresses in the concrete, and the moment of inertia should be determined with respect to the concrete area.

273. Example.—Calculate the reactions and draw influence lines for the floor system shown in Fig. 69. One-pound loads will be applied at points 1, 2, and 3 and all reactions determined. For the outside beams $I = 700$ in.^{4} and for the inner beams $I = 442$ in.^{4} Then from eq. (9) $N = 0.6314$.

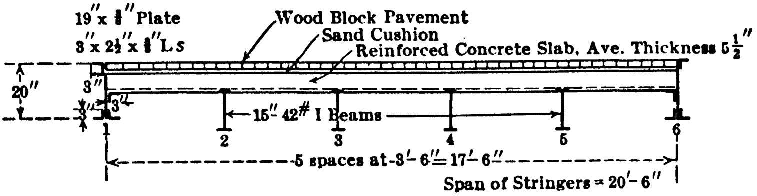

19″ x ⅜″ Plate
8″ x 2½″ x ⅜″ Ls
Wood Block Pavement
Sand Cushion
Reinforced Concrete Slab, Ave. Thickness 5½″
20″
3″
3″
3″
15″ 42# I Beams
1
2
3
4
5
6
5 spaces at 3′-6″ = 17′-6″
Span of Stringers = 20′-6″

FIG. 69.

Taking into account the load distribution due to the pavement, it will be assumed that the effective width of the slab is 0.7 of the span of the stringers. Then from eq. (19), $k = \dfrac{l^3}{60\,E\,I}$. For an inner beam, $l = 20.5$ ft.; $E = 30{,}000{,}000$ and $I = 442$ in.^{4} Then $k_i = 0.00001871$ in. From eq. (9), $a = \dfrac{k_i\,E_T\,I_T}{d^3}$. For concrete assume $E_T = 3{,}000{,}000$. Neglecting the reinforcement $I_T = \frac{1}{12}\,b\,h^3$. With $b = 0.7 \times 20.5$ ft. $= 172$ in., and $h = 5\tfrac{1}{2}$ in., $I_T = 2{,}380$ in.^{4} From Fig. 69, $d = 42$ in., the stringer spacing. Then

$$c = \frac{(0.00001871)\,(3{,}000{,}000)\,(2{,}380)}{(42)^3} = 1.8.$$

Substituting $N = 0.6314$ and $a = 1.8$ in the condition equations of Table B, the equations for the case under consideration are as given in Table 1.

TABLE I

CONDITION EQUATIONS

Eq. No.	R_2	R_3	R_4	R_5	ABSOLUTE TERMS.		
					Load at No. 1	Load at No. 2	Load at No. 3
1	547	321	275	170	136.4	276	321
2	321	719	422	275	102.3	321	449
3	275	422	719	321	68.2	275	422
4	170	275	321	547	34.1	170	275

Solving these equations, the values of the reactions are as given in Table II.

TABLE II

VALUES OF REACTIONS

	R_1	R_2	R_3	R_4	R_5	R_6
Load at No. 1......	+ 0.791	+ 0.232	+ 0.059	− 0.0145	− 0.0311	− 0.0364
Load at No. 2......	+ 0.368	+ 0.300	+ 0.232	+ 0.116	+ 0.0330	− 0.0488
Load at No. 3......	+ 0.095	+ 0.230	+ 0.326	+ 0.255	+ 0.118	− 0.0238

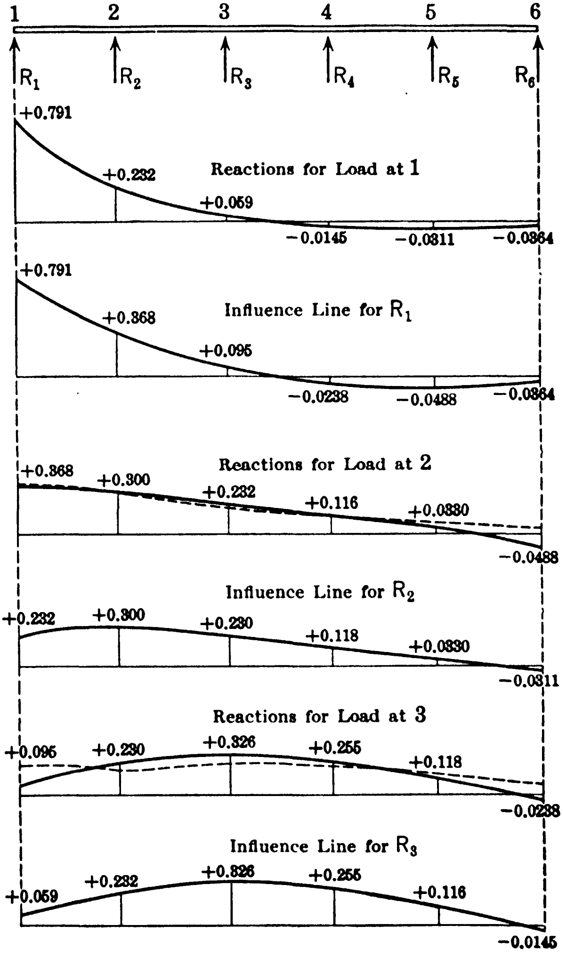


FIG. 70.

In Fig. 70 are shown results of the foregoing calculations in the form of reaction curves and also influence lines.

The floor system shown in Fig. 69 was designed by the Ohio Highway Commission for a highway bridge near Columbus, Ohio. Prof. C. T. Morris of Ohio State University made a series of tests on the floor of this bridge in order to determine the load distribution on the stringers due to concentrated loading. These tests are reported in Bull. No. 28 of the Ohio Highway Commission. The results obtained from these tests are shown by the dotted lines on the curves for reactions for loads at 2 and 3, Fig. 70. Note the close agreement between the observed and calculated values in most cases.

CHAPTER VII

SECONDARY STRESSES*

274. Primary and Secondary Stresses.—It has been assumed heretofore in the analysis of stresses that all members of a truss are free to turn at the joints. It has also been assumed that all members are straight, that the joints lie in the gravity axes of the members, and that all external loads, including the weight of the members, are applied at the joints only. The stresses so calculated are the *axial* or *direct stresses.* They may also be called the *primary stresses* in the truss.

These assumed conditions are not realized in practice. The joints offer more or less resistance to the turning of the members; the members themselves are not straight; the joints are often eccentric; and, finally, the weights of the individual members must be carried to the joints by the members themselves acting as beams. By reason of these modifying causes the members of the truss are in general subjected not only to axial or direct stresses, but also to bending moments, involving bending and shearing stresses. In a complete analysis of stresses it is convenient to calculate, first, the primary or axial stresses under the usual assumptions, and afterward to calculate separately the effects of the modifying conditions here mentioned, the resulting stresses being called *secondary stresses.* It will be found that the most important of the secondary stresses are the bending stresses; the shearing stresses, may, however, be important in some cases. There are also small secondary axial stresses, which act to modify slightly the primary stresses already found. These are also very small and need not usually be considered.

275. Secondary Stresses Due to Rigidity of Joints. Nature of the Problem.—If the members of a truss were free to turn at the joints, the longitudinal deformations which are produced in the various members by their axial stress would, in general, cause a slight change in angle between all the members at all the joints. In the truss *a k*

---

* See Chapter VIII for application of slope deflection method of calculating secondary stresses.

Fig. 1, for example, suppose the nature of the stresses in the various members to be as indicated by the signs. In the triangle *a b c* the member *a c* will be shortened and the members *c b* and *a b* will be lengthened. This will cause a decrease in the angle at *b* and an increase or decrease in the angles at *a* and *c*, depending upon the relative changes in *a b* and *c b*. If the changes in the lengths of these members are known, it is possible to calculate what the changes in the several angles will be. So, also, for the other angles, it is evident that if the changes in lengths of all the members are known, the changes in all the angles can be determined.

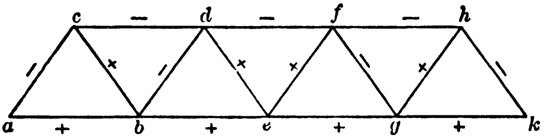


FIG. 1.

Now suppose the members are rigidly connected at all the joints, by riveting or otherwise, so that the angles between the various members cannot change. The lengths of the members will still be changed as before, from the axial stresses, and the vertices of the triangles will move as before. This movement will now force the members of a triangle to bend, as represented diagrammatically in Fig. 2, thus producing bending moments in the members which are a maximum near the joints. The secondary stresses arising from these moments are, in many cases, of large amount, and require careful consideration. It is generally possible and sufficient to so design a structure as to

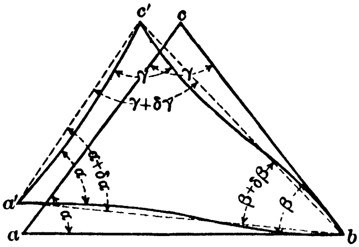


FIG. 2.

keep these stresses within low limits, and then to neglect them in the calculations, but in many special cases and in large and important structures they will require calculation.

In Fig. 2, let *a*, *b*, and *c* represent the original positions of the three joints, and *a′*, *b*, and *c′* the positions after the members are stressed. Let α, β, and γ represent the original angles at *a*, *b*, and *c*, and $\delta\,\alpha$, $\delta\,\beta$, and $\delta\,\gamma$ the changes in these angles if the members were free to turn at the joints. The angles between the dotted straight lines joining *a′*, *b*, and *c′*, will then be equal to $\alpha + \delta\,\alpha$, $\beta + \delta\,\beta$, and $\gamma + \delta\,\gamma$, respectively. In proceeding with a solution of the problem, the changes of angle $\delta\,\alpha$, $\delta\,\beta$, and $\delta\,\gamma$ are first found from the known changes of lengths of the sides of the triangle due to axial stresses. From these changes of angle the bending moments in the members are then calculated. The process requires the simultaneous consideration of all the members and angles of the truss and is, therefore, somewhat tedious, although simple in application if careful attention is paid to signs and the work is well systematized.

276. Calculation of the Changes of Angle in Any Triangle in Terms of the Changes in the Lengths of the Members.—Let *a b c*, Fig. 3, be

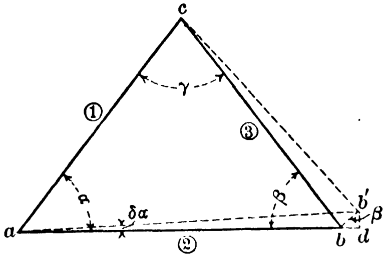


FIG. 3.

the original form of a triangle. Number the members 1, 2, 3, as shown and let

s_1, s_2, s_3 = stress intensity in each member;

l_1, l_2, l_3 = length of each member;

E = modulus of elasticity.

Then the change in length of any member will be in general $\frac{s\,l}{E}$.

Consider the angle α and the change caused in this angle by changes of length in the various members, the joints being frictionless and the members remaining straight. Each member may be considered separately and the results added (the deformations being small). It will be assumed that each member increases in length, an increase being considered as a positive deformation.

Suppose, first, that $a\,b$ is elongated by an amount $\frac{s_2\,l_2}{E}$, shown as $b\,d$ in the figure. Assuming a and c to stand fast, the point b will move to b', on a line $b\,b'$ drawn perpendicular to $b\,c$ and intersecting a line $d\,b'$ perpendicular to $a\,b$. Then $\delta\,\alpha = -\frac{b'\,d}{l_2}$. But $b'\,d = b\,d\,\cot\beta = \frac{s_2\,l_2}{E}\cot\beta$. Hence $\delta\,\alpha = -\frac{s_2}{E}\cot\beta$. In the same manner it is found that for a change of length in member $a\,c$, $\delta\,\alpha = -\frac{s_1}{E}\cot\gamma$. For a change in member $b\,c$, let point b (Fig. 4) stand fast and let $c\,e$ $\left(= \frac{s_3\,l_3}{E}\right)$ represent the elongation of $b\,c$. The position of c will be at c', the intersection of the perpendiculars $c\,c'$ and $e\,c'$. Then $\delta\,\alpha = \frac{c\,c'}{a\,c} = \frac{c\,e}{a\,c\sin\gamma} = \frac{s_3\,l_3}{E\,a\,c\sin\gamma}$. But $a\,c\sin\gamma = a\,k = \frac{b\,c}{\cot\beta + \cot\gamma}$, hence we derive by substitution, $\delta\,\alpha = \frac{s_3}{E}(\cot\beta + \cot\gamma)$.

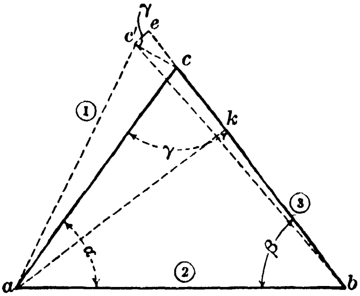


FIG. 4.

The total change in α due to changes of length in all three members, may then be expressed by the convenient formula,

$$\delta\alpha = \frac{s_3 - s_2}{E} \cot\beta + \frac{s_3 - s_1}{E} \cot\gamma. \quad . \quad . \quad . \quad (1)$$

Likewise, we have for the other angles

$$\delta\beta = \frac{s_1 - s_3}{E} \cot\gamma + \frac{s_1 - s_2}{E} \cot\alpha. \quad . \quad . \quad . \quad (2)$$

and

$$\delta\gamma = \frac{s_2 - s_1}{E} \cot\alpha + \frac{s_2 - s_3}{E} \cot\beta. \quad . \quad . \quad . \quad (3)$$

Note that in each case the numerator is the value of s for the side opposite, minus that for the side adjacent, times the cotangent of the included angle.

In calculations of the changes of angles in a triangle, a convenient check is the relation

$$\delta\alpha + \delta\gamma + \delta\beta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

277. The Deflection Angles of a Beam Subjected to Given Moments Applied at the Two Ends.—Consider the beam *A B*, Fig. 5, subjected to the moments M_1 and M_2, and shears V_1 and V_2, but sustaining no intermediate loads. Assume a counter-clockwise direction of moment to be positive.

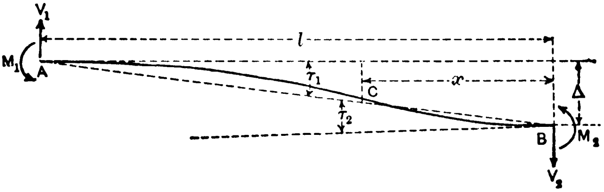


FIG. 5.

From $\Sigma M = 0$ at A and B, we have,

$$V_1 = V_2 = \frac{M_1 + M_2}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

At any point C, distant x from B, the bending moment is

$$M_x = M_2 - V_2 x = M_2 - (M_1 + M_2)\frac{x}{l}. \quad . \quad . \quad . \quad (6)$$

The deflection Δ, at B, with respect to the tangent at A, considered as positive downward, will be equal to $-\int_B^A \frac{M\,x\,d\,x}{E\,I}$ (see Art. 3), or

$$\Delta = -\frac{1}{E\,I}\int_0^l \left[M_2\,x - (M_1 + M_2)\frac{x^2}{l}\right] d\,x$$

$$= \frac{l^2}{6\,E\,I}(2\,M_1 - M_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The value of the deflection angle τ_1 is Δ/l, or

$$\tau_1 = \frac{l}{6\,E\,I}(2\,M_1 - M_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Likewise

$$\tau_2 = \frac{l}{6\,E\,I}(2\,M_2 - M_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eqs. (8) and (9) may be solved for M_1 and M_2 in terms of τ_1 and τ_2, giving

$$M_1 = \frac{2\,E\,I}{l}(2\,\tau_1 + \tau_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$M_2 = \frac{2\,E\,I}{l}(2\,\tau_2 + \tau_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In a truss, distorted as indicated in Fig. 2, the members are subjected to bending moments and shears like the beam in Fig. 5. In addition to the forces here considered, there exist also the primary or direct stresses, which have been omitted in Fig. 5. So long as the distortion is small and the joints concentric, these direct stresses have little effect upon the bending moments, the lever arm being small, but where the distortion is large they exert a considerable influence thereon, especially in the case of compression members. It is possible to include this effect in the analysis, but such a solution is very laborious and not in general practicable. The methods of such exact treatment are considered in a subsequent article. For the present, the effect of the direct stresses upon the bending moments will be neglected. Where the joints are eccentric, the direct stresses cause large bending moments even though the deflection of the member is small. This case will be fully considered.

278. Notation and Conventional Scheme for Signs.—In calculations, such as here discussed, it is important that a convenient notation and a scheme for signs be adopted and that it be closely adhered to throughout. It will be assumed that when the distortions of the members of a triangle are as shown in Fig. 6, the bending moments in the members at the ends are all positive. The values of τ are also to be considered as all positive.

The joints being numbered, the various τ's and M's will be designated by subscripts as shown. The lengths and moments of inertia of the members will be denoted in the same way, but $I_{23} = I_{32}$, $l_{23} = l_{32}$, etc. Fig. 7 shows a truss in which all members are represented as having positive moments at their ends and all values of τ are positive.

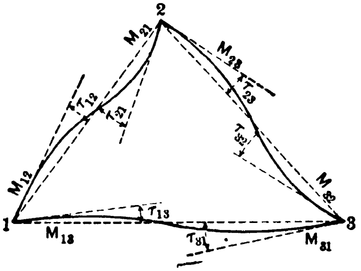


FIG. 6.

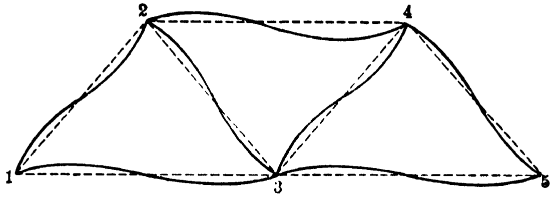


FIG. 7.

279. Values of the Deflection Angles τ in Terms of the Changes of Angle $\delta\alpha$, etc.—Consider any joint n of any structure, Fig. 8, and let the straight lines n-1, n-2, n-3, etc., represent the lines joining the several joints after distortion. The full lines show the bent forms of the several members. The angles τ_{n1}, τ_{n2}, τ_{n3}, and τ_{n4} represent the deflection angles of the several members at joint n. Let α_1, α_2, and α_3 be the original angles between the members 1 n 2, 2 n 3, and

3 *n* 4. After distortion, the angles between the straight lines joining the apexes will be respectively, $\alpha_1 + \delta\alpha_1$, $\alpha_2 + \delta\alpha_2$, and $\alpha_3 + \delta\alpha_3$, as shown. Then if we select one of the deflection angles, as τ_{n1}, as a

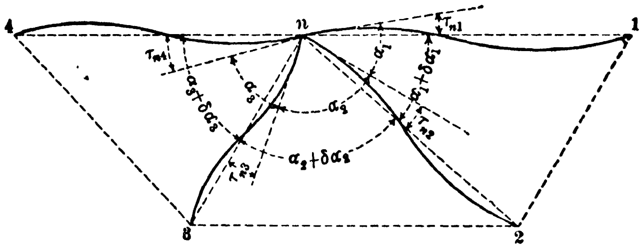


FIG. 8.

reference angle, we may express the other values of τ at joint *n* in terms of τ_{n1} and the changes of angle $\delta\alpha_1$, etc., as follows:

$$\tau_{n2} = \tau_{n1} + \delta\alpha_1$$
$$\tau_{n3} = \tau_{n1} + \delta\alpha_1 + \delta\alpha_2$$
$$\tau_{n4} = \tau_{n1} + \delta\alpha_1 + \delta\alpha_2 + \delta\alpha_3$$

and so on, for any number of members. Or, in general, we may write for any joint *n*

$$\tau_{nm} = \tau_{n1} + \Sigma_1^m \delta\alpha, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where τ_{nm} represents any value of τ, τ_{n1} is the reference angle selected, and $\Sigma_1^m \delta\alpha$ is the sum of all angular changes up to the member *n m* in question.

280. Selection of the Reference Deflection Angle.—It will be convenient at each joint of the structure to select a certain angle τ as the reference angle, and to express all the other deflection angles at the joint in terms of this reference angle and the changes of angle $\delta\alpha$, as given in eq. (12). In the following analysis the reference angle will be selected as indicated in Fig. 9, namely, as the deflection angle of the first member encountered in passing around a joint toward the *right*, beginning on the outside of the truss. This deflection angle may be denoted by the use of a single subscript number, the same as that of the joint, as shown in Fig. 9. For an interior joint any convenient angle may be taken, but it must be definitely designated in each case.

281. The Moments at Any Joint in Terms of the Deflection Angles τ.—If the axes of the several members are concentric so that the primary stresses cause no bending moment, then we may write $\Sigma M = 0$

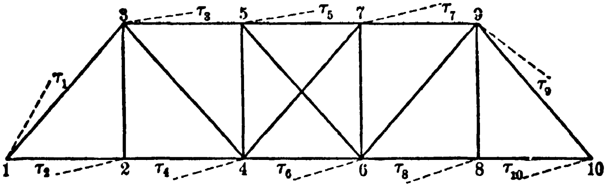


FIG. 9.

for the bending moments at the ends of the members intersecting at any joint. Thus in Fig. 8, we have

$$M_{n1} + M_{n2} + M_{n3} + M_{n4} = 0.$$

Expressing these moments in terms of the several deflection angles τ, as in eq. (10), we have

$$\frac{2\,E\,I_{n1}}{l_{n1}}(2\,\tau_{n1} + \tau_{1n}) + \frac{2\,E\,I_{n2}}{l_{n2}}(2\,\tau_{n2} + \tau_{2n}) + \frac{2\,E\,I_{n3}}{l_{n3}}(2\,\tau_{n3} + \tau_{3n})$$
$$+ \frac{2\,E\,I_{n4}}{l_{n4}}(2\,\tau_{n4} + \tau_{4n}) = 0. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In eq. (13) each term contains a quantity I/l. For convenience, let this be represented by K, using a subscript to indicate the member in question. Then eq. (13) becomes (after dividing by $2\,E$)

$$K_{n1}(2\,\tau_{n1} + \tau_{1n}) + K_{n2}(2\,\tau_{n2} + \tau_{2n}) + \text{etc.} = 0 \quad . \quad . \quad (14)$$

A similar equation may be written out for each of the other joints of a structure, thus giving as many equations as there are joints. Since the values of τ in these equations are all unknown, and since there are twice as many unknowns as there are members in the truss, the equations can not be solved in their present form.

Now, as shown in Art. 279, each τ can be expressed in terms of the reference angle τ_n for the joint in question, and the known values of $\delta\,\alpha_1$, $\delta\,\alpha_2$, etc. In the several equations will then appear as many different reference angles as there are joints, thus making the number of unknowns equal to the number of equations. These equations may then be solved for the values of the reference angles τ_n, whence the values of all the τ's are obtained from eq. (12).

Substituting for the τ's in eq. (14) values given by eq. (12) and collecting terms, we have,

$$2\,\tau_n\,[K_{n1}+K_{n2}+K_{n3}+\text{etc.}]+2\,[K_{n2}\,\delta\angle_1+K_{n3}\,(\delta\angle_1+\delta\angle_2)+\text{etc.}]$$
$$+\left(K_{1n}\,\tau_1+K_{1n}\sum\nolimits_{\tau_1}^{1n}\delta\angle\right)+\left(K_{2n}\,\tau_2+K_{2n}\sum\nolimits_{\tau_2}^{2n}\delta\angle\right)+\text{etc.}=0 \quad (15)$$

In this equation $\delta\angle$ represents the angular changes δa_1, δa_2, etc. of Art. 297, and $\sum_{\tau_2}^{2n}\delta\angle$, etc., represents the sum of all angular changes from the reference angle τ_2 up to member $2\,n$. Expressed in general form, eq. (15) can be written,

$$2\,[(\Sigma\,K)\,\tau_n+\Sigma\,(K\;\Sigma\;\delta\;\angle)]$$
$$+\left[\left(K_{mn}\,\tau_m+K_{mn}\sum\nolimits_{\tau_m}^{mn}\delta\;\angle\right)+\begin{pmatrix}\text{similar terms for}\\ \text{other members}\end{pmatrix}\right]=0 \quad (16)$$

Eq. (16) is in convenient form for tabulation, as explained in Art. 284.

If the axes of the members are eccentric, so that the primary stresses cause a turning moment at the joint, then if M_n represents this moment, we have $\Sigma\,M+M_n=0$, and eq. (16) becomes,

$$2\,[(\Sigma\,K)\,\tau_n+\Sigma\,(K\;\Sigma\;\delta\;\angle)]$$
$$+\left[\left(K_{mn}\,\tau_m+K_{mn}\sum\nolimits_{\tau_m}^{mn}\delta\;\angle\right)+\begin{pmatrix}\text{similar terms for}\\ \text{other members}\end{pmatrix}\right]+\frac{M_n}{2E}=0 \quad (17)$$

282. The Moment and Fibre Stress in Terms of τ.—After the values of the τ's have been obtained, the bending moments can be obtained from the general equation (see eq. 10).

$$M_{nm}=2\,E\,K_{nm}\,(2\,\tau_{nm}+\tau_{mn}) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

From the general expression $f=M\,c/I$, we have from eq. (18)

$$f_{nm}=\frac{2\,E\,c}{l}\,(2\,\tau_{nm}+\tau_{mn}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (18a)$$

In these equations M_{nm} and f_{nm} are respectively the bending moment and fibre stress at joint n in member nm, and c = distance of fibre from neutral axis.

283. Omission of E in the Calculations.—Where the value of E is the same throughout the structure, as is usually the case, this quantity may generally be omitted from the calculations, thus saving labor and giving quantities more convenient to handle. The values of $\delta\alpha$, etc., from eqs. (1), (2) and (3), will then be E times too great and the values of τ resulting from the solution of eqs. 14 also E times too great. Then in calculating M and f from (18) and (18a), the quantity E is again omitted, thus giving correct values of M and f. Where a moment M_n, due to eccentricity or external force, is included, as in eq. (17), the term to be added is $M_n/2$ instead of $M_n/2E$. Whenever, for any reason, true values of τ are needed, the values given by the foregoing process must be divided by E.

284. **Arrangement of the Calculations.**—In calculations such as here considered, it is important to reduce the work to as nearly a mechanical basis as possible, and systematic and convenient methods of tabulation are important. Assuming the primary stresses to be already determined, the process of calculating secondary stresses may be considered in five steps: (1) Calculation of the changes of angle $\delta\ \alpha$, etc.; (2) tabulation of the values of $\Sigma\ \delta\ \alpha$, etc., or $\Sigma\ \delta\ \angle$, as hereafter indicated; (3) formulation of the equations, one for each joint; (4) solution of the equations; and (5) calculation of the several individual values of τ and of the bending moments or fibre stresses.

The most important part of the process is the convenient arrangement of the values of τ so that equation (16) may be readily written out. The solution of these equations is not a long process. The method of arrangement here proposed is most readily explained by means of a numerical example.

Take the truss of Fig. 10 and assume loads and dimensions such that the unit stress s, in members 1-2 and 2-4 is 8,000 lbs. per sq. in., in 1-4, 12,000 lbs. per sq. in., and in 2-3, 15,000 lbs. per sq. in. Further-

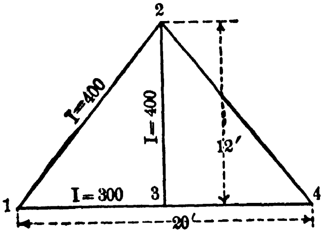


FIG. 10.

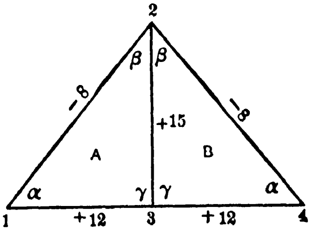


FIG. 11.

more, suppose that the members have values of I as shown, taken about axes at right angles to the plane of the truss.

This truss is symmetrical about the center and all deformations and stresses are similar on the two sides. This condition permits the calculations to be greatly shortened, but for purposes of illustration all members and all joints will be considered as if unsymmetrical, and the effect of symmetry considered later.

285. (*a*) *Calculation of Values of* $\delta \angle$.—Omitting E, the general formula for angular change is, from (eq. 1):

$$\delta \alpha = (s_3 - s_2) \cot \beta + (s_2 - s_1) \cot \gamma,$$

where s_3 is the unit stress in member opposite, and s_2 and s_1 are unit stresses in the members adjacent. Tension is plus and compression minus. The angles may be taken in any order, but where the several triangles are similar it is desirable to give the same letter to equal angles. A separate sketch of the truss is made (Fig. 11), on which the angles are lettered and the unit stresses written (shown here in thousands of pounds). The triangles are conveniently lettered A, B, C, etc.

The values of the cotangents are as follows:

$$\cot \alpha = 0.8333 \qquad \cot \beta = 1.20 \qquad \cot \gamma = 0$$

Then, by triangles, the values of $\delta \alpha$, etc., are calculated as follows:

$$\text{Triangle } A. \begin{cases} \delta \alpha = (15 - 12) \times 0 + (15 + 8) \times 1.2 = + 27.6 \\ \delta \beta = (12 - 15) \times 0 + (12 + 8) \times .8333 = + 16.666 \\ \delta \gamma = (- 8 - 12) \times .8333 + (- 8 - 15) \times 1.2 = - 44.266 \end{cases}$$

Triangle B. Same as for triangle A.

286. (*b*) *Calculation of the Values of* $\Sigma \, \delta \angle$.—The reference angles are shown in Fig. 12. The values of $\delta \alpha$, etc., are conveniently tabulated as given on p. 393, arranged in groups by joints.

In Column (2) the internal angles at the several joints are stated in regular order, beginning with the angle first met with in passing around the joint as before described. The first side of the first angle listed at each joint is therefore the member whose deflec-

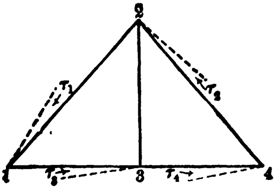


FIG. 12.

VALUES OF δ, $\angle$, τ, AND M.

Joint.	$\angle$	$\delta \angle$	$\Sigma \delta \angle$	Member.	$K = \frac{I}{l}$	$K \times \Sigma \delta \angle$	τ	$2\tau_{nm} + \tau_{mn}$	Moments.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
				12	2.14	0	− 30.6	− 44.5	− 190.5
1	213	+ 27.6	+ 27.6	13	2.50	+ 69.0	− 3.0	+ 38.2	+ 191.0
					4.64	+ 69.0			
				24	2.14	0	− 16.7	− 2.8	− 12.0
2	423	+ 16.666	+ 16.666	23	2.78	+ 46.33	0	0	0
	321	+ 16.666	+ 33.332	21	2.14	+ 71.33	+ 16.7	+ 2.8	+ 12.0
					7.06	+ 117.66			
				31	2.50	0	+ 44.2	+ 85.4	+ 42.7
3	132	− 44.27	− 44.27	32	2.78	− 122.9	0	0	0
	234	− 44.27	− 88.54	34	2.50	− 221.3	− 44.2	− 85.4	− 42.7
					7.78	− 344.2			
				43	2.50	0	+ 3.0	− 38.2	− 191.0
4	342	+ 27.6	+ 27.6	42	2.14	+ 59.06	+ 30.6	+ 44.5	+ 190.5
					4.64	+ 59.06			

tion angle is the reference angle of that joint. Thus at joint 2, member 2–4 is the one whose deflection angle is the reference angle τ_2. Column (3) contains the values of $\delta\,\alpha$, etc., as obtained in Art. 276. These are called $\delta \angle$. Column (4) contains the summation for each joint of the values of Column (3), up to any given angle. Thus at joint 2, the value of $\delta \angle$ for $\angle$ 423, is + 16.666, and for 321 is + 16.666. The sum of these is + 33.332. If a third angle existed here its $\delta \angle$ would then be added in Column (4) to the value of + 33.332 for $\angle$ 321, etc.

In Column (5) are listed the several members meeting at the joint, given in the same order as the angles, the joint number being stated first in each case. In Column (6) are the several values of $K = I/l$ for each member, and in Column (7) are the products of Columns (4) and (6).

Columns (6) and (7) are important, as they contain, in convenient form, the values of $K\tau$ for all members, in the following manner: For member 1–2 the value of τ is τ_1, and $K_{12}\,\tau_{12} = 2.14\ \tau_1$, the coefficient of which quantity is given in Column (6), opposite member 1–2. For member 1–3 the value of τ is τ_{13}, which is equal, by eq. (12), to $\tau_1 + \delta\ (213), = \tau_1 + 27.6$. Then $K_{13}\,\tau_{13} = 2.50\ [\tau_1 + 27.6] = 2.50\ \tau_1 + 69.0$, as given in Columns (6) and (7) opposite member 1–3, the angle τ_1 being added to the coefficient of Column (6). Again, at joint 2, τ_2 is the reference angle and is the value of τ for member 2–4; and $K_{24}\,\tau_{24} = 2.14\ \tau_2$, as in Column (6). Then $\tau_{23} = \tau_2 + \delta\ (423) = \tau_2 + 16.666$, and $K_{23}\,\tau_{23} = 2.78\ \tau_2 + 46.33$; also $\tau_{21} = \tau_2 + \delta\,(423) + \delta\,(321) = \tau_2 + \Sigma\,\delta\,\angle = \tau_2 + 33.332$, and $K_{21}\,\tau_{21} = 2.14\,\tau_2 + 71.33$, as given in Column (6) and (7). Likewise, the value of $K_{34}\,\tau_{34}$ (joint 3) is read at once as $2.50\,\tau_3 - 221.3$, etc., etc.

287. (*c*) *Formulation of Equations.*—We will now proceed to formulate an equation for each joint after eq. (16).

Joint 1.—For this joint, eq. (16) becomes

$$2\,(K_{12}\,\tau_{12} + K_{13}\,\tau_{13}) + K_{21}\,\tau_{21} + K_{31}\,\tau_{31} = 0.$$

The quantity in parenthesis is the sum of all the values of $K\tau$ for joint 1. These are conveniently summed up in the table, Columns

(6) and (7), and the result there given is $4.64\,\tau_1 + 69.0$. This is to be multiplied by two. The values of $K_{21}\,\tau_{21}$ and $K_{31}\,\tau_{31}$ are obtained directly from the table opposite members 2–1 and 3–1 of joints 2 and 3. They are $2.14\,\tau_2 + 71.33$ and $2.50\,\tau_3$. Adding the absolute terms we have the equation

$$9.28\,\tau_1 + 2.14\,\tau_2 + 2.50\,\tau_3 = -\,209.33 \quad . \;\; . \;\; . \;\; . \quad (a)$$

Joint 2.—The equation for this joint is

$$2\,(K_{24}\,\tau_{24} + K_{23}\,\tau_{23} + K_{21}\,\tau_{21}) + K_{42}\,\tau_{42} + K_{32}\,\tau_{32} + K_{12}\,\tau_{12} = 0$$

The quantity in parenthesis is the summation opposite joint 2 in Columns (6) and (7), and is $7.06\,\tau_2 + 117.66$. The other values of $K\,\tau$ are found opposite the respective members, 4–2, 3–2, and 1–2. In formulating the equation the terms may be conveniently written down as follows:

$$\begin{aligned}
(2\,\Sigma\,K\,\tau) &= 14.12\,\tau_2 + 235.3 \\
(K_{42}\,\tau_{42}) &= 2.14\,\tau_4 + 59.06 \\
(K_{32}\,\tau_{32}) &= 2.78\,\tau_3 - 122.9 \\
(K_{12}\,\tau_{12}) &= 2.14\,\tau_1.
\end{aligned}$$

Whence by adding absolute terms we have

$$14.12\,\tau_2 + 2.14\,\tau_4 + 2.78\,\tau_3 + 2.14\,\tau_1 = -\,171.5. \quad . \;\; . \quad (b)$$

Joint 3.—Noting that the list of members in Column (5), opposite joint 3, may be strictly followed in finding the various single terms $K\,\tau$, we may write out at once the quantities for joint 3 as follows:

$$\begin{array}{lr}
15.56\,\tau_3 - & 688.4 \\
2.50\,\tau_1 + & 69.0 \\
2.78\,\tau_2 + & 46.33 \\
2.50\,\tau_4 & \\
\hline
 & -\,573.1
\end{array}$$

whence

$$15.56\,\tau_3 + 2.50\,\tau_1 + 2.78\,\tau_2 + 2.50\,\tau_4 = +\,573.1 \quad . \quad (c)$$

Joint 4.—In the same manner,

$$\begin{array}{l}
9.28\,\tau_4 + 118.1 \\
2.50\,\tau_3 - 221.3 \\
2.14\,\tau_2 \\
\hline
\qquad\quad - 103.2
\end{array}$$

whence

$$9.28\,\tau_4 + 2.50\,\tau_3 + 2.14\,\tau_2 = +\,103.2 \qquad \ldots \qquad (d)$$

288. (*d*) *Solution of the Equations.*—Arranging the equations in systematic order, we have

$$\begin{array}{lrrrrl}
(a) & 9.28\,\tau_1 + & 2.14\,\tau_2 + & 2.50\,\tau_3 & & = -\,209.3 \\
(b) & 2.14\,\tau_1 + & 14.12\,\tau_2 + & 2.78\,\tau_3 + & 2.14\,\tau_4 & = -\,171.5 \\
(c) & 2.50\,\tau_1 + & 2.78\,\tau_2 + & 15.56\,\tau_3 + & 2.50\,\tau_4 & = +\,573.1 \\
(d) & & 2.14\,\tau_2 + & 2.50\,\tau_3 + & 9.28\,\tau_4 & = +\,103.2.
\end{array}$$

The solution may be made in any convenient manner, but the order of elimination used here will generally be found convenient for longer problems where it is a matter of some importance.

From (*a*), (*b*), and (*c*), we get

$$\begin{array}{lrrrrl}
(a') & \tau_1 + & .231\,\tau_2 + & .269\,\tau_3 & & = -\;\;22.55 \\
(b') & \tau_1 + & 6.60\;\;\tau_2 + & 1.299\,\tau_3 + & \tau_4 & = -\;\;80.14 \\
(c') & \tau_1 + & 1.112\,\tau_2 + & 6.224\,\tau_3 + & \tau_4 & = +\,229.2
\end{array}$$

Then subtracting (*a*′) from (*b*′) and (*a*′) from (*c*′), we have

$$\begin{array}{lrrrl}
(e) & 6.37\;\;\tau_2 + & 1.030\,\tau_3 + & \tau_4 & = -\;\;57.59 \\
(f) & .881\,\tau_2 + & 5.955\,\tau_3 + & \tau_4 & = +\,251.8.
\end{array}$$

Then from (*e*), (*f*), and (*d*) we get, in a similar manner,

$$\begin{array}{lrrrl}
(e') & \tau_2 + & .1617\,\tau_3 + & .157\,\tau_4 & = -\;\;\;9.04 \\
(f') & \tau_2 + & 6.760\;\;\tau_3 + & 1.135\,\tau_4 & = +\,286.0 \\
(d') & \tau_2 + & 1.168\;\;\tau_3 + & 4.337\,\tau_4 & = +\;\;48.2 \\
\hline
(g) & & 6.598\;\;\tau_3 + & .978\,\tau_4 & = +\,295.0 \\
(h) & & 1.006\;\;\tau_3 + & 4.180\,\tau_4 & = +\;\;57.26.
\end{array}$$

Then from (*g*) and (*h*)

$$\begin{array}{lrl}
(g') & \tau_3 + \ .148\,\tau_4 = & + \ 44.68 \\
(h') & \tau_3 + 4.155\,\tau_4 = & + \ 52.93 \\
\hline
 & 4.007\,\tau_4 = & + \ 12.25 \\
 & \tau_4 = & + \ \ 3.06.
\end{array}$$

Then, substituting back in (*g*′), (*e*′), and (*a*′), we get the following values:

$$\begin{aligned}
\tau_3 &= + \ 44.23 \\
\tau_2 &= - \ 16.67 \\
\tau_1 &= - \ 30.63.
\end{aligned}$$

289. (*e*) *Calculation of the Several Values of τ and of the Bending Moments.*—The values of τ_1, τ_2, τ_3, and τ_4 are inserted in Column (8) of the table, opposite the first member of the respective joint. The other values of τ are found by adding to these, the values of $\Sigma\ \delta \angle$, given in Column (4). Thus $\tau_{13} = \tau_1 + 27.6 = -\ 3.0$; $\tau_{23} = \tau_2 + 16.67 = 0$; $\tau_{21} = \tau_2 + 33.33 = +\ 16.67$, etc. Note that values of τ in opposite halves of the truss are alike but of opposite sign.

The bending moments are found from eq. (18),

$$M_{nm} = 2\ K\ (2\ \tau_{nm} + \tau_{mn}).$$

In Column (9) are given values of $2\ \tau_{nm} + \tau_{mn}$ for each member. Thus for 1–2 the value is

$$2\ \tau_{12} + \tau_{21} = -\ 61.2 + 16.7 = -\ 44.5;$$

for member 1–3 the value is

$$2\ \tau_{13} + \tau_{31} = -\ 6.0 + 44.2 = +\ 38.2, \text{ etc.}$$

The values of the moments are then equal to $2\ K$ times the values in Column (9). The results are given in Column (10). The slight differences between the values of M_{12} and M_{13}, and between M_{43} and M_{42} are due to inaccuracies in the calculations.

Fibre stresses are readily determined from the moments, or from eq. (18*a*), but the calculations will not be carried further in this problem. From the signs of the moments and the adopted scheme, illustrated in Fig. 8, the true directions of bending are found to be as indicated in Fig. 13.

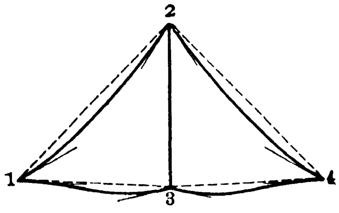


FIG. 13.

290. *Effect of Symmetrical Conditions in Shortening the Calculations.* —Where the structure and the load are symmetrical, as in this case, it is necessary to carry the equations only as far as the centre of the truss, as the values of τ on the right will be equal to the corresponding values on the left, but of opposite sign. Where a vertical member is placed at the centre, as member 2–3, we have, from symmetry, $\tau_{23} = 0$, and $\tau_{32} = 0$. But from the table, joint 2, Column (4), we have $\tau_{23} = \tau_2 + 16.666$, whence $\tau_2 = -16.666$. Likewise $\tau_{32} = 0 = \tau_3 - 44.27$, whence $\tau_3 = +44.27$. Then a single equation, eq. (*a*), Art. 287, is sufficient to determine τ_1. In the preceding analysis the work has been fully carried out as would be necessary for unsymmetrical conditions.

291. **The Secondary Stresses in a Pratt Truss.**—To fully determine the maximum fibre stresses in all members of a truss, due to both primary and secondary stresses, it is necessary to calculate the secondary stresses throughout, for a single load at each joint. The results can then be combined with the primary stresses in any desired manner and the true maximum values determined. Correct maximum results will not always be reached by adding the maximum secondary stress to the maximum primary stress, as these maxima may not occur simultaneously. Combined influence lines furnish a ready method of exact calculation for any kind of loading. To illustrate these points, and at the same time the application of the pre-

ceding methods to a larger problem, a complete solution will be made of a six-panel Pratt truss.

The general dimensions are shown in Fig. 14, and the various elements needed in the calculations are given in Table A. The joints are numbered consecutively as shown, for convenience in the work of formulating and solving the equations. In selecting the order of numbering the same order may be adopted as is followed in drawing

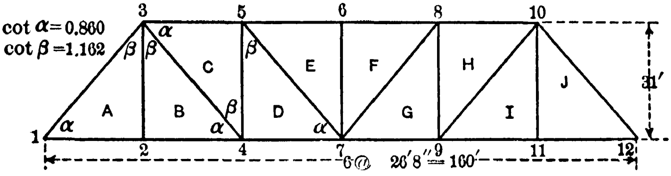


FIG. 14.

TABLE A.

Member.	Sectional Area sq. in.	Length Inches.	I	c Inches.	$K = \frac{I}{l}$	$\frac{c}{l}$	Make-up of Section.
1–2	29.44	320	1218	9.12	3.80	.0285	4 L's 4″ × 4″ × $\frac{3}{8}$ 2 Pls. 18″ × $\frac{1}{2}$″
2–4	29.44	320	1218	9.12	3.80	.0285	4 L's 4″ × 4″ × $\frac{3}{8}$″ 2 Pls. 18″ × $\frac{1}{2}$″
4–7	45.48	320	1907	9.12	5.96	.0285	4 L's 4″ × 4″ × $\frac{5}{8}$″ 2 Pls. 18″ × $\frac{3}{4}$″
1–3	58.49	490.7	4490	9.54 14.08	9.15	.0194 .0287	1 Cov. Pl. 26″ × 9/16″ 2 L's $3\frac{1}{2}$″ × $3\frac{1}{2}$ × $\frac{1}{2}$″ 2 L's 5″ × $3\frac{1}{2}$″ × $\frac{5}{8}$″ 2 Webs. 22″ × $\frac{5}{8}$″
3–5	52.35	320	3978	9.19 14.43	12.43	.0287 .0451	1 Cov. Pl. 26″ × 9/16″ 2 L's $3\frac{1}{2}$″ × $3\frac{1}{2}$″ × $\frac{3}{8}$″ 2 L's 5″ × $3\frac{1}{2}$″ × $\frac{1}{2}$″ 2 Webs. 22″ × 9/16″
5–6	52.35	320	3978	9.19 14.43	12.43	.0287 .0451	
3–2	16.00	372	94.8	5.4	0.255	.0145	4 L's 5″ × $3\frac{1}{2}$″ × $\frac{1}{2}$″
3–4	29.42	490.7	805.4	7.5	1.64	.0153	2 [’s 15″ — 50 lb.
5–4	26.48	372	750.2	7.5	2.016	.0202	2 [’s 12″ — 45 lb.
5–7	20.58	490.7	358.6	6.0	.731	.0122	2 [’s 12″ — 35 lb.
6–7	14.70	372	288.0	6.0	.774	.0161	2 [’s 12″ — 27 lb.

a stress diagram, beginning at joint 1. The triangles are lettered in regular order.

To make a complete solution, the stresses will be separately determined for joint loads at 7, 9, and 11, and to simplify the calculations these joint loads will be so taken as to give the same reaction at the left end in each case. Joint loads will therefore be assumed as follows: 1,000 lbs. at 7, 1,500 lbs. at 9, and 3,000 lbs. at 11. For all three cases, the primary stresses in the left half of the truss will be the same. The final values of secondary stress can readily be reduced to the same

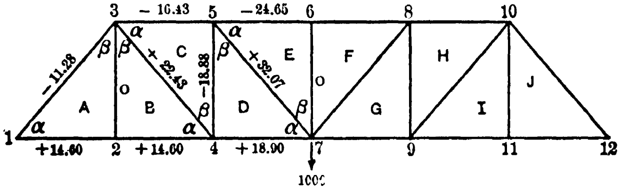


FIG. 15.

TABLE B1.

CALCULATION OF δ ∠, CASE I.

Triangle.	Angle.	Factor of cot α (.8602).	Factor of cot β (1.162).	δ ∠.
A	1	....	0+11.28	+13.11
	2	−11.28−14.60	−11.28−0	−35.37
	3	+14.60+11.28	....	+22.26
B	2	+22.43−14.60	+22.43−0	+32.80
	3	+14.60−22.43	....	− 6.74
	4	....	0−22.43	−26.06
C	3	....	−18.88−22.43	−48.00
	4	−16.43−22.43	....	−33.42
	5	+22.43+16.43	+22.43+18.88	+81.42
D	4	+32.07−18.90	+32.07+18.88	+70.55
	5	+18.90−32.07	....	−11.33
	7	....	−18.88−32.07	−59.22
E	5	....	0−32.07	−37.27
	6	+32.07+24.65	+32.07−0	+86.05
	7	−24.65−32.07	....	−48.78

basis by dividing by the joint load assumed in each case. The order of procedure will be the same as in the previous problem, but all three cases will be carried along simultaneously.

292. (*a*) *Calculation of Changes of Angles* ($\delta \angle$).—There being but two different angles in this truss (other than 90°), they will be called α and β in general, as shown in Fig. 14.

Case I. Load of 1,000 *lbs. at Joint* 7.—The unit stresses s are first computed. These are conveniently shown on a diagram of the truss,

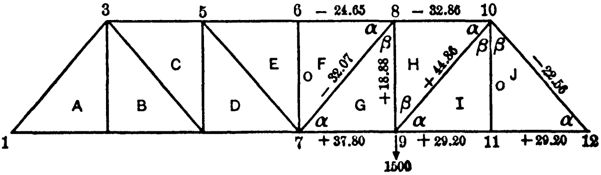


Fig. 16.

TABLE B2.

CALCULATION OF $\delta \angle$, CASE II.

Triangle.	Angle.	Factor of cot α (.8602).	Factor of cot β (1.162)	$\delta \angle$.
A to *E*....	Same as	in Table B1.		
F....	6	$-32.07+24.65$	$-32.07-0$	$-\ 43.64$
	7	$-24.65+32.07$	....	$+\ \ 6.38$
	8	....	$0+32.07$	$+\ 37.26$
G....	7	....	$+18.88+32.07$	$+\ 59.20$
	8	$+37.80+32.07$	....	$+\ 60.10$
	9	$-32.07-37.80$	$-32.07-18.88$	-119.30
H....	8	$+44.86+32.86$	$+44.86-18.88$	$+\ 97.05$
	9	$-32.86-44.86$	....	$-\ 66.85$
	10	....	$+18.88-44.86$	$-\ 30.20$
I	9	....	$0-44.86$	$-\ 52.12$
	10	$+29.20-44.86$	....	$-\ 13.48$
	11	$+44.86-29.20$	$+44.86-0$	$+\ 65.60$
J	10	$+29.20+22.56$	....	$+\ 44.52$
	11	$-22.56-29.20$	$-22.56-0$	$-\ 70.74$
	12	....	$0+22.56$	$+\ 26.22$

NOTE.—Values of $\delta \angle$ for triangles I and J are twice the values for triangles A and B of Table B1.

Fig. 15. Then in Table B1 are given in tabular form the calculations of the changes of angle, arranged by triangles. Each angle is denoted by the apex number of the triangle in question. The values of $\delta\ \angle$ for the right half are the same as those for the left half and need not be recorded here.

Case II. Load of 1,500 *lbs. at Joint* 9.—The necessary calculations are shown in Fig. 16 and Table B2. Up to triangle *E* the results are the same as in Table B1, as the stresses are the same.

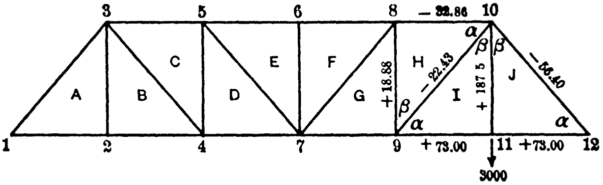


FIG. 17.

TABLE B3.

CALCULATION OF $\delta\angle$, CASE III.

Triangle.	Angle.	Factor of cot α (.8602).	Factor of cot β (1.162).	$\delta\angle$.
A to *G*...	Same as in	Table B2.		
H....	8	−22.43+32.86	− 22.43− 18.88	− 39.01
	9	−32.86+22.43	....	− 8.98
	10	....	+ 18.88+ 22.43	+ 47.99
I	9	....	+ 187.5+ 22.43	+244.0
	10	+73.00+22.43	....	+ 82.09
	11	−22.43−73.00	− 22.43− 187.5	−326.09
J....	10	+73 00+56.40	....	+111.3
	11	−56.40−73.00	− 56.40−187.5	−394.8
	12	....	+187.5 + 56.40	+283.5

Case III. Load of 3,000 *lbs. at Joint* 11.—The values for triangles *A* to *G* will be found in Tables B1 and B2. The calculations for the remaining angles are given in Fig. 17 and Table B3.

293. (*b*) *Calculation of* $\Sigma\ \delta\ \angle$ and $K\ \Sigma\ \delta\ \angle$ *at the Several Joints.*—The calculation and tabulation of these quantities for all three cases are readily combined in Table C, p. 406. The values of K are the same for all cases. Furthermore, in all respects, Case II is the same as Case I up to joint 5, and Case III the same as Case II up to joint 7. The values for Case I need not be carried beyond joint 7 because of symmetry of conditions.

294. (*c*) *Formulation of Equations.*—From Table C, the equations are written out as explained in the preceding example, using the values of Columns (3), (7), (10), and (13). Thus for Case I, the equation for joint 3 is obtained as follows, from Columns (3) and (7):

$$\begin{array}{lrlrl} & 2 \times 23.48\,\tau_3 & - \; 2 \times & 390.1 & \text{(See summations for joint 3)} \\ + & 12.43\,\tau_5 & + & 408.0 & \text{(See member 5–3 under joint 5)} \\ + & 1.64\,\tau_4 & - & 42.7 & \text{(See member 4–3 under joint 4)} \\ + & 0.26\,\tau_2 & - & 9.2 & \text{(See member 2–3 under joint 2)} \\ + & 9.15\,\tau_1 & & & \text{(See member 1–3 under joint 1)} \end{array}$$

whence

$$9.15\,\tau_1 + .26\,\tau_2 + 46.96\,\tau_3 + 1.64\,\tau_4 + 12.43\,\tau_5 = +\,424.1.$$

The list of members under joint 3, Column (2), is used for finding the various terms in the table after the first. For Cases II and III the equations for joints 1 to 5 are identical with those for Case I.

For joints beyond No. 5, the equations will be different for Cases I and II, and beyond joint 7 they will be different for each case; but it will be noted that the coefficients of τ are functions of K only and are the same for *all cases*. Hence the equations, where different, will differ only in the absolute terms. The equations for all three cases can then conveniently be arranged in a single table. Table *D* contains all the equations for all the cases thus formulated.

295. (*d*) *Solution of Equations.*—The solution of the equations for all three cases may be carried along together as far as necessary. The same general method of elimination is used as in Art. 288. The entire work is given in Table E, the final values of τ for the three cases being given at the foot of the table.

The process is as follows: Eqs. (1), (2), and (3) of Table D, containing τ_1, are each divided by the coefficient of τ_1, giving eqs. (1′), (2′), and (3′) of Table E. Eq. (1′) is then subtracted from (2′) and from (3′), giving (*a*) and (*b*), and eliminating τ_1. These two

TABLE C.

CALCULATION OF $\Sigma \delta \angle$ AND K $\Sigma \delta \angle$

Joint. (1)	Member. (2)	K (3)	Angle. (4)	Case I (Load at 7).			Case II (Load at 9).			Case III (Load at 11).		
				$\delta \angle$. (5)	$\Sigma \delta \angle$. (6)	K × $\Sigma \delta \angle$. (7)	$\delta \angle$ (8)	$\Sigma \delta \angle$. (9)	K × $\Sigma \delta \angle$. (10)	$\delta \angle$. (11)	$\Sigma \delta \angle$. (12)	K × $\Sigma \delta \angle$. (13)
1	13	9.15	....	....	....	....		Same as I.			Same as II.	
	12	3.80	312	+13.11	+ 13.11	+ 49.82						
		12.95				+ 49.82						
2	21	3.80	....	....	....	....						
	23	.26	123	−35.37	− 35.37	− 9.20						
	24	3.80	324	+32.80	− 2.57	− 9.76						
		7.86				− 18.96						
3	35	12.43	....	....	....	....						
	34	1.64	534	−48.00	− 48.00	− 78.72						
	32	.26	432	− 6.74	− 54.74	− 14.23						
	31	9.15	231	+22.26	− 32.48	− 297.2						
		23.48				− 390.15						
4	42	3.80	....	....	....	....						
	43	1.64	243	−26.06	− 26.06	− 42.73						
	45	2.02	345	−33.42	− 59.48	− 120.15						
	47	5.96	547	+70.55	+ 11.07	+ 65.95						
		13.42				− 96.93						
5	56	12.43	....	....	....	....						
	57	.73	657	−37.27	− 37.27	− 27.21						
	54	2.02	754	−11.33	− 48.60	− 98.15						
	53	12.43	453	+81.42	+ 32.82	+ 408.0						
		27.61				+ 282.6						
6	68	12.43	....	....	....	....						
	67	.77	867	+86.05	+ 86.05	+ 66.25	−43.64	−43.64	− 33.60			
	65	12.43	765	+86.05	+172.10	+2139.2	+86.05	+42.41	+527.2			
		25.63				+2205.4			+493.6			

7	74	5.96	....	....	....	....	....	....	....			
	75	.73	475	−59.22	−59.22	−43.25	−59.22	−59.22	−43.25			
	76	.77	576	−48.78	−108.00	−83.15	−48.78	−108.00	−83.15			
	78	.73	678	−48.78	−156.78	−114.45	+6.38	−101.62	−74.18			
	79	5.96	879	−59.22	−216.00	−1287.5	+59.20	−42.42	−252.8			
		14.15				−1528.4			−453.3			
8	8 10	12.43	....	....	....	....	....	....	....			
	89	2.02	10.8.9	....	....	....	+97.05	+97.05	+196.0	−39.01	−39.01	−78.8
	87	.73	987	....	....	....	+60.10	+157.15	+114.7	+60.10	+21.09	+15.4
	86	12.43	786	....	....	....	+37.26	+194.41	+2416.5	+37.26	+58.35	+725.5
		27.61							+2727.2			+662.1
9	97	5.96	....	....	....	....	....	....	....	....	....	....
	98	2.02	798	....	....	....	−119.30	−119.30	−241.0	−119.30	−119.30	−241.0
	9 10	1.64	8.9.10	....	....	....	−66.85	−186.15	−305.3	−8.98	−128.28	−210.4
	9 11	3.80	10.9.11	....	....	....	−52.12	−238.27	−905.5	+244.00	+115.72	+439.7
		13.42							−1451.8			−11.7
10	10 12	9.15	....	....	....	....	....	....	....	....	....	....
	10 11	.26	12.10.11	....	....	....	+44.52	+44.52	+11.57	+111.31	+111.31	+28.9
	10 9	1.64	11.10.9	....	....	....	−13.48	+31.04	+50.90	+82.09	+193.40	+317.3
	10 8	12.43	9.10.8	....	....	....	−30.20	+.84	+10.44	+47.99	+241.39	+3002.
		23.48							+72.91			+3348.2
11	11 9	3.80	....	....	....	....	....	....	....	....	....	....
	11 10	.26	9.11.10	....	....	....	+65.60	+65.60	+17.05	−326.09	−326.09	−84.8
	11 12	3.80	10.11.12	....	....	....	−70.74	−5.14	−19.53	−394.8	−720.89	−2739.4
		7.86							−2.48			−2824.2
12	12 11	3.80	....	....	....	....	....	....	....	....	....	....
	12 10	9.15	11.12.10	....	....	....	+26.22	+26.22	+239.9	+283.5	+283.5	+2593.
		12.95							+239.9			+2593.

TABLE D.

EQUATIONS FOR CASES I, II, AND III.

No. of Joint or Eq.	First Member of Equation.	Absolute Terms.		
		Case I.	Case II.	Case III.
1	$25.90\,\tau_1+3.80\,\tau_2+9.15\,\tau_3$ .... =	$+197.6$	....	....
2	$3.80\,\tau_1+15.72\,\tau_2+.26\,\tau_3+3.80\,\tau_4$ .... =	$+2.33$	....	....
3	$9.15\,\tau_1+.26\,\tau_2+46.06\,\tau_3+1.64\,\tau_4+12.43\,\tau_5$ ... =	$+424.1$	....	....
4	$3.80\,\tau_2+1.64\tau_3+26.84\,\tau_4+2.02\,\tau_5+5.96\,\tau_7$ ... =	$+380.5$	....	....
5	$12.43\,\tau_3+2.02\,\tau_4+55.22\,\tau_5+12.43\,\tau_6+.73\,\tau_7$ .. =	-2541.0	-929.0	....
6	$12.43\,\tau_5+51.26\,\tau_6+.77\,\tau_7+12.43\,\tau_8$ .... =	....	-3320.5	-1629.5
7	$5.96\,\tau_4+.73\,\tau_5+.77\,\tau_6+28.30\,\tau_7+.73\,\tau_8+5.96\,\tau_9$ =	....	$+786.8$	$+886.1$
8	$12.43\,\tau_6+.73\,\tau_7+55.22\,\tau_8+2.02\,\tau_9+12.43\,\tau_{10}$ .. =	....	-5149.7	-4011.0
9	$5.96\,\tau_7+2.02\,\tau_8+26.84\,\tau_9+1.64\,\tau_{10}+3.80\,\tau_{11}$ .. =	....	$+2909.5$	$+37.7$
10	$12.43\,\tau_8+1.64\,\tau_9+46.96\,\tau_{10}+.26\,\tau_{11}+9.15\,\tau_{12}$. =	....	-97.5	-8904.2
11	$3.80\,\tau_9+.26\,\tau_{10}+15.72\,\tau_{11}+3.80\,\tau_{12}$ .... =	....	$+898.9$	$+5170.8$
12	$9.15\,\tau_{10}+3.80\,\tau_{11}+25.90\,\tau_{12}$ .... =	....	-460.3	-2446.6

equations are then divided by the coefficient of τ_2, giving (a') and (b'). Then in Table D, eq. (4) is also divided by the coefficient of τ_2, giving (4′) of Table E. Then, as before, (a') is subtracted from (b') and also from (4′), giving (c) and (d) and eliminating τ_2. Then τ_3 is eliminated in the same way. To eliminate τ_4 we use eqs. (e), (f), and No. 7, instead of No. 6, as the latter does not contain τ_4 while the former does. Then eq. (6) is next used in eliminating τ_5, after which the equations are used in regular order. Finally the values of τ_{12}, for Cases II and III, are obtained in eq. (u'). Having these, the values of τ_{11} are most readily obtained from eq. (s'), then τ_{10} from (q'), etc. The several values are given at the foot of the table. The process for Case I is carried only so far as to eliminate τ_4, giving eq. (g), from which (g') is obtained, which contains τ_5, τ_6, and τ_7. From symmetry we know that the member 6–7 has no moment at either end, hence $\tau_{67} = 0$, and $\tau_{76} = 0$. From Table C, joint 6, we have $\tau_{67} = \tau_6 +$

TABLE E

"Solution of Equations

No of Eq	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}	Absolute Terms		
													Case I	Case II	Case III
1' 2' 3'	1 1 1	.147 4.137 .0284	.353 .068 5.130	 1.000 .1792	 1.358								+ 7.63 + .61 + 46.35		
(a) (b)		3.990 - .1186	- .285 4.777	1.000 .1792	 1.358								- 7.02 + 38.72		
a' b' 4'		1 -1 1	- .071 40.30 .432	.251 1.511 7.063	 11.45 .532		 1.568						- 1.70 +326.7 +100.13		
(c) (d)			40.229 .503	1.762 6.812	11.45 .532		 1.568						+324.94 +101.89		
c' d' 5'			1 1 1	.044 13.54 .163	.285 1.058 4.445	 1	 3.117 .059						+ 8.07 + 202.5 -204.4	+ 8.07 + 202.5 - 74.7	
(e) (f)				13.496 .119	.773 4.16	 1	3.117 .059						+194.4 -212.5	+194.4 - 82.8	
e' f' 7'				1 1 1	.057 34.96 .1225	 8.41 .129	.231 .496 4.747	 .122					+ 14.4 +1786.0	+ 14.4 -696.0 +132.1	+ 14.4 - 696.6 + 148.7
(g) (h)					34.90 .0655	8.41 .129	.265 4.516	 .122					-1800.4	-710.4 +117.7	-710.4 +134.3
g' h' 6'					1 1 1	.241 1.972 4.125	.007 68.90 .062	 1.87 1	 15.26				- 51.58	- 20.36 +1790. -267.2	- 20.36 +2049. - 131.1
(i) (j)						1.731 3.884	68.89 .054	1.87 1	15.26					+1816.4 - 246.8	+2069.4 - 110.7
i' j' 8'						1 1 1	39.82 .0139 .0588	1.081 .257 4.445	8.815 .1625	 1				+1049.5 - 63.55 -414.5	+1195.5 - 28.5 - 322.7
(k) (l)							39.81 .0449	.824 4.188	8.815 .1625	 1				+1113.0 -351.0	+1224.0 -294.2
k' l' 9'							1 1 1	.020 93.30 338	.221 3.670 4.503	 22.27 .275	 .637			+ 27.96 -7817. +488.	+ 30.75 -6555. - 6.32
(m) (n)								93.28 .318	3.398 4.282	22.27 .275	 .637			-7845. +460.	-6585.8 - 24.43
m' n' 10'								1 1 1	.036 13.465 .131	.238 .864 3.777	 2.006 .021	 .736		- 84.10 +1447. - 7.85	- 70.60 - 70.83 - 723.5
(o) (p)									13.429 .095	.626 3.538	2.006 .021	 .736		+1531.1 + 76.25	- 6.25 -652.9
o' p' 11'									1 1 1	.0466 37.04 .0684	.149 .219 4.135	 7.703 1		+ 114.0 + 798. + 230.7	- .46 -6835. +1363.5
(q) (r)										36.99 .0218	.070 3.986	7.703 1		+684. +1122.7	-6835 +1364.
q' r' 12'										1 1 1	.0019 18.28 .4153	.208 45.90 2.830		+ 10.50 +5627. - 50.30	- 184.8 +6255.0 - 267.2
(s) (t)											18.28 .4134	45.69 2.622		+5068.5 - 68.80	+6273.5 - 82.4
s' t'											1 1	.250 6.54		30.70 -166.4	+ 343.3 - 199.3
(u)												6.09		-197.1	-542.6
u'												1		- 32.35	- 89.1

Values of τ

Case	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}
I	+ 1.21	+ 1.69	+17.47	- 8.75	-31.67	-86.05	+108.0					
II	+4.54	-4.40	+10.59	+13.56	-10.91	-39.44	+ 6.30	- 94.01	+ 107.04	+ 25.17	+ 38.79	- 32.35
III	+3.65	-2.30	+12.23	+ 5.61	-15.48	-21.60	+41.83	- 29.04	- 47.33	- 160.9	+365.6	-89.10

86.05, whence $\tau_6 = -86.05$. Also under joint 7, $\tau_{76} = \tau_1 - 108.0$, whence $\tau_7 = +108.0$. These values, substituted in eq. (g'), Case I, determine τ_5. Then τ_4 is obtained from eq. (e'), etc.

296. (*c*) *Calculation of Individual Deflection Angles and Fibre Stresses.*—Table E gives the values of the reference angle τ_n for each joint. The various angles, τ, for each member, are now obtained from Table C by adding to the reference angle for the joint, the value of $\Sigma\,\delta\,\angle$ for the member in question. Thus for member 5–4, Case I, the value of τ_{54} is $-31.7 - 48.6 = -80.3$; and for this member, Case II, it is $-10.9 - 48.6 = -59.5$. For the other end of this member, $\tau_{45} = -8.75 - 59.48 = -68.2$, and $+13.56 - 59.48 = -45.9$, respectively, for Cases I and II. These values of τ are given in Table F, for all members, and for all three cases. In following columns of the same table are given the values of $2\,\tau_{nm} + \tau_{mn}$ for use in the formula for fibre stress, eq. (18), $f_{nm} = \frac{2\,c}{l}\,(2\,\tau_{nm} + \tau_{mn})$. Thus for member 5–6 the value of this quantity is $2 \times \tau_{56} + \tau_{65}$, etc. Finally, the fibre stress for 1,000-lb. joint loads is obtained by multiplying these values by the quantity $\frac{2\,c}{l}$ for Case I, by $\frac{2\,c}{1.5\,l}$ for Case II, and by $\frac{2\,c}{3\,l}$ for Case III.

In regard to signs, the sign of τ indicates the direction of inclination at the joint, as shown in Fig. 9. Assuming all values positive, then the members will be bent toward the *right* with respect to the joint as centre as in Fig. 8. A plus sign for f will then signify a tensile stress in the fibre first met with in passing around the joint toward the right, the upper fibre in members on the right of the joint and the lower fibre in members on the left, etc. On account of the different effects of loads on different joints, it is desirable to use the sign for f in this way until the necessary combinations are made. For unsymmetrical members, such as the top chord and end post, two values of f are determined, the first being the stress in the fibre corresponding to the one referred to above, namely, in the fibre first met with in passing around the joint The second value is the stress in the opposite fibre.

TABLE F.

VALUES OF DEFLECTION ANGLES τ, AND FIBRE STRESSES f.

(The fibre stresses are for 1,000 lb. joint loads.)

Joints.	Member.	$\frac{c}{l}$	CASE I (LOAD AT 7).			CASE II (LOAD AT 9).			CASE III (LOAD AT 11).		
			Load = 1,000 lbs.			Load = 1,500 lbs.		Load = 1,000 lbs.	Load = 3,000 lbs.		Load = 1,000 lbs.
			τ	$2\tau_{nm}+\tau_{mn}$	f	τ	$2\tau_{nm}+\tau_{mn}$	f	τ	$2\tau_{mn}+\tau_{mn}$	f
1	13	.0194 .0287	+ 1.2	− 12.6	−0.49 +0.72	+ 4.5	− 12.9	−0.33 +0.50	+ 3.6	− 13.0	−0.17 +0.25
	12	.0285	+14.3	+ 30.3	+1.73	+17.6	+ 30.9	+1.18	+16.8	+ 31.2	+0.59
2	21	.0285	+ 1.7	+ 17.7	+1.01	− 4.4	+ 8.8	+0.33	− 2.3	+ 12.2	+0.23
	23	.0145	−33.7	−104.6	−3.04	−39.8	−123.7	−2.39	−37.7	−117.8	−1.14
	24	.0285	− 0.9	− 10.5	−0.60	− 7.0	− 0.4	− .02	− 4.9	− 4.1	−0.08
3	35	.0287 .0451	+17.5	+ 36.2	+2.07 −3.26	+10.6	+ 3.1	+1.65 −2.59	+12.2	+ 41.8	+0.80 −1.26
	34	.0153	−30.5	− 95.8	−2.94	−37.4	− 87.3	−1.78	−35.8	− 92.0	−0.94
	32	.0145	−37.3	−108.3	−3.14	−44.2	−128.1	−2.48	−42.5	−122.7	−1.19
	31	.0287 .0194	−15.0	− 28.8	−1.65 +1.12	−21.9	− 39.2	−1.50 +1.02	−20.2	− 36.9	−0.71 +0.48
4	42	.0285	− 8.8	− 18.5	−1.05	+13.6	+ 20.2	+0.77	+ 5.6	+ 6.4	+0.12
	43	.0153	−34.8	−100.1	−3.07	−12.5	− 62.4	−1.27	−20.5	− 76.7	−0.78
	45	.0202	−68.2	−216.7	−8.75	−45.9	−151.4	−4.08	−53.9	−171.8	−2.31
	47	.0285	+ 2.3	+112.6	+6.42	+24.6	+ 55.6	+2.12	+16.7	+ 75.2	+1.43
5	56	.0287 .0451	−31.7	+ 22.6	+1.30 −2.05	−10.9	− 18.8	−0.72 +1.13	−15.5	− 10.2	−0.20 +0.31
	57	.0122	−68.9	− 89.0	−2.17	−48.2	−149.3	−2.43	−52.8	−122.9	−1.00
	54	.0202	−80.3	−228.8	−9.24	−59.5	−164.9	−4.44	−64.1	−182.0	−2.45
	53	.0451 .0287	+ 1.2	+ 19.9	+1.78 −1.14	+21.9	+ 54.4	+3.27 −2.00	+17.3	+ 46.9	+1.41 −0.90

6	68	.0287 .0451	− 86.0	−140.3	− 8.00 +12.65	− 39.4	+ 21.5	+0.82 −1.30	− 21.6	− 13.7	− 0.26 + 0.41
	67	.0161	0	0	0	− 83.1	−267.9	−5.76	− 65.2	−196.7	− 2.11
	65	.0451 .0287	+ 86.0	....	+12.65 − 8.06	+ 3.0	− 5.0	−0.30 +0.19	+ 20.8	+ 26.1	+ 0.78 − 0.50
7	74	.0285	+108.	+218.3	+12.44	+ 6.3	+ 37.2	+1.42	+ 41.8	+100.3	+ 1.91
	75	.0122	+ 48.8	+ 28.7	+ 0.70	− 52.9	−154.0	−2.50	− 17.4	− 87.5	− 0.71
	76	.0161	0	0	0	−101.7	−287.2	−6.17	− 66.2	−197.6	− 2.12
	78	.0122	....	....	....	− 95.3	−127.5	−2.07	− 59.8	−127.5	− 1.04
	79	.0285	....	....	....	− 36.1	+ 34.8	+1.32	− 0.6	− 48.5	− 0.92
8	8 10	.0287 .0451	....	....	....	− 94.0	−162.0	−6.20 +9.74	− 29.0	+ 16.4	+ 0.31 − 0.49
	89	.0202	....	....	....	+ 3.0	− 6.2	−0.16	− 68.0	−302.7	− 4.08
	87	.0122	....	....	....	+ 63.1	+ 31.0	−0.50	− 8.0	− 75.8	− 0.62
	86	.0451 .0287	....	....	....	+100.4	+161.4	+9.71 −6.18	+ 29.5	+ 37.4	+ 1.13 − 0.72
9	97	.0285	....	....	....	+107.0	+178.0	+6.77	− 47.3	− 95.3	− 1.81
	98	.0202	....	....	....	− 12.3	− 21.5	−0.58	−166.6	−401.3	− 5.40
	9 10	.0153	....	....	....	− 79.1	−102.0	−2.08	−175.6	−324.7	− 3.31
	9 11	.0285	....	....	....	−131.2	−223.7	−8.51	+ 68.4	+502.4	+ 9.55
10	10 12	.0194 .0287	....	....	....	+ 25.2	+ 44.1	+1.14 −1.69	−166.9	−139.4	− 1.80 + 2.67
	10 11	.0145	....	....	....	+ 69.7	+243.8	+4.71	− 55.6	− 71.7	− 0.69
	10 9	.0153	....	....	....	+ 56.2	+ 33.3	+0.68	+ 26.5	−122.6	− 1.25
	10 8	.0451 .0287	....	....	....	+ 26.0	− 42.0	−2.53 +1.61	+ 74.5	+119.9	+ 3.61 − 2.30
11	11 9	.0285	....	....	....	+ 38.8	− 53.6	−2.04	+365.6	+799.6	+15.20
	11 10	.0145	....	....	....	+104.4	+278.5	+5.38	+ 39.5	+ 23.4	+ 0.23
	11 12	.0285	....	....	....	+ 33.7	+ 35.0	+1.33	−355.3	−799.7	−15.20
12	12 11	.0285	....	....	....	− 32.4	− 31.1	−1.18	− 89.1	−533.5	−10.13
	12 10	.0287 .0194	....	....	....	− 6.2	+ 12.8	+0.47 −0.33	+194.4	+221.9	+ 4.25 − 2.88

TABLE G.

PRIMARY AND SECONDARY STRESSES FOR 1,000-LB. JOINT LOADS.

Member.		Secondary Stresses for 1,000 lbs. at					Maximum Primary Stress.	Total Secondary Stress for Maximum Primary.	Per cent Secondary of Maximum Primary.
		2	4	7	9	11			
End Post	13 T*	− 2.88	−0.33	− 0.49	−0.33	−0.17	−56.40	− 4.20	7.5
	13 B	+ 4.25	+0.49	+ 0.72	+0.50	+0.25			
	31 T	− 1.80	+1.14	+ 1.12	+1.02	+0.48			
	31 B	+ 2.67	−1.69	− 1.65	−1.50	−0.71		− 2.88	5.1
Top Chords	35 T	− 2.30	+1.61	+ 2.07	+1.65	+0.80	−65.72		
	35 B	+ 3.61	−2.53	− 3.26	−2.59	−1.26		− 6.03	9.2
	53 T	+ 0.31	−6.20	− 1.14	−2.08	−0.90		−10.01	15.2
	53 B	− 0.40	+9.74	+ 1.78	+3.27	+1.41			
	56 T	− 0.72	−6.18	+ 1.30	−0.72	−0.20	− 74.00	− 6.52	8.8
	56 B	+ 1.13	+9.71	− 2.05	+1.13	+0.31			
	65 T	− 0.26	+0.80	− 8.06	+0.10	−0.50		− 7.81	10.6
	65 B	+ 0.41	−1.30	+12.65	−0.30	+0.18			
Bottom Chords	12	+10.13	+1.18	+ 1.73	+1.18	+0.59	+73.10	+14.81	20.3
	21	+15.20	−1.33	+ 1.01	+0.33	+0.23		+15.44	21.2
	24	−15.20	+2.04	− 0.60	−0.02	−0.08	+73.10	−13.86	19.0
	42	− 9.55	+8.51	− 1.05	+0.77	+0.12		− 1.20	1.6
	47	+ 1.81	−6.77	+ 6.42	+2.12	+1.43	+75.70	+ 5.01	6.6
	74	+ 0.92	−1.32	+12.44	+1.42	+1.91		+15.37	20.3
Hip Vertical	23	− 0.23	−5.38	− 3.04	−2.39	−1.14	+62.50	−12.18	19.5
	32	+ 0.69	−4.71	− 3.14	−2.48	−1.19		−10.83	17.3
Second Vertical	45	+ 5.40	+0.58	− 8.75	−4.08	−2.31	−37.76	−15.14	40.1
	54	+ 4.08	+0.16	− 9.24	−4.44	−2.45		−16.13	42.7
Middle Vertical	76	+ 2.12	+6.17	0	−6.17	−2.12	0	− 8.29	
	67	+ 2.11	+5.76	0	−5.76	−2.11		− 7.87	
Main Diagonal	34	+ 1.25	−0.68	− 2.94	−1.78	−0.94	+74.77	− 6.34	8.5
	43	+ 3.31	+2.08	− 3.07	−1.27	−0.78		− 3.04	4.1
2d Diagonal	57	+ 0.62	+0.50	− 2.17	−2.43	−1.00	+64.14	− 5.60	8.7
	75	+ 1.04	+2.07	+ 0.70	−2.50	−0.71		− 2.41	3.8

* *T* = top fibre; *B* = bottom fibre.

297. (*f*) *Total Secondary Stresses and Relation to Primary Stresses.*—From Table F the fibre stress f, in any member, due to 1,000 lbs., at any joint, can be obtained. Thus for member 2–4, load at 2, the stress is the same as in member 11–9, for load at 11, but with opposite sign, etc. In Table G, are brought together the values of the primary and secondary stresses at each end of each member of one-half of the truss, for a load of 1,000 lbs. at each joint. There is also given the total secondary stress which occurs for the same loading as the maximum primary stress. This is given both in pounds per sq. in. and also as percentage of the maximum primary stress. For the end post and top chord sections both top and bottom fibre stresses are given [indicated by "T" and "B"]. For the other members, only one value is given, this value being the stress in the fibre first met in passing around the joint in clockwise rotation. These signs correspond to those of Table F. As these members are all symmetrical, there will exist in all cases equal stresses of opposite sign from those given. The sign is therefore important only in combining results for loads on several joints.

It will be seen that for some members the secondary stresses are not a maximum when the primary stresses are a maximum, but inasmuch as the primary stress is generally much larger than the secondary stress, the total is generally a maximum when the primary stress is a maximum. On this basis the percentages in Table G are calculated. The secondary stresses in the chord members are generally of like sign for all loads, so that these stresses are a maximum, or nearly so, for full load. This relation is apt to be disturbed for loads applied near the member in question. Thus for member 4–7 all secondary stresses are plus except for load at 4, which gives a negative value. Again, in member 2–4, all values are negative except for load at 4.

In the web members the loading for maximum primary stress causes nearly the maximum secondary stress. In 4–5, for example, the maximum primary requires joints 7–11 to be loaded. This loading gives also maximum secondary stress. In member 7–5 the same is true, excepting for the small stress of + .70 for load at 7. In 4–3 there is also one exception to this rule. In the hip-vertical a fully loaded structure gives nearly the maximum secondary stress.

The true relations can best be seen by the use of influence lines.

Such lines have been drawn in Fig. 18 for some of the most important stresses. In these diagrams the primary stresses and the secondary stresses for both ends have been shown for members 1–2, 3–5, 4–7, and 4–5. The shaded areas show the secondary stresses at one end,

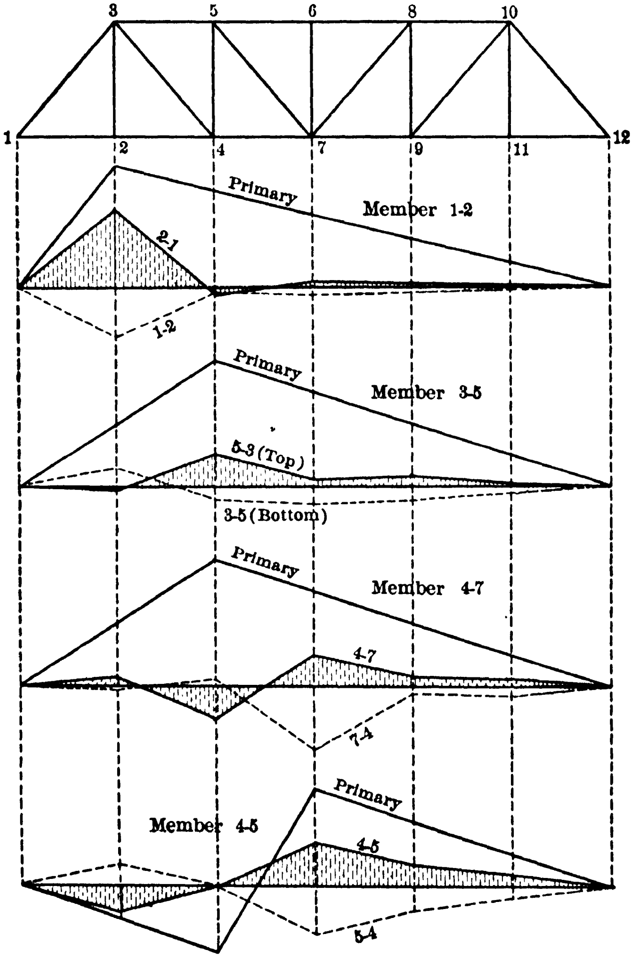


FIG. 18.

and the dotted lines those at the opposite end, the ordinates to the latter being plotted below the axis when of like sign to the primary stresses. The combined influence line for the sum of the two stresses may be used, if desired, for getting the exact maximum fibre stress for any kind of loading. Sufficiently accurate results for most purposes will, however, be obtained by the assumption of equal joint loads, as in Table G, as the information desired is generally the percentage relation of the secondary stress to the primary stress, and this will be but slightly affected by the nature of the loading. Fig. 19 shows the nature of bending in the various members for a fully loaded structure.

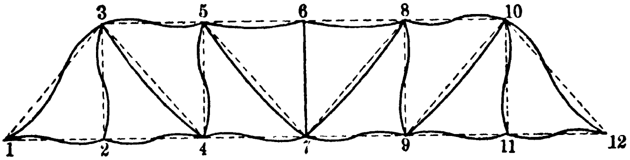


FIG. 19.

The analysis here made assumes the loads applied at lower joints. For upper-joint loads the results will be somewhat different owing to the difference in stress in the vertical posts. For dead load, therefore, the results apply only for that portion applied at the lower joints, but as the load applied at the upper joints is generally small, compared to the total load, it is sufficiently accurate for most purposes to assume the dead load applied at the loaded chord. The greatest difference will occur at the end where the influence of the hip vertical is of primary importance.

298. General Remarks on the Foregoing Method of Calculation.—The calculation of secondary stresses, even of a small truss, has generally been considered by engineers a very laborious operation, requiring many days' time of a skilled computor. A systematic arrangement of the calculations, as illustrated by the preceding problem, will, to a large extent, obviate this objection to such calculations. By the method here given, a good computor, after becoming familiar with the process, can make a complete analysis by joint loads of an ordinary truss in less than two days' time. For larger structures of single intersection the labor involved increases about in proportion to the number of panels. The number of equations is increased, but in their solution the number of operations is increased only in the same proportion as the number of equations. Additional cases of loading, required by more numerous joints, require somewhat more work, but it will be noted from Tables B and C that each new case involves only a small portion of the structure.

Very often sufficient information can be obtained by calculating the secondary stresses for a fully loaded structure. Such a loading gives the maximum stresses in all chord members, the end post, and generally in the hip vertical. This being a symmetrical load, only one-half of the truss need be considered, corresponding to Case I.

The order of elimination adopted in Table E is to be noted, especially with reference to the subtraction of one equation from another. Thus, referring to the group (c'), (d'), and ($5'$), eq. (c') is subtracted from (d'), and also from ($5'$). This procedure is convenient and maintains the relative accuracy as far as possible.

Again, assuming equations ($4'$), ($5'$), ($6'$), etc., to be correct, it will be found that the order of procedure here adopted is such that any error made in the calculations tends to become smaller as the work progresses, so that no great accuracy is necessary to secure satisfactory results. Furthermore, after the value of τ_{12} is obtained and the others are found by substituting back in (s'), (q'), (o'), etc., it is seen that here, also, any error of calculation tends to become less on account of the small values of the coefficients in these particular equations. These general relations are brought about from the fact that in the equation of equilibrium at any joint, given in Table D, the coefficient of the τ for the particular joint is relatively large, equal to twice the sum of all the other coefficients.

299. Solution by Successive Approximations.—The general relation of the values of the coefficients mentioned above enables the method of successive appropriations to be used. In such a method approximate values are first found by omitting from each equation all values of τ excepting the one relating to the particular joint. Thus from eq. (1), we have $\tau_1 = 197.6/25.9$; and from eq. (2), $\tau_2 = 2.33/15.72$, etc. The approximate values thus found are then substituted in the several equations and more accurate values determined, etc.* This method requires about the same amount of work as the direct method, but has the advantage of enabling mistakes in calculation to be readily discovered. Secondary stress calculations need not be made with a high degree of precision.

---

* See Art. 377 for example.

300. Approximate Methods of Calculation for a Limited Number of Joints.—When, for a particular loading, it is desired to determine the secondary stresses in a certain part of the structure, or in certain members only, very satisfactory results can be reached by neglecting the effect of the joints that are remote from the joint or joints in question. Results will be more accurate as more joints are included in the calculations, entire accuracy requiring the consideration of all the joints. It is generally sufficient, however, to include only one joint in each direction beyond those under consideration. Thus, in Fig. 20, if the stresses at joint 5, only, are desired, we may neglect all joints beyond 3, 4, 6, and 7. The small influence of τ_8 on τ_5 may be noted by referring to eqs. (g'), (f'), and (k') of Table E. The value of τ_8 appears to the extent of 2 per cent in τ_7 and 26 per cent in τ_6. Then τ_7 appears to the extent of 0.7 per cent in τ_5, and τ_6 to the extent of 24 per cent. Hence τ_8 comes into τ_5 to the extent of about 6 per cent.

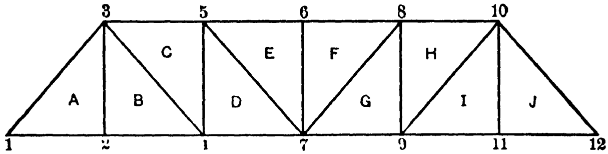


FIG. 20.

EXAMPLES.—(1) As an example, this approximate method will be applied to joint 5 of the truss analyzed in the preceding articles, assuming the loading of Case I, that is, 1,000 lbs. at joint 7. Limiting our attention to joints 3, 4, 5, 6, and 7, we proceed to formulate the five equations. If this problem were to be solved from the beginning it would be necessary to calculate the values of $\delta \angle$ for triangles C, D, and E, and for the angles at 3, 4, 6, and 7 of triangles A, B, F, and G, so that in Table C, the values of $\delta \angle$ for all the joints considered would be known. Assuming this done the equations are written from Table C as follows.

Joint 5.—This equation is the same as given in Table D, and is used in complete form.

Joint 4.—This equation is made up in the same manner as before, except that the value of τ_{24} is omitted (including both τ_2 and the absolute term). The several terms are derived as follows:

from joint 4, $(\Sigma K + \Sigma \delta \angle)$	$26.84\,\tau_4$	$-\ 193.9$
from joint 3, member 3-4	$1.64\,\tau_3$	$-\ 78.7$
from joint 5, member 5-4	$2.02\,\tau_5$	$-\ 98.1$
from joint 7, member 7-4	$5.96\,\tau_7$	

whence the equation

$$26.84\,\tau_4 + 1.64\,\tau_3 + 2.02\,\tau_5 + 5.96\,\tau_7 = +\,370.7.$$

In the same manner we write out the equations for joints 3, 6, and 7 as follows:

$$\begin{aligned}
&(3) \quad 46.95\,\tau_3 + 12.43\,\tau_5 + 1.64\,\tau_4 &&= +\,415.0\\
&(6) \quad 51.26\,\tau_6 + .77\,\tau_7 + 12.43\,\tau_5 &&= -\,4327.7\\
&(7) \quad 28.3\,\tau_7 + 5.96\,\tau_4 + .73\,\tau_5 + .77\,\tau_6 &&= +\,2951.8
\end{aligned}$$

Solving these equations as before, we get $\tau_5 = -\,33.5$ as compared to a value of $-\,31.7$ by the exact process.

(2) As another illustration take joint 11, Case II, as this value is relatively large. Consider only joints 9, 10, 11, and 12. The equations are written out as before. They are:

$$\begin{aligned}
&(9) \quad 26.84\,\tau_9 + 1.64\,\tau_{10} + 3.80\,\tau_{11} &&= -\,293.9\\
&(10) \quad 46.96\,\tau_{10} + 1.64\,\tau_9 + .26\,\tau_{11} + 9.15\,\tau_{12} &&= -\,8994.2\\
&(11) \quad 15.72\,\tau_{11} + 3.80\,\tau_9 + .26\,\tau_{10} + 3.80\,\tau_{12} &&= +\,5180.\\
&(12) \quad 25.90\,\tau_{12} + 9.15\,\tau_{10} + 3.80\,\tau_{11} &&= -\,2446.6
\end{aligned}$$

Solving, we get $\tau_{11} = +\,366.1$ as compared to $+\,365.6$ by the exact process.

The method of approximation here used assumes $\tau = 0$ at the ends of the members where the joints are not considered. This is equivalent to assuming these members to have one-half as much moment at the remote ends as at the joints considered. The results are probably more accurate than if the remote ends be assumed as hinged, and the equations are more easily formulated.

301. Effect of a Collision Strut upon the Secondary Stresses in the End Post.—Suppose in the truss analyzed in the preceding articles, a collision strut 2-*a* be used (Fig. 21), having a moment of inertia of

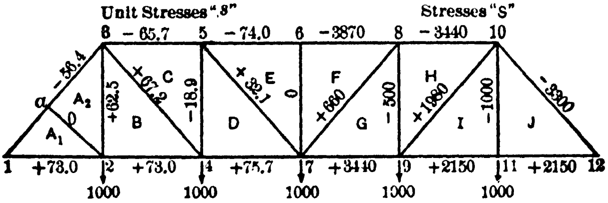


FIG. 21.

50 in^4. Its effect upon the secondary stresses in the member 1–3 will be determined for a load of 1,000 lbs. at each joint. The angle 1–*a*–2 is 90°.

Fig. 21 gives in the right half the primary stresses *S*, and in the left half the unit stresses *s*. The stress in the member *a*–2 will be zero. The unit stresses in all members, excepting 3–4, 5–4, and 5–7, may be taken from Table G, Art. 297. The calculations of $\delta\ \angle$'s requires the consideration of the two triangles, A_1 and A_2, and of member a–2.

The loading being symmetrical, it will not be necessary to carry the equations beyond joint 5, the values of τ_6 and τ_7 being obtained from the condition that both τ_{67} and $\tau_{76} = 0$. There will therefore be six equations, including that for joint a. These equations are as follows:

$$
\begin{array}{llllllll}
(1) & 50.64\,\tau_1 + & 21.52\,\tau_a + & 3.80\,\tau_2 & & & & = -\ 7{,}710 \\
(a) & 21.52\,\tau_1 + & 75.30\,\tau_a + & .206\,\tau_2 + & 15.92\,\tau_3 & & & = -\ 14{,}400 \\
(2) & 3.80\,\tau_1 + & .206\,\tau_a + & 16.12\,\tau_2 + & .26\,\tau_3 + & 3.80\,\tau_4 & & = +\ 2{,}372 \\
(3) & 15.92\,\tau_a + & .26\,\tau_2 + & 60.49\,\tau_3 + & 1.64\,\tau_4 + & 12.43\,\tau_5 & & = +\ 2{,}518 \\
(4) & 3.80\,\tau_2 + & 1.64\,\tau_3 + & 26.84\,\tau_4 + & 2.02\,\tau_5 + & 5.96\,\tau_7 & & = +\ 2{,}782 \\
(5) & 12.42\,\tau_3 + & 2.02\,\tau_4 + & 55.22\,\tau_5 + & 12.43\,\tau_6 + & .73\,\tau_7 & & = -\ 8{,}194
\end{array}
$$

It will be noted that, except for members 1–3 and a–2, the left-hand members of the equations are the same as in Table D.

The solution of these equations gives the following results: $\tau_1 = -81.2$; $\tau_a = -194.1$; $\tau_2 = +153.7$; $\tau_3 = +121.6$; $\tau_4 = +524$; $\tau_5 = -150.7$. The stresses at joints 1, a, and 3 are as follows:

TABLE H.

STRESSES WITH COLLISION STRUT.

Member.		Secondary Stress.	Primary Stress.	Per cent Secondary to Primary.
End Post	1–a	$-4.3\ B$	-56.4	7.6
	a–1	$-28.1\ T$		50
	a–3	$-28.1\ T$		50
	3–a	$-24.8\ B$		44
Top Chord..........	3–5	$-27.8\ B$	-65.7	42
Bottom Chord.......	1–2	$+10.2\ B$	$+73.0$	14

The very large secondary stresses of 44 per cent to 50 per cent, at joints a and 3 are very significant and show the detrimental effect of the collision strut upon the end post. Without the strut the maximum secondary stresses in this member, given in Table G, Art. 323, are only 7.5 per cent and 9.2 per cent. This case illustrates the general effect of secondary members attached to large primary members in such a

TABLE I.

VALUES OF $\Sigma\delta\angle$, $K\Sigma\delta\angle$, and τ.

Joint. (1)	$\angle$ (2)	$\delta\angle$ (3)	$\Sigma\delta\angle$ (4)	Member. (5)	$K=\frac{I}{l}$ (6)	$K\times\Sigma\delta\angle$ (7)	τ (8)
1......	312	+138.2	+138.2	13	9.15	....	+ 16.22
				12	3.80	+ 525.	+154.4
					12.95	+ 525.	
2....				21	3.80	....	+ 76.6
	123	−249.5	−249.5	23	.255	− 63.6	−172.9
	324	+ 0.57	−249.0	24	3.80	− 948.	−172.4
					7.855	−1011.6	
3....				35	12.43	....	− 28.51
	534	−100.0	−100.0	34	1 64	− 164.	−128.5
	432	+ 4.95	− 95.0	32	.255	− 24.22	−123.5
	231	+111.3	+ 16.3	31	9.15	+ 149.1	− 12.2
					23.475	− 39.12	

				42	3.80	....	+ 69.6
4....	243	− 5.52	− 5.52	43	1.64	− 9.05	+ 64.1
	345	−114.4	−119.92	45	2.016	− 241.8	− 50.3
	547	+ 21.68	− 98.24	47	5.96	− 585.5	− 28.7
					13.416	− 836.35	
				56	12.43	....	−111.7
5....	657	− 37.27	− 37.27	57	.731	− 27.23	−149.0
	754	+ 37.52	+ 0.25	54	2.016	+ 0.50	−111.4
	453	+214.4	+214.65	53	12.43	+2669.	+103.
					27.607	+2642.3	
				68	12.43	....	−128.5
6 ...	867	+128.47	+128.47	67	.77	+ 99.0	0
	768	+128.47	+256.94	65	12.43	+3194.	+128.5
				74	5.96	....	+150.4
7....	475	− 59.20	− 59.20	75	.73	− 43.2	+ 91.20
	576	− 91.20	−150.40	76	.77	− 115.7	0
	678	− 91.20	−241 60	78	.73	− 176.3	− 91.20
	879	− 59.20	−300.80	79	5.96	−1792.	−150.40

way as to cause them to deflect out of line. Other illustrations are given in Art. 342.

302. Effect of Eccentric Joints.—If the axes of the members at any joint do not intersect in a single point, then the primary stresses produce a moment M_n which must be balanced by the secondary bending moments at the joint. This moment, M_n, is taken account of as explained in Art. 281 and eq. (17), by adding to the moment equation of the joint the term $\frac{M_n}{2E}$, or if E is neglected, the term is $\frac{M_n}{2}$. A positive value of this moment is when the tendency of rotation about the joint is clockwise.

Suppose in the Pratt truss of Art. 291, the joints in the top chord and end post are eccentric by 1 inch, the intersection of the axes of the web members being 1 inch below the gravity axis in each case. The secondary stresses in the end post and top-chord members will be determined for the case of a symmetrical loading of 1,000 lbs. at each lower joint. Only five equations will be needed, and the coefficients of τ will be the same as those given in Table D. The absolute terms will be different and will include the moments at the joints due to eccentricity.

The total stresses, S, and the unit stresses, s, are the same as given in Fig. 21. The detailed calculation of the angular changes will not be given, but the results are shown in Table I, together with the values of $\Sigma\ \delta\ \angle$ and $K\ \Sigma\ \delta\ \angle$, as in Art. 293.

The moment of the primary stresses at joint 5 is equal to the horizontal component of the stress in 5–7, multiplied by 1 inch, $= 500 \times \frac{320}{372} \times 1 = 430$, and is negative in sign. The term to be added to the *left* side of the equation is therefore $- 430/2 = - 215$. The absolute term for joint 5 is then made up as follows, from Column (7), Table I:

Joint 5 (2 × 2642.3)	+ 5,285
Member 6–5	+ 3,194
" 7–5	− 43
" 4–5	− 242
External moment	− 215
Total	+ 7,979

At joint 3 we may conveniently take the moment centre at the intersection of the diagonal and hip vertical. The moment of the top-chord stress is then − 3,440, and of the end post is + 3,300, giving a net moment of − 140 in.-lbs. The term − 70 is then to be added to the absolute term for the equation for joint 3. At joint 1, assuming the intersection of the axis of 1–2, and the reaction, to be 1 inch below the gravity axis of 1–3, the moment will equal the stress in 1–3 times 1 inch, = − 3,300 in.-lbs., giving the large term − 1,650 to add to the equation for joint 1. The resulting equations are as follows:

(1) $25.90\,\tau_1 + 3.80\,\tau_2 + 9.15\,\tau_3 = +451$

(2) $3.80\,\tau_1 + 15.72\,\tau_2 + .26\,\tau_3 + 3.80\,\tau_4 = +1{,}522$

(3) $9.15\,\tau_1 + .26\,\tau_2 + 46.96\,\tau_3 + 1.64\,\tau_4 + 12.43\,\tau_5 = -2{,}448$

(4) $3.80\,\tau_2 + 1.64\,\tau_3 + 26.84\,\tau_4 + 2.02\,\tau_5 + 5.96\,\tau_7 = +2{,}784$

(5) $12.43\,\tau_3 + 2.02\,\tau_4 + 55.22\,\tau_5 + 12.43\,\tau_6 + .73\,\tau_7 = -7{,}979$

Also $\tau_{67} = 0 = \tau_6 + 128.5$

and $\tau_{76} = 0 = \tau_7 - 150.4$.

The solution of these equations results in values of τ as given in Table I. From these values we find fibre stresses and percentage secondary stresses to primary stresses as follows:

TABLE J.

SECONDARY STRESSES WITH ECCENTRICITY.

Member.		Secondary Stress.	Primary Stress.	Percentage Secondary to Primary.	Percentage without Eccentricity.
End Post	13	$-1.16\,B$	-56.4	2.0	7.5
	31	$-0.47\,B$		0.8	5.1
Top Chord	35	$-4.15\,B$	-65.7	6.3	9.2
	53	$-10.18\,T$		15.5	15.2
	56	$-5.45\,T$	-74.0	7.4	8.8
	65	$-8.34\,T$		11.2	10.6

In the last column are given the values from Table G, Art. 297, which are the percentages without eccentricity. It will be seen that the effect of eccentricity is small except in the end post, where it happens to be of such sign to reduce the stresses. Generally speaking, the effect of eccentricity is to increase the secondary stresses.

303. Effect of Pin-Connections.—If certain members are pin-connected the maximum moment that can develop at the ends of such members is limited to the moment required to turn the member on the pin. If, on the assumption of rigid joints, the resulting moment is less than this limiting value, then the member will not turn and the moments so calculated are correct. If, however, the calculated moment is greater than such limit, then the member turns and the actual bending moment is equal to the turning moment. Whether a member will turn about the pins depends upon the stiffness of the member, size of pin, and coefficient of friction. A general analysis will assist in judging what is likely to happen.

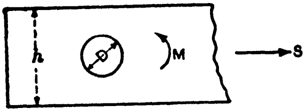


Fig. 22.

Fig. 22 represents the end of a pin-connected member.

Let S = total direct stress;
A = gross sectional area of the body of the member;
s = working stress on gross area;
f = fibre stress due to bending at the joint,
D = diameter of the pin;
k = coefficient of friction;
h = depth of member, assumed as symmetrical:
I = moment of inertia of member;
r = radius of gyration.

Under the stress S, the turning moment required to overcome friction is $M = \frac{SkD}{2} = \frac{sAkD}{2}$. The resulting fibre stress is

$$f = \frac{Mh}{2I} = \frac{skDh}{4r^2}, \qquad \text{(19)}$$

whence the ratio of the stress f to the working stress s is

$$f/s = \frac{kDh}{4r^2}. \qquad \text{(20)}$$

For rectangular sections, such as eye-bars, $r = .29\ h$, and for built-up sections r will generally range from $.35\ h$ to $.4\ h$. If we take $k = .2$, we then have approximately,

$$f/s = .55 \frac{D}{h} \text{ to } .30 \frac{D}{h}. \qquad (21)$$

For eye-bars D/h is generally not less than ¾, whence for such members $f/s = .55 \times .75 = .41$; that is, the secondary stress f must be about 40 per cent of the working stress in order to turn the bar. Such a large secondary stress in the case of narrow sections, as eye-bars, is unusually great and not likely to occur. Even with a coefficient of friction as low as 0.15 the necessary bending moment would not be likely to develop. Hence we may conclude that eye-bars generally do not receive sufficient bending to cause turning on the pins.

In the case of built-up members the ratio of D/h will generally be much smaller than for eye-bars and may indeed be quite small. A ratio of 1/3 represents an ordinary value. This gives $f/s = \frac{1}{3} \times .30 = .10$ (assuming $r = .4\ h$), that is, a turning on the pins will occur when the secondary stress is 10 per cent of the working stress on *gross* section.

This analysis assumes the pressure on the pin equal to the total stress in the member. For built-up members, continuous beyond the pin, as the usual top chord section, the pin pressure at intermediate joints is only the increment of chord stress and much less than the total stress. In such a case a turning on the pin will be brought about at much lower bending stresses than calculated above. Since a secondary stress of 10 per cent may be considered a minimum, it may be concluded that with a coefficient of friction of 1/5 to 1/4, built-up members will, in general, turn upon the pins.

In the calculation of secondary stresses, therefore, in structures partly or wholly pin-connected, it may generally be assumed that eye-bars do not turn and that built-up members will turn. In the latter case a turning moment equal to the moment of friction will develop, which may be taken account of by treating it as an exterior moment at a known value. But for most purposes such moment may be neglected and the pin-ended built-up member omitted in the calculations, without, however, ignoring the fact that in some cases the friction of turning should be estimated, as for example, at the hinges of an arch.

The value of the coefficient of friction between a pin and its bearing is a matter about which little information is available. It is probable that in new work, with pins well painted, the friction is considerably less than 0.2, but in old work where rusting has taken place the friction is greater. An occasional turning on the pins tends to prevent the friction becoming excessively great through rust, while, on the other hand, a fixed condition of the joint, such as is likely to be the case with eye-bars, tends to increase the resistance to turning by a rusting of the parts. It has often been noted in the removal of old structures that the pins are very firmly fixed in the eye-bar heads, evidently not having turned for many years.

In calculating secondary stresses where some of the members are to be considered free to turn, this condition is conveniently allowed for by simply omitting such member from the equations of moments at the joints. If the member turns, it is equivalent to a member whose moment of inertia is zero, or whose $K = 0$; whence the value of $K\tau$ for the member drops out. In the calculation of the changes of angle, $\delta\alpha$, etc., such members should, however, be considered, the same as others that are rigidly connected.

304. *Top Chord of Pin-Connected Pratt Truss.*—In a pin-connected truss the top chord is generally made continuous and the posts and diagonals pin-connected. In such a case it may be assumed that all members turn at the joints excepting the top-chord sections. At the hip joint the top chord is also generally pin-connected. The calculation of the secondary stresses in the top chord in such a case is a simple problem.

As an example take the Pratt truss analyzed in preceding articles and assume a load of 1,000 lbs. at each lower joint. The top-chord stresses will be calculated, (*a*) with pins concentric and (*b*) with pins eccentric.

(*a*) *Without Eccentricity.*—The values of $\delta \angle$ and of $\Sigma \delta \angle$ are calculated as before, but such calculation need be made for the three upper joints, 3, 5, and 6, only. The necessary values are given in Table K on the following page.

TABLE K.

VALUES OF $\Sigma\delta\angle$, $K\Sigma\delta\angle$, and τ.

Joint. (1)	$\angle$ (2)	$\delta\angle$ (3)	$\Sigma\delta\angle$ (4)	Member. (5)	K (6)	$K\Sigma\delta\angle$ (7)	τ (8)
3....				35	12.43		− 43.0
					12.43		
				56	12.43		−128.8
5....	657	− 37.27	− 37.27	57	0		
	754	+ 37.52	+ 0.25	54	0		
	453	+214.4	+214.65	53	12.43	+2669	+ 85.9
					24.86	+2669	
				68	12.43		−128.5
6....	867	+128.47	+128.47	67	0		
	765	+128.47	+256.94	65	12.43	+3194	+128.5
					24.86	+3194	

At joint 3 the only member considered is 3–5 and the equation is

$$24.86\,\tau_3 + 12.43\,\tau_5 = -2669. \qquad . \quad . \quad . \quad . \quad . \quad (a)$$

At joint 5, members 5–3 and 5–6 enter, and the equation is

$$49.72\,\tau_5 + 12.43\,\tau_3 + 12.43\,\tau_6 = -8532. \qquad . \quad . \quad (b)$$

At joint 6 we have $\tau_{67} = 0$, whence $\tau_6 = -128.5$.

Then from (*a*) and (*b*) we derive,

$$\tau_3 = -43.0 \text{ and } \tau_5 = -128.8.$$

The remaining values of τ are calculated in the table in the usual way. The resulting stresses are given in Table L on the following page.

(*b*) *With Eccentricity.*—Assuming the pins eccentric by 1 inch, the moment of the primary stresses at joint 3 is − 3440, and at joint 5 is − 430. It is zero at 6. Hence the absolute terms of equations (*a*) and (*b*) become − 2669 + 1720 = − 949, and − 8532 + 215 = − 8317, respectively. From these we derive $\tau_{35} = +33.6$; $\tau_{53} = +71.1$; $\tau_{56} = -143.6$ and $\tau_{65} = +128.5$. The resulting fibre stresses are given in Table L.

TABLE L.

SECONDARY STRESSES IN PIN-CONNECTED TOP CHORD.

Member.	Without Eccentricity.		With Eccentricity.	
	Secondary Stress.	Percentage of Primary Stress.	Secondary Stress.	Percentage of Primary Stress.
35	0	0	$-12.5\,B$	18.7
53	$-7.4\,T$	11.2	$-10.1\,T$	15.0
56	$-7.4\,T$	10.0	$-9.1\,T$	12.3
65	$-7.4\,T$	10.0	$-6.5\,T$	8.8

305. Effect of Bending Moments in the Members Due to Transverse Loads Applied Between Joints.—Suppose the beam *A B*, Fig. 23,

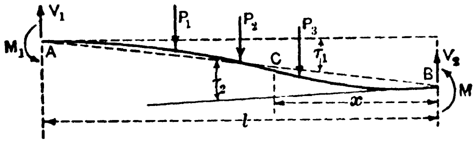


FIG. 23.

to support certain transverse loads, P_1, P_2, etc., and at the same time to be subjected to certain end moments, M_1 and M_2. If the beam were freely supported at *A* and *B*, the transverse loads would cause a bending moment at any point *C*, distant x from *B*, calculated as for a simple beam. Call this moment M'. The true bending moment in the beam at point *C*, under the actual conditions, will then be equal to the moment due to M_1 and M_2, plus M', or, from eq. (6), Art. 277,

$$M_x = M_2 - (M_1 + M_2)\frac{x}{l} + M'. \quad . \quad . \quad . \quad (22)$$

The deflection at $B = \Delta = -\int \frac{M\,x\,d\,x}{EI}$. Substituting from eq. (22) we derive

$$\Delta = \frac{l^2}{6\,EI}(2\,M_1 - M_2) - \int \frac{M'\,x\,d\,x}{EI} \quad . \quad . \quad . \quad . \quad (23)$$

and

$$\tau_1 = \Delta/l = \frac{l}{6\,EI}(2\,M_1 - M_2) - \frac{1}{EI\,l}\int_0^l M'\,x\,d\,x. \qquad (24)$$

Likewise

$$\tau_2 = \frac{l}{6\,E\,I}\,(2\,M_2 - M_1) + \frac{1}{E\,I\,l}\int_0^l M'\,(l - x)\,d\,x \quad . \quad (25)$$

in which x is to be measured from B. From these equations we derive the values of M_1 and M_2, as follows:

$$M_1 = \frac{2\,E\,I}{l}\,(2\,\tau_1 + \tau_2) - \frac{2}{l^2}\int_0^l M'\,(l - 3\,x)\,d\,x \quad . \quad (26)$$

$$M_2 = \frac{2\,E\,I}{l}\,(2\,\tau_2 + \tau_1) + \frac{2}{l^2}\int_0^l M'\,(3\,x - 2\,l)\,d\,x. \quad . \quad (27)$$

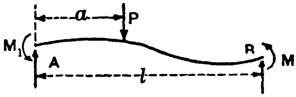


FIG. 24.

In these equations the quantity M' is the bending moment in a simply supported beam due to the given external loads; its sign is to be considered as positive when it tends to cause bending in the same sense as the moment M_2, such as would be caused by loads acting *downward* in Fig. 23.

For a single load, P, applied a distance a from A (Fig. 24), the values of the terms involving M' in eqs. (26) and (27) reduce to $-P\cdot\frac{a\,(l-a)^2}{l^2}$ and $-P\,\frac{a^2\,(l-a)}{l^2}$, respectively. From Art. 6 we note that these values are the same as the end moments of a beam fixed at the ends. Therefore we may write in general

$$M_1 = \frac{2\,E\,I}{l}\,(2\,\tau_1 + \tau_2) \pm (M_A \text{ for fixed ends}). \quad . \quad . \quad (28)$$

In order to harmonize signs with the convention adopted in Art. 278, we will confine our attention to the left end of the member, Fig. 24, where the direction of bending is positive. The last term of eq. (28) may be looked upon as an external moment, in the same manner as a moment due to eccentricity (Art. 281), and to be taken as positive when it acts clockwise around the joint, as at A, Fig. 24. Therefore, if we let M^f denote the moment at the joint due to transverse forces, considering the beam as fixed, we have in general,

$$M_{nm} = \frac{2\,E\,I}{l}\,(2\,\tau_{nm} + \tau_{mn}) + M^f_{mn} \quad . \quad . \quad . \quad (29)$$

and

$$f = 2\,(2\tau_{nm} + \tau_{mn})\,\frac{c}{l} + M^f_{mn}\,\frac{c}{I} \quad . \quad . \quad . \quad (30)$$

In applying these equations to a problem of secondary stresses the quantities M' are to be calculated separately. Then in place of eq. (16), Art. 308, we have

$$2\,(K_{n1}\,r_{n1} + K_{n2}\,r_{n2} + \text{etc.}) + K_{1n}\,r_{1n} + K_{2n}\,r_{2n} + \text{etc.}$$
$$+\frac{M'_{n1}}{2E} + \frac{M'_{n2}}{2E} + \text{etc.} = 0 \quad . \quad . \quad . \quad . \quad . \quad (31)$$

there being terms $\frac{M'}{2E}$ for each member that is subjected to transverse forces, the sign being interpreted as previously shown.

306. Secondary Stresses in Top Chord Due to Weight of Member.—As an illustration of the application of the preceding article the stresses in the top chord of the truss previously analyzed will be determined so as to include the effect of its own weight. In this case it will be desirable to assume a joint load corresponding to the actual load on the structure. A joint load of 75,000 lbs. will, therefore, be taken, applied at each lower joint. The top chord will be assumed as pin-connected, as in Art. 304, and equations written out for joints 3 and 5.

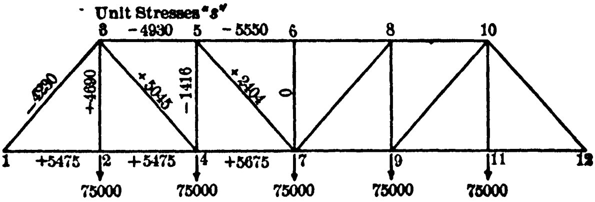


Fig. 25.

(*a*) *Without Eccentricity.*—Fig. 25 gives the unit stresses, s, and Table M, the values of $\Sigma\ \delta\ \angle$, etc.

The end moments of members 3–5 and 5–6, due to their own weight, must be determined, on the assumption of fixed ends. This moment is $\frac{1}{12} w\, l^2$, where w is the load per unit length. Taking the weight of 3–5 and 5–6 at 178 lbs. per foot, the moment at each end is 126,600 in.-lbs. At joint 3 this is positive and hence 126,600/2 = 63,300 is to be added to the left side of the equation (31). For joint 5 nothing is added as the values of M_f for the two members 5–6 and 5–3 are equal but of opposite signs.

TABLE M.

TOP CHORD, WEIGHT INCLUDED.

Values of $\Sigma\,\delta\,\angle$, $K\,\Sigma\,\delta\,\angle$, and τ.

Joint. (1)	$\angle$ (2)	$\delta\,\angle$ (3)	$\Sigma\,\delta\,\angle$ (4)	Member. (5)	K (6)	$K\,\Sigma\,\delta\,\angle$ (7)	τ (8)
3				35	12.43	....	−6140
					12.43		
				56	12.43	....	−8920
5	657	− 2795	− 2795	57	0		
	754	+ 2815	+ 20	54	0		
	453	+16090	+16110	53	12.43	+200,000	+7190
					24.86	+200,000	
				68	12.43	....	−9635
6	867	+9635	+ 9635	67	0		
	765	+9635	+19270	65	12.43	+239,600	+9035
					24.86	+239,600	

The resulting equations are

$$24.86\,\tau_3 + 12.43\,\tau_5 = -\,263{,}300$$
$$12.43\,\tau_3 + 49.72\,\tau_5 + 12.43\,\tau_6 = -\,639{,}600.$$

Then as before, $\tau_{67} = 0$, and $\tau_6 = -\,9{,}635$. The resulting values of τ are given in Table M.

Having the values of τ the fibre stresses at the joints are calculated by means of eq. (30). This requires the term $M' \times \frac{c}{I}$ to be added to the stress as previously calculated. The results are given in Table N.

The effect of the weight of the member is to reduce the secondary stresses at the joints. This is due to the fact that, with pin-connections, the only secondary stresses arising from rigidity of joints in the top chord are due to the bending of the chord as a whole into a slight curve corresponding to the deflection of the entire structure. This produces small compressive stresses in the upper fibres throughout, as shown in Art. 304, Table L. When the effect of the weight of the members

TABLE N.

SECONDARY STRESSES IN TOP CHORD, INCLUDING EFFECT OF WEIGHT OF MEMBER.

Member.	Without Eccentricity.		With Eccentricity.	
	Secondary Stress.	Percentage of Primary Stress.	Secondary Stress.	Percentage of Primary Stress.
35	0		−938 *B*	19.0
53	−180 *T*	3.7	−382 *T*	7.8
56	−180 *T*	3.2	−309 *T*	5.6
65	−300 *T*	5.4	−237 *T*	4.2

is added, this reduces the stresses at the joint but increases those at the centres of the members.

At the centres of the members the stresses may be found as follows: Consider member 5–6. The moment at the centre of a fixed beam being $\frac{1}{24} w\,l^2$, and at the ends $-\frac{1}{12} w\,l^2$, the centre moment exceeds the end moment by $\frac{1}{8} w\,l^2$, which is the centre moment in a simple beam. In the present example, this moment is 3/2 × 126,600 = + 189,900 in.-lbs., giving a fibre stress in the top fibres of − 438 lbs. per sq. in. From Table M, the stress at one end is − 180 and at the other end is − 300, hence at the centre it will be $\frac{-180-300}{2}$ − 438 = − 678 lbs. per sq. in., or 11 per cent of the primary stress. Fig. 26 (*a*) shows the variation in upper fibre stress throughout.*

(*b*) *With Eccentricity.*—The results obtained with an eccentricity of 1 inch are given in Table N. Eccentric pins in this case have little effect except at joint 3. Fig. 26 (*b*) shows the fibres stresses in this case.

307. Secondary Stresses Due to Fixed Supports.—If no hinge is provided at the support, as is quite common in the case of small trusses, and as sometimes occurs in other cases, the joint in question will be restrained and the secondary stresses will generally be much increased thereby. In calculating the secondary stresses where any given joint is thus restrained the equation for the joint in question must include an external moment, due to the action of the support. This moment is unknown, but the deflection angle of the joint is fixed and may be calculated by other means, thus leaving the external moment as the unknown quantity in the equation for the joint, in place of the deflection angle.

---

* This particular problem can be more easily solved by treating the top chord as a continuous girder with a uniform load of 178 lbs. per ft. and adding the resulting stresses to those found in Art. 304, but the foregoing is given as an illustration of the general method for such cases.

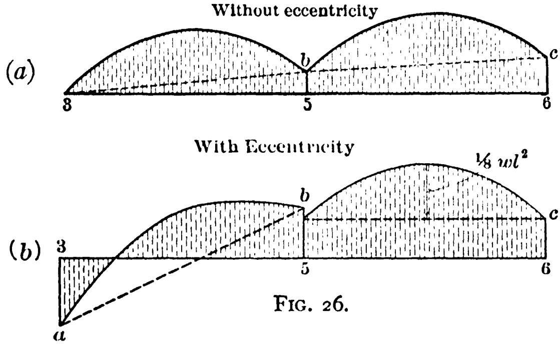


FIG. 26.

Thus in Fig. 27, suppose joint 1 be fixed by the support, thus developing an external moment equal to M_1. The value of $\delta \angle$ for angle 312 is calculated as usual. The deflection angle of 1–3 is τ_{13}, the reference angle of the joint, and that of 1–2 is $\tau_{13} + \delta$ (312). The deflection angle of 1–2 is now to be directly determined by calculating the deflection of joint 2 by the usual formula $\Delta = \Sigma \frac{S u l}{E A}$. If this deflection is Δ_2 then the deflection angle $\tau_{12} = \Delta_2/d$. This angle then becomes known and thence the reference angle τ_{13} or τ_1. The equation of the joint is

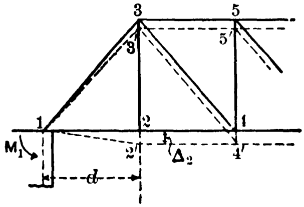


FIG. 27.

$$2\,(K_{13}\,\tau_{13} + K_{12}\,\tau_{12}) + K_{31}\,\tau_{31} + K_{21}\,\tau_{21} - \frac{M_1}{2\,E} = 0. \quad . \quad (32)$$

Written out in the usual manner, this equation will involve τ_1, τ_2, τ_3, and M_1, but, as shown above, τ_1 is known, hence the unknowns are τ_2, τ_3, and M_1. The unknowns are thus the same in number as in the usual case and are readily determined in the solution of the equations for the entire structure. Unless especially desired, the value of M_1 need not be determined, as τ_1 is known from the deflection of joint 2. In this case, after determining τ_1, we proceed at once to the equations for joints 2 and 3, in which τ_1 will appear as a known quantity.

If E is to be generally omitted in the calculations it should also be omitted in the calculation of Δ_2, giving $\Delta_2 = \Sigma \frac{S u l}{A}$.

EXAMPLE.—Suppose the Pratt truss of Art. 291 be supported by a fixed shoe so that no turning can take place at this joint. The resulting secondary stresses for a load of 1,000 lbs. at each joint will be approximately determined by considering only joints 1, 2, and 3.

The first step is to calculate the deflection of joint 2 for the given load by the formula $\Sigma \frac{S u l}{A}$. The result is 159,300 units. The deflection angle τ_{12} is therefore 159,300/320 = + 498. Calculation of the values of $\Sigma\ \delta\ \angle$ and $K\ \Sigma\ \delta\ \angle$ are given in Table O.

From the table we have $\tau_{12} = \tau_1 + 138.2$. But $\tau_{12} = 498$, hence $\tau_1 = 498 - 138 = +360$. The equations for joints 2 and 3 are

$$\begin{aligned} 15.7\,\tau_2 + 3.80\,\tau_1 + .26\,\tau_3 &= +1522. \\ .26\,\tau_2 + 9.15\,\tau_1 + 46.96\,\tau_3 &= +141.8. \end{aligned}$$

Substituting the value of $\tau_1 = 360$, we derive $\tau_2 = +11.04$, $\tau_3 = -67.2$, and thence the values of τ given in the table. The moments at joint 1 are:

$$\begin{aligned} M_{12} &= 2\,K_{12}\,(2\,\tau_{12} + \tau_{21}) = +\ 7{,}480 \text{ in.-lbs.}, \\ M_{13} &= 2\,K_{13}\,(2\,\tau_{13} + \tau_{31}) = +\ 12{,}240 \text{ in.-lbs.} \end{aligned}$$

For equilibrium of the joint the external moment furnished by the support must be equal to $-(M_{12} + M_{13}) = -19{,}720$ in.-lbs. The minus sign signifies a left-handed moment.

The fibre stresses in the members 1–2 and 1–3 are 57.4 and 38.4, or 79 per cent, and 68 per cent, respectively, of the primary stresses. These large values show the importance of hinged bearings at the supports, and in connection with the discussion of Art. 303 indicate that there must be some turning upon the pins under these conditions.

The effect of a fixed shoe upon the pressure on the masonry may be noted. The negative moment developed at the support is 19,720 in.-lbs. The reaction is 2,500 lbs. This result, therefore, gives an eccentricity of bearing of $\frac{19{,}720}{2{,}500} = 7.9$ inches.

TABLE O.

EFFECT OF FIXED SUPPORT.

Values of $\Sigma\,\delta\,\angle$, $K\,\Sigma\,\delta\,\angle$, and τ.

Joint. (1)	$\angle$ (2)	$\delta\angle$ (3)	$\Sigma\delta\angle$ (4)	Member. (5)	K (6)	$K\Sigma\delta\angle$ (7)	τ (8)
				13	9.15	....	+360
1....	312	+138.2	+ 13.82	12	3.80	+ 525	+498
					12.95	+ 525	
				21	3.80	....	+ 11.0
2....	123	−249.5	−249.5	23	.26	− 63.6	−238.5
	324	+ 0.6	−248.9	24	3.80	− 948.	−238.0
					7.86	−1011.6	
				35	12.43	....	− 67.2
	534	−100.0	−100.0	34	1.64	− 164.0	−167.2
3....	432	+ 4.9	− 95.0	32	.26	− 24.2	−162.2
	231	+111.3	+ 16.3	31	9.15	+ 149.1	− 50.9
					23.48	− 39.1	

In certain cases the stresses which would be produced by rigid bearings would be much greater than in this example. In a cantilever bridge, for example, the normal changes of angle at the main pier *B*, Fig. 28, is unusually great, as the deformations of both arms of the cantilever contribute to this change when the span *B C* is loaded. In calculating the stresses in this case, the change of angle at *B* is first determined by calculating the deflection of joint *D*, as in the example given above. This determines the deflection angles at *B*. The equations are then written out for a sufficient number of joints surrounding joint *B* to secure the desired accuracy.

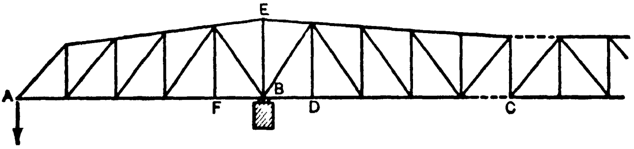


FIG. 28.

308. Calculation of Changes of Angle, $\delta\alpha$, etc., in Figures of Four or More Sides.—The method of calculation of the changes of angle, $\delta\alpha$, etc., at a joint, explained in Art. 276, relates to triangular figures. Frequently, a case will arise where the space formed by contiguous members of the truss is of four or even five sides, as occurs in a double intersection truss, or one in which sub-panels are employed.

In such a case, additional members may be assumed as inserted so as to divide the space into triangles. The deformation of such imaginary members may then be determined by calculating the relative deflection of the two joints which they connect, assuming the sectional areas of the inserted members to be zero, or by utilizing some obvious relation between the angular changes at a joint which will involve the change of length of the imaginary member.* The latter method is generally the simpler and will be illustrated by two common cases.

309. *The Baltimore Truss with Sub-ties.*—Fig. 29 represents any panel of a Baltimore truss. Member $m\,O$ is not supposed to be present. The figure $M\,O\,o\,m$ is therefore a quadrilateral, and the problem is to calculate the angle changes at the four corners of this figure. To do this, suppose a member $m\,O$ to be inserted, dividing the quadrilateral into the triangles A and B. Let s_8 = unit stress of this added member. Its value will be found by equating to zero the sum of the angle changes at the interior joint O.

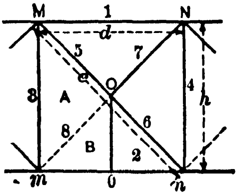


FIG. 29.

Let s_1, s_2, s_3, etc., denote the unit stresses in the several members of the panel as numbered in the figure. Beginning with triangle A and

---

[*] A method suggested by Mr. D. L. Evans, Seattle, Washington.

proceeding counter-clockwise around the joint O, we have, then, from Art. 276, the following expression:

$$\left[(s_3 - s_8)\frac{h}{d} + (s_3 - s_5)\frac{h}{d}\right] + \left[(s_2 - s_8)\frac{d}{h}\right] + \left[(s_2 - s_6)\frac{d}{h}\right]$$

$$+\left[(s_4 - s_6)\frac{h}{d} + (s_4 - s_7)\frac{h}{d}\right] + \left[(s_1 - s_5)\frac{d}{h} + (s_1 - s_7)\frac{d}{h}\right] = 0. \quad (33)$$

From this we derive

$$s_8 = 2\,(s_1 + s_2)\frac{d^2}{c^2} + 2\,(s_2 + s_4)\frac{h^2}{c^2} - (s_5 + s_6 + s_7). \quad . \quad (34)$$

The angle changes at m and O of triangles A and B may now be determined, using the above value of s_8, and thus the desired changes in the angles $M\ m\ O$ and $M\ O\ o$. Then in writing the moment-equations at the several points, the imaginary member 8 is omitted.

In a truss with inclined chords, a general expression for the unknown stress in the imaginary member is cumbersome, and numerical values should be used for the cotangents in the general equation corresponding to eq. (33).

The foregoing method is also applicable to a K-truss and, in general, to any case where the imaginary member connects at one end to an interior joint.

310. *The Double Triangular Truss.*—In the double triangular truss, Fig. 30, the imaginary members needed are $B\ b$, $C\ c$, $D\ d$, etc. The unit stress in $B\ b$ is first determined, then $C\ c$, $D\ d$, etc. If the diagonals are connected at their intersections, the desired relation can be obtained by equating to zero the angle changes at these joints. If not connected, we may use the relation that the change of angle $A\ a\ b$

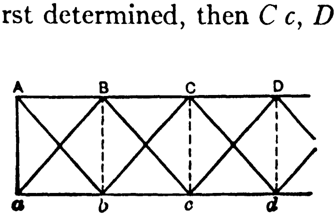


FIG. 30.

of triangle $A\ a\ b$ is equal to the sum of the changes of angles $A\ a\ B$ of triangle $A\ a\ B$ and $B\ a\ b$ of triangle $B\ a\ b$.

Writing out the several angle changes we have (Fig. 31):

$$\left[(s_5 - s_3)\frac{h}{d} + (s_5 - s_2)\frac{d}{h}\right] = (s_1 - s_4)\frac{d}{h} + (s_6 - s_4)\frac{h}{d}$$

whence

$$s_6 = -(s_1 + s_2)\frac{d^2}{h^2} + (s_4 + s_5)\frac{c^2}{h^2} - s_3. \quad . \quad . \quad (35)$$

Having found s_6 for $B\ b$, the similar value for $C\ c$ is found from the quadrilateral $B\ C\ b\ c$, and so on.

The unit deformations in the several verticals being known, the changes of angles at the various joints are readily found by the usual method.

This problem may also be solved by assuming the diagonals connected at their intersections and equating to zero the angle changes at these intersection joints. This method is valid whether or not the diagonals are actually connected, as the primary stresses in the several members are not affected by such connections.

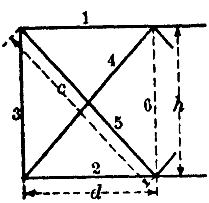


FIG. 31.

311. Relative Amounts of Secondary Stresses in Trusses of Different Types.—The examples of the preceding articles serve to give some notion of the amount of secondary stress likely to exist in the ordinary Pratt truss. However, as this stress will vary greatly with the proportions of the structure, no single set of values can be applied generally. If close results are desired they must be worked out for the particular truss at hand.

Owing to the somewhat lengthy calculations involved, it is probable that secondary stresses will not as a rule be determined for each individual case, except in large and important structures; and that the allowance necessary to be made for such stresses will be based on results obtained on typical designs similar to the structures in question. Calculations of such typical structures will not only serve to show the amount necessary to allow for secondary stresses, but will give valuable information regarding the relative merits of different designs so far as this feature is concerned. In such calculations the information needed is generally the ratio of secondary stress to primary stress, when the latter is a maximum, as the total is then usually a maximum. Where the secondary stresses are very high, however, it may happen that the maximum total is dependent more upon the secondary than upon the primary stress. Such cases require special consideration.

It is not the purpose of this work to give the analysis of a large number of particular trusses. Such analyses can readily be made in any office, and of such structures as may be most important for the office concerned. A little familiarity with the method will enable any good computor to make a complete analysis of an ordinary truss in two or three working days, or less. It is thought, however, that a study may profitably be made here of types of trusses and proportions, by the use of certain approximate methods, making general assumptions as to relative sections and unit stresses. In this way it is proposed to study five of the most common types of trusses: (1) The Pratt system; (2) the Warren system without verticals; (3) the Warren system with verticals; (4) the double Warren system without verticals; and (5) the double Warren system with verticals.

312. *General Method of Analysis.*—In the method of investigation here used it will be assumed that the truss is of uniform construction for several panels, that is, that all of several consecutive panels are exactly alike. It will also be assumed that the unit stresses are, likewise, the same in these several panels, so that both the primary and the secondary stresses will be the same in the different panels. Then, by assuming various proportions for panel length, sectional areas, and unit stresses, the general effect of a variation in these elements can be seen and some notion obtained of the relative merits of various types of trusses.

313. *The Pratt Truss System.*—The Pratt system will be investigated by applying this method of analysis to a number of trusses in which the ratio of height to panel length, the relative moments of inertia, and the relative primary stresses in the various members will be varied. To illustrate the method of analysis, detailed calculations will be made of one case, with three different assumptions as to primary stresses. The results of this analysis together with several other cases are tabulated in Table Q.

A truss will be assumed whose ratio of height to panel length is 4 : 3, Fig. 32. The moment of inertia of all the web members will be assumed to be equal and that of the chords to be four times that of the webs, the same for top and bottom chords. The absolute values of l and I need not be known in getting the values of τ for the various members; only relative values are required (assuming joints to be concentric). We may then, for brevity, let $l_{24} = 3$; $l_{34} = 4$, and

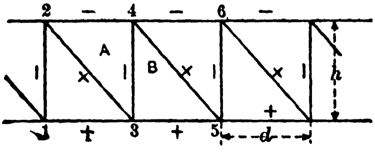


Fig. 32.

$l_{23} = 5$. Also $I_{24} = I_{13} = 4$, and $I_{23} = I_{34} = 1$. As to unit stresses, three conditions will be assumed:

(*a*) stress in chords = 10, stress in webs = 10;
(*b*) stress in chords = 5, stress in webs = 10;
(*c*) stress in chords = 10, stress in webs = 0.

Positive moment and positive shear are assumed, giving compression in the upper chord and verticals, and tension in the lower chord and diagonals.

Case (*a*), assuming equal stresses in chords and webs, corresponds to a condition of loading which produces about equal stresses in these members, as for example, at sections near the end of a simple span; case (*b*), with one-half as much stress in chords as webs, corresponds to a section of truss or a loading such that the shear is a maximum, but the moment is about one-half the maximum, as might occur in sections near the centre of a simple span under partial load; and case (*c*), with zero stress in the webs, corresponds to a condition of maximum moment and zero shear. These conditions will fairly represent the range necessary for a good knowledge of the relations of secondary and primary stresses, not only for ordinary simple trusses, but for such structures as stiffening trusses of suspension bridges and the like. The results thus obtained are particularly applicable to long trusses of numerous panels.

In the calculation of the secondary stresses, under the assumed conditions, it will be necessary to consider two joints only, such as joints 3 and 4, since $\tau_2 = \tau_4 = \tau_6$ and $\tau_1 = \tau_3 = \tau_5$.

TABLE P.

GENERAL CASE OF PRATT SYSTEM.

Values of $\Sigma \delta \angle$ and $K \Sigma \delta \angle$.

Joint.	Member.	K	$\angle$	CASE (*a*).			CASE (*b*)			CASE (*c*).		
				$\delta \angle$	$\Sigma \delta \angle$	$K \Sigma \delta \angle$	$\delta \angle$	$\Sigma \delta \angle$	$K \Sigma \delta \angle$	$\delta \angle$	$\Sigma \delta \angle$	$K \Sigma \delta \angle$
3	31	1.33	....	....	....	....	....	....	....	....	....	....
	32	.20	132	−26.7	−26.7	− 5.33	−26.7	−26.7	− 5.33	0	0	0
	34	.25	234	−15.0	−41.7	−10.42	−11.2	−37.9	− 9.48	−7.5	− 7.5	− 1.87
	35	1.33	435	+26.7	−15.0	−20.00	+30.4	− 7.5	−10.00	−7.5	−15.0	−20.00
		3.11				−35.75			−24.81			−21.87
4	46	1.33	....	....	....	....	....	....	....	....	....	....
	45	.20	645	−26.7	−26.7	− 5.33	−26.7	−26.7	− 5.33	0	0	0
	43	.25	543	0	−26.7	− 6.67	− 3.7	−30.4	− 7.60	+7.5	+ 7.5	+ 1.87
	42	1.33	342	+41.7	+15.0	+20.00	+37.9	+ 7.5	+10.00	+7.5	+15.0	+20.00
		3.11				+ 8.00			+ 2.93			+21.87

The changes in angle at joints 3 and 4 are found from a consideration of triangles A and B, in the same manner as explained in previous examples. These values, and the calculation of $K \Sigma \delta \angle$ for the three cases, (*a*), (*b*), and (*c*), are given in Table P.

In formulating the equations for joints 3 and 4, note that $\tau_{13} = \tau_{35}$, $\tau_{31} = \tau_{53}$, $\tau_{23} = \tau_{45}$, etc.; and $\tau_1 = \tau_3 = \tau_5$; $\tau_2 = \tau_4 = \tau_6$. The resulting equations for the two joints are as follows, for all three cases

	(*a*)	(*b*)	(*c*)
$8.88\,\tau_3 + .45\,\tau_4 =$	$+ 103.5$	$+ 72.5$	$+ 61.9$
$.45\,\tau_3 + 8.88\,\tau_4 =$	$- 20.2$	$+ 10.7$	$- 61.9$

Solving we have

	(*a*)	(*b*)	(*c*)
$\tau_3 =$	$+ 11.80$	$+ 8.13$	$+ 7.34$
$\tau_4 =$	$- 2.88$	$+ 0.79$	$- 7.34$

From these values the several values of τ are calculated, from which the bending moments may be determined. For present purposes, however, it will be more useful to calculate the quantity $2\ (2\ \tau_{nm} + \tau_{mn})$, which, after multiplying by c/l, will be the fibre stress. Since, however, the ratio of c/l has not been assumed, the calculations will not be carried further. For convenience we may call the quantity $2\ (2\ \tau_{nm} + \tau_{mn})$, the *coefficient* of secondary stress. This coefficient, for the three cases above considered, is given in Table Q, with other values for other assumed proportions. Each member will have a separate coefficient for each end, but the larger of the two values only is given in the table.

Coefficients of Secondary Stress for Pratt Trusses of Various Proportions. Following the methods illustrated in the preceding article, twenty cases have been investigated and the results tabulated in Table Q. Two different ratios of height to panel length have been assumed. In part of the trusses the values of I for the two chords have been taken to be the same, and in part the value of I for the lower, or tension chord, has been taken at one-half that of the upper chord. The value of I for the webs has been taken at one-quarter and one-half that of the upper chord. Three relations of stress intensity have also been taken. The results are given as "Coefficients of Secondary Stress" in the last four columns of the table. As explained above, these coefficients, when multiplied by the value of c/l for any member, will be the secondary fibre stress. Its relation to the primary stress is readily determined by referring to the assumed values of the latter in Cols. (5) and (6).

An examination of these coefficients brings out clearly several general relations. In the first place it is seen that the maximum secondary stresses occur when both the chords and the webs are fully stressed.

The secondary stresses for maximum chord stress and minimum web stress are very small.

Comparing with respect to size of web members it is seen that as the web members increase in size their stress decreases and that in the chords increases. It must be remembered, however, that the values given in the table are to be multiplied by c/l to get actual fibre stress, so that with slender web members the high values of the coefficients may not mean very high fibre stress.

Comparing with respect to relative size of the two chords, when these are equal the coefficients are about equal, but a decrease in size of the lower chord increases considerably its coefficient.

Comparing on the basis of panel length, the ratio of 4:4 gives a smaller coefficient for web members than the 4:3 ratio, but somewhat larger for the chords. For the same height of truss and same width of members the ratio c/l for the chords will be three-fourths as great for the 4:4 panel as for the 4:3 panel. Taking this into account, the secondary fibre stresses in the chords of the 4:4 panel, under full load, will be about 80 per cent as large as in the 4:3 panel. In the webs they are from 70 to 75 per cent as large.

Again, since the coefficients hold good no matter what the actual values of c/l may be, so long as the assumed proportions between the members are maintained, it follows that the secondary stresses are directly proportional to the widths of the members and inversely proportional to their lengths.

For compression members the value of c/l will usually range from about $\frac{1}{30}$ to $\frac{1}{80}$, corresponding to values of l/r of from 40 to 100. A value of $\frac{1}{40}$ applied to the chord coefficient gives secondary stresses

TABLE Q.

SECONDARY STRESSES IN PRATT TRUSSES. (FIG. 32.)

Ratio of Height to Panel Length. $h:d$	Moments of Inertia.			Primary Stress Intensity.		Coefficients of Secondary Stress.			
	Top Chord.	Bot. Chord.	Webs.	Chords.	Webs.	Top Chord.	Bottom Chord.	Vertical.	Diagonal.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
4:3	4	4	1	10	10	43	41	178	148
				5	10	35	34	178	141
				10	0	16	14	0.3	15
	4	4	2	10	10	63	60	159	127
				5	10	55	53	158	120
				10	0	17	17	0.7	14
	4	4	4	10	10	93	86	132	99
				5	10	84	80	131	92
				10	0	19	19	1	14
	4	2	1	10	10	41	62	171	141
				5	10	34	56	171	134
				10	0	16	17	0	15
	4	2	2	10	10	61	92	150	119
				5	10	40	73	157	119
				10	0	17	18	1	15
4:4	4	4	1	10	10	49	46	153	112
	4	4	2	10	10	70	64	134	92
	4	4	4	10	10	97	86	110	67
	4	2	1	10	10	48	66	146	106
	4	2	2	10	10	64	91	125	85

varying from 15 per cent to about 22 per cent of the primary stress when the latter is a maximum; and applying a ratio of $\frac{1}{60}$ to the webs gives values of 20 per cent to 30 per cent of the maximum primary. These values are significant of the amount of secondary stress due to general truss action and which cannot be avoided. Local effects due to the deformation of hangers and other secondary members may easily add 100 per cent to the general values here given.

314. *The Warren Truss System.*—Calculations similar to those above given for the Pratt truss have been made for the Warren truss system, both with and without verticals. The results are given in Table R. Proportions and unit stresses have been varied as shown in the table. Where verticals are used their moment of inertia has been taken at $\frac{1}{20}$th of the top chord. The actual ratio may be considerably less than this, but even at this value the bending resistance of these members has but very little effect upon the stresses in the main members. The stresses in the verticals themselves are dependent mainly upon the ratio of their width to length.

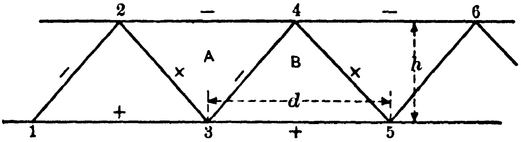


FIG. 33.

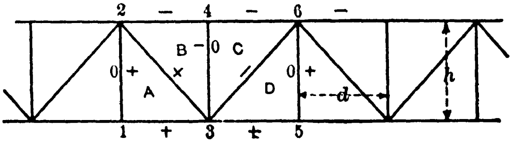


FIG. 34.

Both tension and compression verticals are assumed to exist, but only one set is assumed to be stressed in any particular case, the other set being used merely to support the opposite chord.

Here, as in the case of the Pratt truss, it is seen that the maximum secondary stresses occur when both chords and webs are fully stressed.

TABLE R.

SECONDARY STRESSES IN WARREN TRUSSES. (FIGS. 33, 34.)

Ratio of Height to Panel Length. $h:d$	MOMENTS OF INERTIA.			PRIMARY STRESS INTENSITY.		COEFFICIENTS OF SECONDARY STRESS.		
	Top Chord.	Bottom Chord.	Webs.	Chords.	Webs.	Top Chord.	Bottom Chord.	Webs.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
4 : 6	4	4	1	10	10	48	48	76
				5	10	33	33	69
	4	4	2	10	10	60	60	65
				5	10	45	45	57
	4	2	1	10	10	47	62	73
				5	10	32	47	65
	4	2	2	10	10	56	77	62
				5	10	42	62	53
4 : 4	4	4	1	10	10	42	42	108
	4	4	2	10	10	57	57	93
	4	2	1	10	10	41	59	103
	4	2	2	10	10	54	81	87

With Verticals.

			Diag.	Vertical.	Chords.	Diagonals.	Verticals.			Diagonals.	Verticals.
4 : 8	4	4	1	0.2	10	10	+10	26	96	75	124
	4	4	2	0.2	10	10	+10	32	101	65	124
	4	2	1	0.2	10	10	+10	26	100	71	125
				0.2	10	10	−10	97	34	71	125
	4	2	2	0.2	10	10	+10	30	100	60	127
				0.2	10	10	−10	97	42	60	127

Comparing the Warren system with the Pratt, for ratio of $h : d = 4 : 6$ in the former and $4 : 3$ in the latter case, it is to be noted that the secondary stresses in the chords are very nearly the same, but in the webs they are much smaller in the Warren system. Comparing the trusses of shorter panel-lengths (ratio $h : d = 4 : 4$), with the longer panel-length (ratio $h : d = 4 : 6$), about the same differences are to be seen as in the Pratt system. The shorter panel has somewhat smaller values for the "coefficient," but when these are multiplied by c/l the fibre stresses will be considerably greater.

The results obtained for the trusses with verticals are very instructive. Where the primary stress in the vertical is plus, the structure is a through bridge, the verticals being in tension; and when this stress is minus the structure is a deck bridge. The effect of the strains in the verticals upon the chords is seen to be very great. The coefficients for the lower chords in the through bridges and those of the upper chord in the deck bridges are all about 100, while for the opposite chord they range from 26 to 42, which are very small values. To secure small secondary stresses in such trusses the sectional areas of the verticals, where supporting joint loads, should therefore be made large.

The stresses in the diagonals are about the same as in the truss without verticals. The coefficients for the verticals are large, but as these are generally narrow members the fibre stresses are not high.

It is to be noted from the result of both Tables Q and R that the larger and stiffer members tend to control the flexure at the joint. A reduction of moment of inertia of any member tends to increase the deflection of that member and reduce that of others. To maintain a balance in the fibre stresses themselves it is necessary to make the width of a member correspond in a measure with its moment of inertia. Wide members of small moment of inertia (thin or unsymmetrical sections, as, for example, a T-section composed of a plate and two angles) are likely to have high secondary stresses; narrow and compact members will have low secondary stresses. This statement does not, of course, take account of the long column action in compression members, the effects of which tend in the opposite direction.

315. *The Double Intersection Triangular Truss.*—In a multiple intersection truss the independent action of the different web systems

may give rise to very severe secondary stresses, especially in the chord members. If one system happens to be loaded considerably heavier than the other for several panels, the joints of the more heavily loaded system will all deflect more than the others and will bend the chords accordingly, producing large secondary stresses. The amount of such action depends upon the amount of the excess load on the one system and this depends, in any case, upon the actual wheel concentrations. If the panel-length corresponds to a half-car length (18 to 20 feet), the conditions are very unfavorable, as a train of heavy cars would give a large excess on the one system throughout the entire structure. A panel-length equal to one-half the length of a locomotive would also give a heavy excess in the case of double-header operation. An exact solution for any given structure and loading can be made by the use of influence lines, but a general analysis of some value may be obtained by assuming a certain percentage of excess on the one system over that on the other.

Consider the truss of Fig. 35 and assume that the load on the full system is 25 per cent greater on the average than that upon the dotted

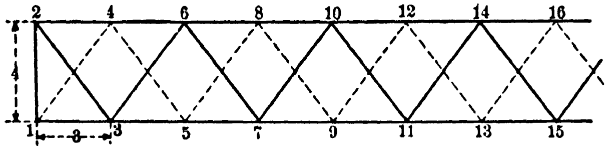


Fig. 35.

system. In view of the very heavy concentrations in use, and the possibility of correspondence of panel length and wheel spacing, such excess is not at all unlikely to occur. The maximum effect produced will be near the centre of the structure where the deflections are a maximum. We will consider a section at joint 13, six panels from the left end, and assume that the joints 3, 7, 11, etc., of the full system are loaded with 25 per cent greater load than the joints 5, 9, 13, etc., of the dotted system. It will also be assumed that the average stress intensity in the chords is 10, that in the webs of the full system is 7.5, and that in the dotted system is 6. We may then proceed to determine the unit deformations along the vertical lines 11–12, 13–14, and 15–16, as explained in Art. 310, eq. (35). In this problem the values of s_1 and s_2 of that equation are equal and of opposite sign, and therefore the term $(s_1 + s_2) \times \frac{d^2}{h^2}$ disappears. We also have $s_4 + s_5 = \pm 1.5$ in all cases. The stress in 1–2 may be taken at -7.5. Then assuming $c^2 = 1.6\,h^2$ approximately, we have, for the deformation of the vertical 3–4,

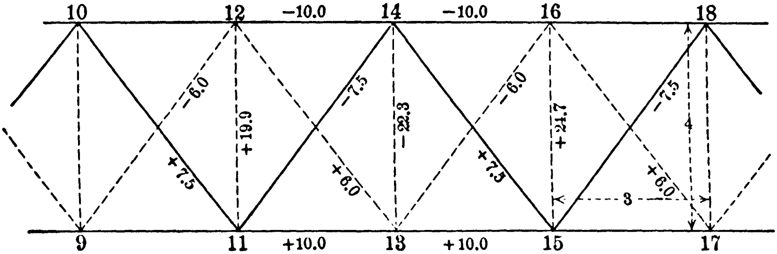


FIG. 36.

$s = +7.5 + 2.4 = +9.9$; then for the vertical 5–6, $s = -9.9 - 2.4 = -12.3$, etc. Finally we have for

11–12, $s = +7.5 + 5 \times 2.4 = +19.9$

13–14, $s = -7.5 - 6 \times 2.4 = -22.3$

15–16, $s = +7.5 + 7 \times 2.4 = +24.7$

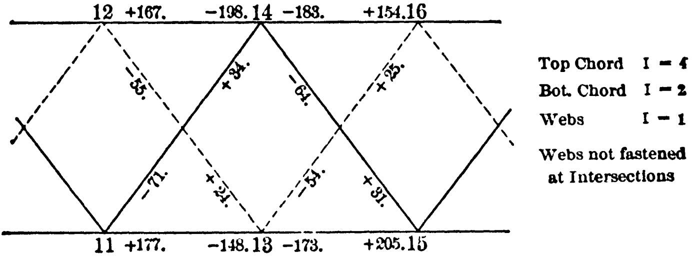


FIG. 37.

The unit strains in all members will therefore be as shown in Fig. 36. It will be assumed, as approximately correct, that the deformations and stresses at joint 12 are the same as at joint 16, those at 11 the same as at 15, etc. Hence in the analysis it will be necessary to write out equations for joints 11, 12, 13, and 14.

Two general cases have been considered. first, with the diagonals not connected at their intersections: and second, with those members rigidly riveted together. In the latter case it is necessary to write out two additional equations for the two interior joints. The results of these calculations are given in Figs. 37, 38, and 39. The trusses of Figs. 37 and 38 differ only in the relative size of the web members, in both cases the webs are not connected at their intersections.

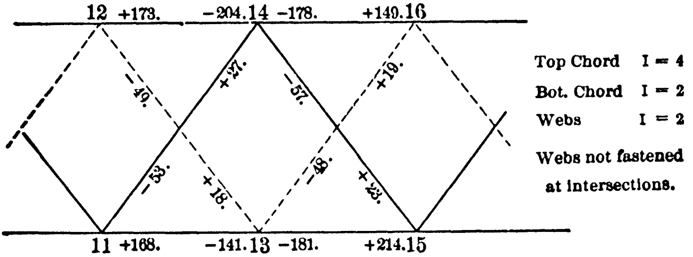


FIG. 38.

Fig. 39 represents a truss in which the diagonals are rigidly connected.

In these diagrams the coefficients of secondary stress are given for each end of all the members of the two panels considered. The most noteworthy feature is the very large stresses in the chord members, these being nearly four times the values given in Table R. The signs

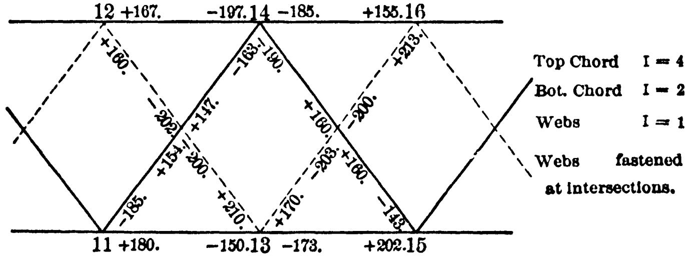


FIG. 39.

of the stresses indicate the downward deflection of joints 11, 14, and 15, relative to the joints adjacent. Assuming a value of c/l of $\frac{1}{40}$, the secondary fibre stresses amount to about 5 units, or 50 per cent of the primary stress. Considering the probability of much larger variations in load than here assumed, it is not unlikely that in double systems the secondary stresses often reach 100 per cent of the primary.

The stresses in the diagonals in Figs. 37 and 38 are not greater than in the single Warren truss, but in Fig. 39 they are relatively high, due to their being fastened at their intersections.

The double Pratt or Whipple truss is evidently subject to about the same effects from concentrated loads as the double Warren.

316. *The Double Triangular Truss with Verticals.*—In the problem of Art. 252, Chapter VI, the equalizing effect of verticals for double triangular trusses was fully demonstrated. Their value in preventing undue secondary stresses is even more important. In the truss there analyzed it was shown that the total load on one vertical, with the truss fully loaded, was only about one-fourth of a panel load. With the sections there used the resulting unit stress would be less than half the usual working stress; and, generally, the maximum stresses in these verticals are not likely to exceed this limit. It will therefore be assumed here that, as an extreme variation, alternate verticals may be stressed to one-half the working stress of the chords, and that the other verticals will be unstressed. One set of diagonals will be assumed as stressed to 7.5 units; the other set will have stresses necessarily dependent upon these, and the unit stresses assumed in the chords and verticals. The correct stresses may be calculated by means of eq. (35), giving the relation between the unit stresses of a rectangular figure. Fig. 40 shows (in parentheses) the unit stresses which have been assumed on this basis. They are, therefore, consistent among themselves.

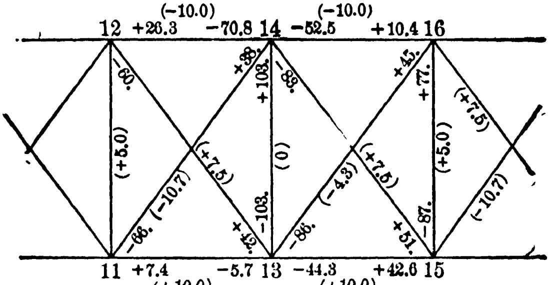


FIG. 40.

Following the usual methods the secondary stresses have been calculated for the two panels of Fig. 40 and the results are shown in the diagram.

The stresses are seen to be very much less than in the double triangular without verticals and to compare very favorably with the single triangular truss. In fact this truss appears to be one of the most satisfactory forms as regards secondary stresses. If the diagonals are rigidly connected at their intersections, the stresses therein will be increased, but those in the chords will be but little changed. The increase in the diagonal stress will, however, be small compared to the effect of fastening in the double truss without verticals.

317. *The Baltimore Truss.*—It has been seen that, other things being the same, the secondary stresses increase with decrease of panel length. Long panels therefore tend generally to reduce such stresses. In this respect the long panels of the Baltimore or Pettit trusses are advantageous. On the other hand the distortion of the secondary members, the hangers, and the sub-ties or sub-struts, tends greatly to increase the secondary stresses in the main members of such trusses unless the distortion of the secondary members is taken account of to some extent in the construction.

Consider the panel of a Baltimore truss as shown in Fig. 41. The members $O\,O'$ and $O\,N'$ are merely supporting struts for the top chord and vertical post. Suppose the structure be fully loaded and that all the members of the panel are stressed well up to their working value. Assume this working stress to be approximately the same in all the members and let s represent this stress.

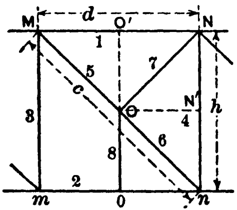


FIG. 41.

The general effect of the strains in the hanger $O\,o$ and the sub-tie $O\,N$ will be to cause the joints o and O' to deflect considerably more than the ends of the respective chord members, and thus to cause considerable bending stress therein. The point N' will also be deflected laterally, bending the post $N\,n$. The exact effect of these strains can be determined only by considering the entire structure, as heretofore, but in this case it will serve our purpose better to calculate the deflections of the joints concerned, and to estimate the bending stresses from this basis.

The deflection of joint o with respect to m and n is calculated by the usual formula, the stress intensity in each member being s. The calculations are as follows:—

Member.	u	l	$\frac{S}{A}$	$\frac{S u l}{A}$
1	$-\frac{d}{2h}$	d	$-s$	$\frac{s d^2}{2h}$
2	0	d	$+s$	0
3	$-\frac{1}{2}$	h	$-s$	$\frac{s h}{2}$
4	$-\frac{1}{2}$	h	$-s$	$\frac{s h}{2}$
5	$+\frac{c}{2h}$	$c/2$	$+s$	$\frac{s c^2}{4h}$
6	0	$c/2$	$+s$	0
7	$+\frac{c}{2h}$	$c/2$	$+s$	$\frac{s c^2}{4h}$
8	$+1$	$h/2$	$+s$	$\frac{s h}{2}$

The deflection is

$$\Delta_0 = \frac{s}{E}\left(\frac{d^2}{2\,h} + \frac{3\,h}{2} + \frac{c^2}{2\,h}\right). \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

Assume that $d = h$, in which case

$$\Delta_0 = \frac{3\,s\,h}{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (37)$$

that is, the deflection of point o relative to m and n is three times the elongation of a tension member of length h under stress s.

The effect of this deflection upon the secondary stresses in $m\,n$ can be estimated on the assumption that $m\,n$ is a beam, fixed at the ends and deflected a distance Δ_0 at the centre. From the table of Art. 7, the deflection in terms of fibre stress is $\Delta = \frac{f\,l^2}{24\,E\,c}$, where c is half the width of a symmetrical section. We therefore have

$$f = \frac{24\,E\,c}{l^2} \times \Delta_0 = 72\,s\,\frac{c}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

that is, the secondary stress is equal to the primary stress s, multiplied by $72\,\frac{c}{l}$. In eq. (38) the value of l is equal to the panel length d.

If eye-bars are used the ratio c/l may be about $\frac{1}{150}$, giving a secondary stress of about 50 per cent. If a riveted chord be used of a ratio of c/l of $\frac{1}{60}$ (corresponding to a 50-foot panel and a depth of chord of 20 in.), the secondary stress reaches the very large value of $\frac{72}{60} = 120$ per cent of the primary stress.

Similar calculations for the top chord and vertical post show that the deflections of the centre points relative to the extremities are $\frac{3}{2}\frac{s\,h}{E}$ in each case or one-half as much as for member $m\,n$. These give values for the secondary stresses, under the same assumption as made above, of $36\ s\frac{c}{l}$. For the usual proportions of top chord this is a large value.

These calculations, while not giving actual values of secondary stresses, show the general extent of the deflections produced and the possible magnitude of such stresses. They are obviously of much importance and merit careful attention, especially with respect to the supporting struts for top chord and vertical post. If these supporting struts are made a little longer than the theoretical lengths, excessive stresses under full load may be largely avoided. Large sections and low unit stresses for hangers also aid in reducing deflections and bending stresses in the chord. If a sub-strut, $m\,O$, be used instead of the sub-tie, $O\,N$, it will be found that, assuming the same unit stresses as before, the deflection of o with respect to $m\,n$ is equal to $\frac{s\,h}{E}$, or one-third the value given by eq. (37). Taking into account the fact that the unit stress in the sub-strut would be smaller than in the sub-tie, the relative deflection would be still smaller. It is also found that the deflection of both the top chord and vertical post are also one-third the values when the sub-tie is used. So far as secondary stresses are concerned, therefore, the sub-strut is much the more favorable arrangement.

318. *The Double Triangular Truss with Sub-Panels.*—An analysis of the truss of Fig. 42 will show the same results as obtained by the use of a sub-strut in Fig. 41. Assuming compression in $a\,b'$ and tension in $b'\,c$, the joint b will deflect a distance $\frac{h\,s}{E}$ with respect to $a\,c$, assuming, as in Art. 317, that the height, h, is equal to a–c, and

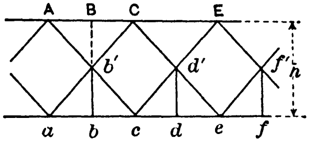
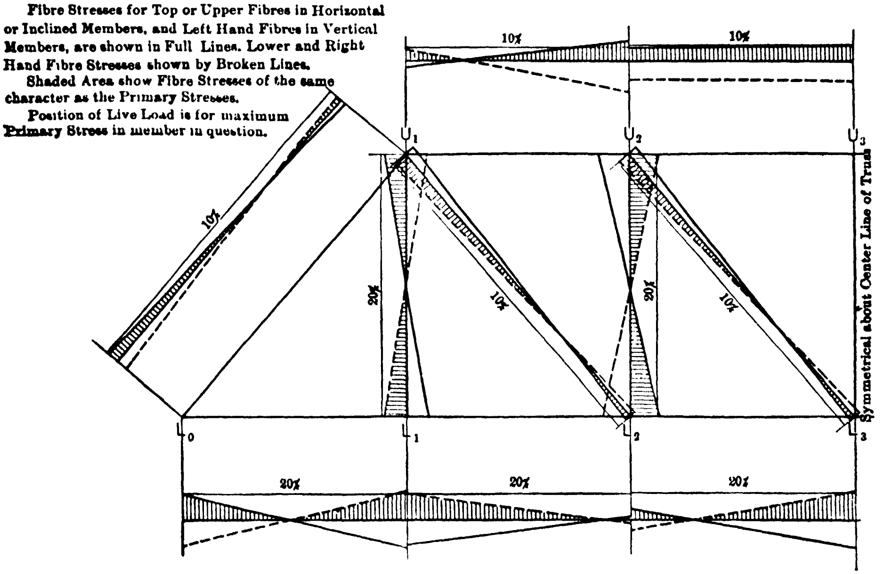


FIG. 42.

that the unit stress, s, is the same for all members. Furthermore, if $B\,b'$ is a supporting strut for the top chord, the deflection of B, with respect to $A\,C$, will be $\frac{h\,s}{2\,E}$, which is the same as in Fig. 41, with the sub-strut $m\,O$. This truss form is, therefore, satisfactory so far as the action of the secondary members is concerned.

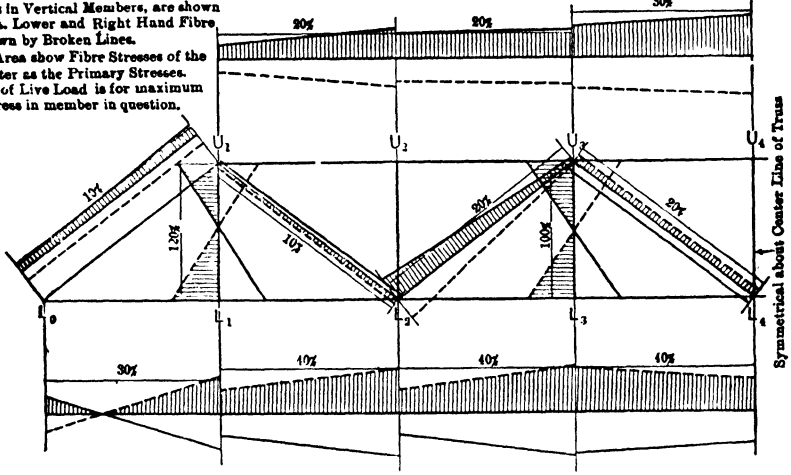

Fibre Stresses for Top or Upper Fibres in Horizontal or Inclined Members, and Left Hand Fibres in Vertical Members, are shown in Full Lines. Lower and Right Hand Fibre Stresses shown by Broken Lines.

Shaded Area show Fibre Stresses of the same character as the Primary Stresses.

Position of Live Load is for maximum Primary Stress in member in question.

FIG. 43. Secondary Stresses in a Riveted Pratt Truss.
6 Panels at 26′ 8″ = 160′ 0″. Height = 31′ 0″.

319. Calculated Stresses in Some Typical Trusses.—*Single Intersection Pratt and Warren Trusses.*—Fig. 43 shows diagrammatically the amount of the secondary stresses in an ordinary Pratt truss of 150 ft. span length and 31 ft. depth. The results are shown as percentages of the primary stresses. Fig. 44 shows the results for a pony riveted truss in which the relative width of the various members is considerably greater than in Fig. 43. The maximum values in the first case amount to about 20 per cent for tension members and about 15 per cent for top chord; and in the latter case about 40 per cent for tension members and 30 per cent for top chord. The difference is due to the greater ratio of width to length of members in the second design. Calculations of other Pratt and Warren truss designs indicate a somewhat greater value reaching 30 to 40 per cent and about $400 \text{ to } 450 \times \left(\frac{\text{width}}{\text{length}}\right)$.



Fibre Stresses for Top or Upper Fibres in Horizontal or Inclined Members, and Left Hand Fibres in Vertical Members, are shown in Full Lines. Lower and Right Hand Fibre Stresses shown by Broken Lines.

Shaded Area show Fibre Stresses of the same character as the Primary Stresses.

Position of Live Load is for maximum Primary Stress in member in question.

FIG. 44. Secondary Stresses in a Riveted Warren Truss.
8 Panels at 13′ 1½″ = 105′ 0″ Height = 10′ 0′

320. *Trusses with Subdivided Panels.*—Fig. 45 gives the results of calculations of a subdivided Pratt or Baltimore truss. The stresses in the lower chord and end post run to 50 and 60 per cent. The high values are due partly to the very short panel length of the loaded chord, making the ratio of width to length large, and partly to the direct effect of the distortion of the suspenders. The top chord

shows only the usual amount of stress. A subdivided double triangulation truss showed results as high as 100 per cent where the panel length was only 8 ft. 4 ins. and depth of chord 24 ins. The stress due to deflection alone would be only from 25 to 30 per cent. The reason for such high values in a truss with sub-panels is clearly brought out in the sketch of Fig. 46, which shows to an exaggerated scale the deflections or movements of the various joints of the truss of Fig. 45.

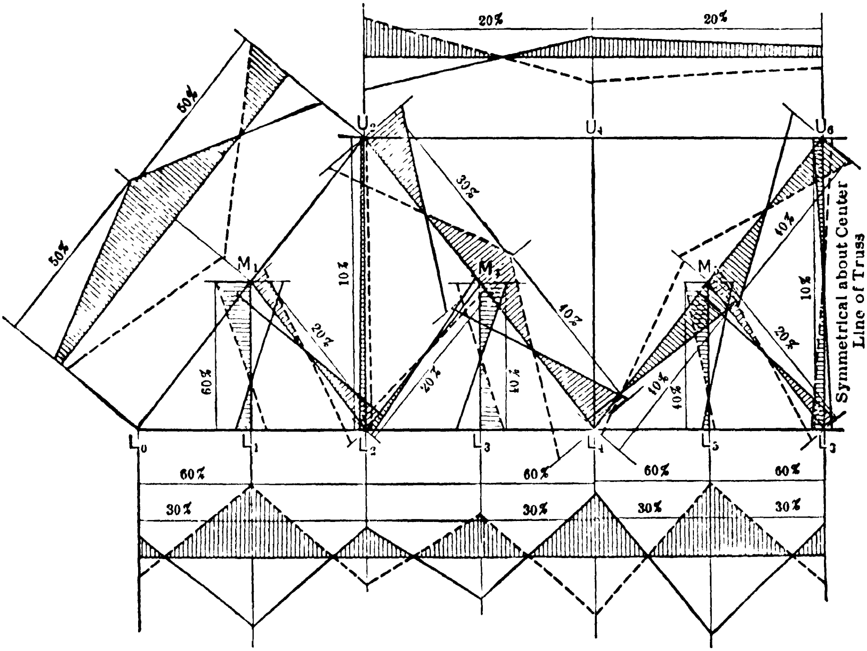


FIG. 45.—Secondary Stresses in a Subdivided Pratt Truss.

This brings out the great effect of the distortion of the hangers and sub-struts on the chord members.

321. *Truss with "K" Type of Bracing.*—Figs. 47 and 48 show the secondary stresses and deflection diagram of a "K" type of truss, for uniform load. This example is noteworthy in the relatively small stresses. This truss is adapted to long spans as it gives a short-panel length without undue inclination of the web members. In this respect it has the advantages of the double intersection truss without its

disadvantages. As compared to the subdivided truss, Figs. 45 and 46, it is much freer from secondary stresses.

322. Secondary Stresses in Longitudinal Members Due to the Action of Lateral Systems.—In Art. 232, the primary stresses in three forms of laterals, such as shown in Figs. 49, 50, and 51, were investigated. In Figs. 49 and 50 the laterals are not affected by the direct stresses in the chords, but in Fig. 51 the lateral diagonals receive considerable direct stress with corresponding stresses in the lateral struts. The secondary stresses in these forms due to the direct stress in the chords will now be determined. The trusses here illustrated represent any truss, in which the longitudinal members are all subjected to stresses of one kind, either compression or tension, such as the upper or lower lateral system of a bridge, or the vertical bent of a trestle tower, or the upper and lower chords of a braced arch.

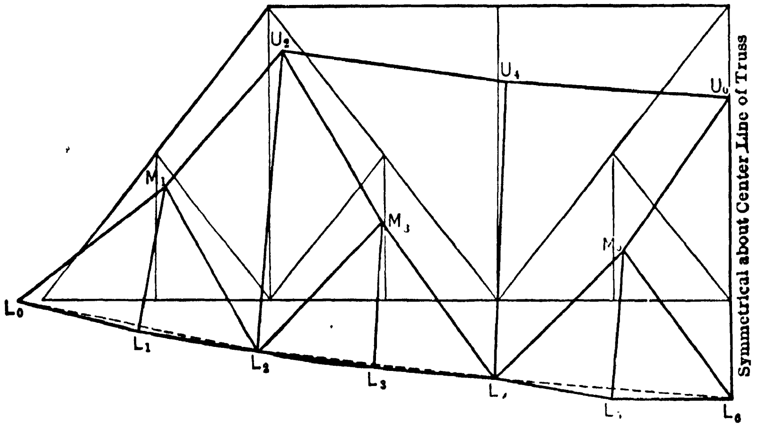


FIG. 46.—Deflection Diagram of a Subdivided Pratt Truss.

The principal feature to be considered is the secondary stresses in the chord members. The laterals being of small section will have small secondary stresses, and their rigidity will have very little influence upon the secondary stresses in the chords. For simplicity, therefore, the laterals will be assumed as pin-ended and omitted in the equations for τ. The moment of inertia of the chord members will be taken as constant and therefore need not appear in the equation. The ratio of depth to panel length will be taken as 3 : 4.

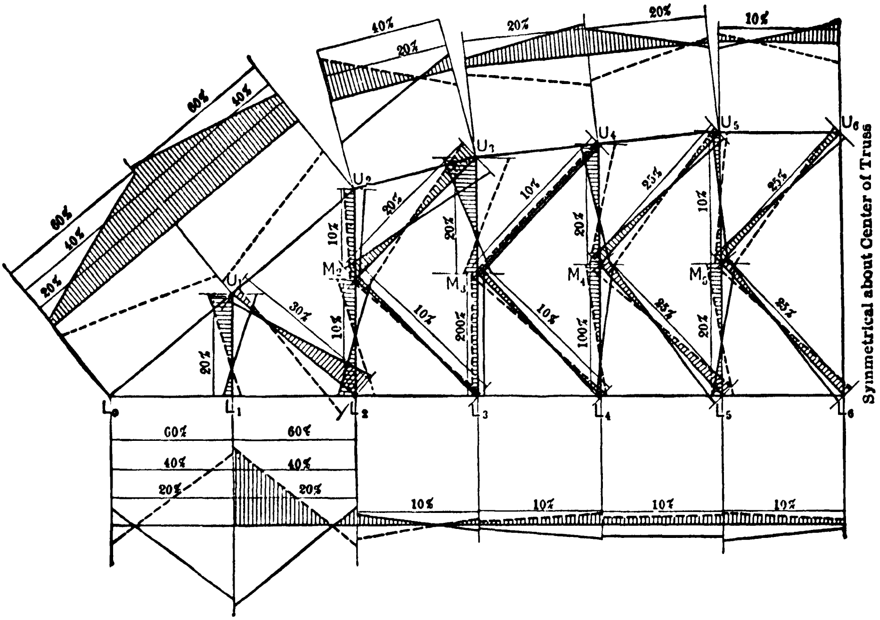


FIG. 47.—Secondary Stresses in a K-Truss.

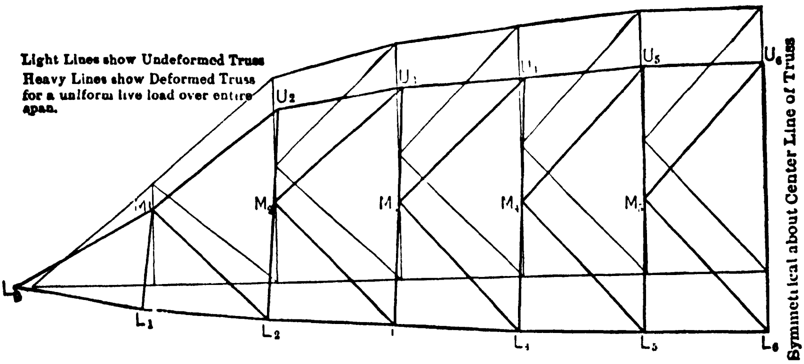

Light Lines show Undeformed Truss
Heavy Lines show Deformed Truss for a uniform live load over entire span.

FIG. 48.—Deflection Diagram of a K-Truss.

323. *The Single Warren System, Fig.* 49.—The direct stress in the laterals is zero, and that in the chords is taken at − 10 per unit area as given in the right half of Fig. 49. This condition causes joints 3 and 4 to deflect downward with respect to joints 2 and 6 or 1 and 5; and, likewise, joints 7 and 8, thus bending the chords into a sinuous line. Carrying out the calculations as usual, we find the secondary stresses to be as given in Fig. 46 by the figures in the left half of the diagram. The stresses are very considerable. For a value of c/l of 1/50 they amount to 14 per cent of the primary stress at joints 5 and 6.

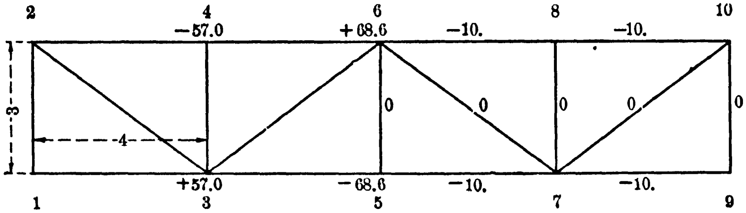


FIG. 49.

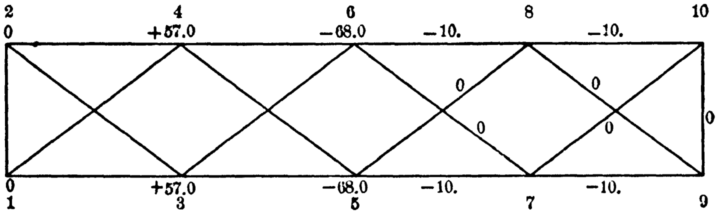


FIG. 50.

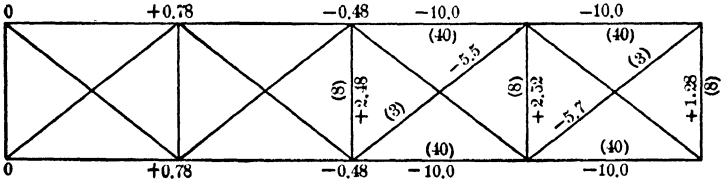


FIG. 51.

The form here considered is commonly used for plate girders. In that case, considering flanges alone, the ratio c/l is large, generally at least $\frac{1}{24}$, under the usual requirements of specifications. At a value of 1/20 the secondary stresses amount to 34 per cent of the primary stresses.

324. *The Double Warren System, Fig.* 50.—Here again the direct stresses in the laterals are zero. By eq. (35), Art. 335, the unit distortion along the line 3–4 is $2\,s\,\frac{d^2}{h^2} = 3.55\,s = 35.5$. That along the line 5–6 will be zero. This gives the same conditions as to bending which prevail in Fig. 49; but the direction of bending is different. Joints 3 and 4 bend outward while joints 5 and 6 stand fast. Each alternate joint will thus deflect outwardly, the other joints standing fast.

The stresses are the same numerically as in Fig. 49, and are given in Fig. 50.

325. *Pratt Truss with Double Diagonals, Fig.* 51.—In this case there will exist compressive stresses in the diagonals and tensile stresses in the verticals. Assuming the sectional areas given in parentheses in Fig. 51, and unit stresses of − 10 in the chords, the unit stresses in diagonals and struts will be as given in the figure. Then for these unit stresses the secondary stresses have been obtained as before, with the results given in the left half of the diagram of Fig. 51. These are seen to be very small, as compared to the stresses in the other forms considered, and show the great merit of this form of bracing. The struts here assumed are of fairly large section, corresponding to four small angles, but a much smaller size of strut than here assumed would be very advantageous as compared to the forms of Figs. 49 and 50, and should always be used if possible.

326. **Effect of Lacing in Compression Members.**—The usual single and double lacing in columns is arranged in essentially the manner shown in Figs. 49 and 50. At the ends of the members large tie plates serve to prevent any lateral distortion. The effect of the diagonal lacing is then similar to that shown in Arts. 323 and 324; the two elements of the column will be bent laterally in a sinuous curve, resulting in secondary bending stresses of considerable amount. Assuming the direct stress in the lacing to be zero and its connections to be rigid, the stresses will be as shown in Figs. 49 and 50. The ratio of c/l in the case of a column will be comparatively large, generally as much as 1/5 or 1/6. With a value of 1/6 the secondary stresses as here calculated would amount to about 100 per cent of the primary stresses.

It is to be noted, however, that in the form of construction here considered, the assumptions above made are hardly justified. The small ratio of c/l makes the members relatively rigid against lateral bending so that the lacing bars will have an appreciable amount of direct stress due to this resistance. Again, the usual riveted connection of the lacing bars is not absolutely rigid as here assumed, and its distortion will materially reduce the bending in the main members. The actual bending effect will tend to be greatest in large columns, and for such members the advantages of the system of bracing shown in Fig. 45 are very pronounced.

327. Effect of Secondary upon Primary Stresses.—The usual primary stresses are calculated on the assumption of frictionless joints.

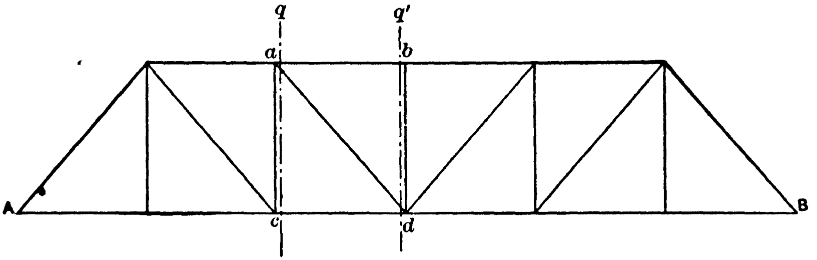


FIG. 52.

The effect of rigid joints is to develop certain bending moments and stresses in the several members, which, as explained in the preceding articles, are called secondary stresses. Obviously, a certain amount of work has been performed by the external loads in producing these secondary stresses, one result of which is to reduce somewhat the primary stresses below the values which would obtain with frictionless joints. Stated in another way, the secondary moments developed render their due proportion of resistance in supporting the external loads.

Assuming the secondary moments to be known, the true primary stresses can readily be determined as follows:—

Suppose the primary stress in $a\,b$, Fig. 51, to be desired. Pass the section q close to joints a and c, and in Fig. 52 represent all external forces and stresses at the cut sections. At each section there will be the primary stress, S, and the secondary moment, M, and shear, V.

The moments and shears are shown acting in a positive direction, according to the notation previously adopted. The secondary moments are supposed to be known, and also the shears, if needed.

To determine stress S_3 take moments about joint a. If M_a is the moment of the external forces (R_1, P_1, and P_2), about joint a, we have

$$S_3 = \frac{M_a + M_{ab} + M_{ad} + M_{cd}}{h}. \quad . \quad . \quad . \quad . \quad (39)$$

(The moments of the secondary shears are zero, as the sections are supposed to be taken very near the joints.)

For stress in $a\,b$ pass the section q' (Fig. 52), close to joints b and d and take moments about d. Fig. 54 shows the forces acting, the

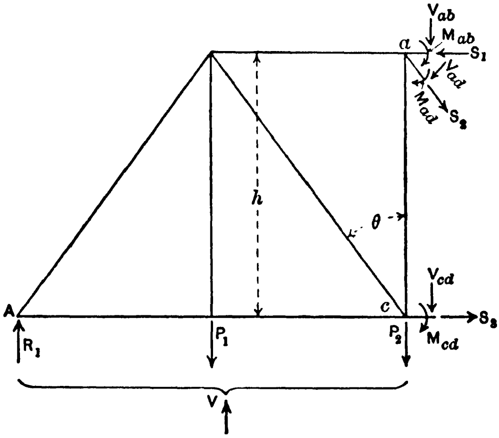


FIG. 53.

secondary moments and shears being again taken as positive. If M_d = external moment about joint d, we have, as before,

$$S_1 = \frac{M_d - M_{ba} - M_{da} - M_{dc}}{h}. \quad . \quad . \quad . \quad . \quad (40)$$

For stress in the member $a\,d$, we have from Fig. 53,

$$\text{Vert. Comp. } S_2 = V - (V_{ab} + V_{cd} + V_{ad} \cos\theta). \quad . \quad (41)$$

The secondary shears are obtained from the moments by the formula

$$V_{ab} = V_{ba} = (M_{ab} + M_{ba})/l, \text{ etc.}$$

The resulting primary stresses will in general be less than those calculated on the assumption of frictionless joints, but the difference is usually so small as to be of no practical consequence. In some cases, however, where the secondary stresses are especially high, the primary stresses are materially reduced thereby. Again, since the secondary stresses are calculated on the basis of certain assumed primary stresses, any change in the latter will affect the former in proportion; that is, if the primary stresses are reduced in any case by 5 per cent the secondary stresses will in turn be reduced by an equal proportion. This error in the secondary stresses is of no practical consequence.

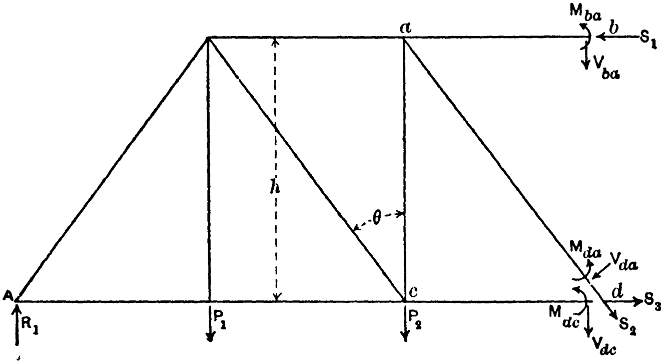


FIG. 54.

EXAMPLE.—As an illustration of the foregoing process, the true primary stresses will be calculated in members 5–6, 4–7, and 5–7 of the truss of Fig. 14, under a load of 1,000 lbs. at each lower joint. The primary stresses under the usual assumptions are given in Fig. 21. The secondary moments are readily found from Table F, using the general formula $M_{nm} = 2K(2\tau_{nm} + \tau_{mn})$. From this we derive the following values needed in the calculation:

$M_{56} = -2,830$; $M_{65} = +3,380$; $M_{47} = +1,045$; $M_{74} = +3,210$; $M_{57} = -\ 268$; $M_{75} = +\ 84$. Then we have

$$S_{47} = 3,440 + \left(\frac{-2,830 + 1,045 - 268}{372}\right) = 3,432.$$

$$S_{56} = 3,870 - \left(\frac{3,380 + 84 + 3,210}{372}\right) = 3,852.$$

We also derive from the moments:

$$V_{56} = +1.7;\ V_{47} = +13.3;\ V_{57} = +0.4.$$

Then for the stress in 5–7,

Vert. comp. 5–7 = 500 − (1.7 + 13.3 + 0.4 × sin θ) = 500 − 15 = 485.

The effect of the secondary stresses in these three cases is to reduce the primary stresses by the following percentages:

In 4–7 by 0.3 per cent.
In 5–6 by 0.5 per cent.
In 5–7 by 0.3 per cent.

It is evident from this example that in trusses of ordinary proportions the effect of the secondary stresses upon the primary stresses is very small. In the case of very shallow trusses with wide members the effect would be considerable.

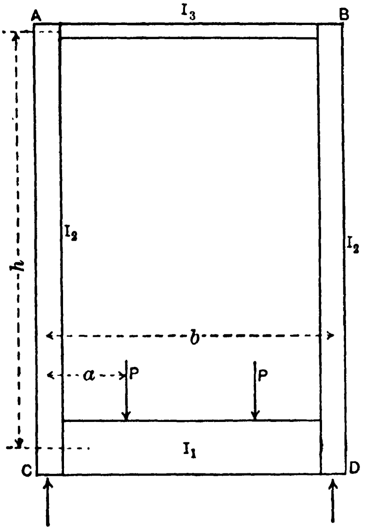


FIG. 55.

The effect here considered takes into account only such secondary stresses as have been heretofore discussed, that is, those stresses due to rigidity of joints. Secondary stresses arising from other causes may have a more important effect.

328. Secondary Stresses in Transverse Frames.—The secondary stresses caused in floor-beams and vertical posts, due to rigid connections, may be calculated by the formulas derived in Chapter VI. Where the top strut, *A B* (Fig. 55), is relatively slender, the posts, *A C* and *B D*, may be considered as hinged at the top joints, in which case the equations of Art. 247 apply. If the transverse or overhead bracing is fairly deep and rigid then the equations of Art. 244 apply. If such sway bracing is very rigid, as in the case of large bridges, the posts may be considered as fixed at the top, and equations of Art. 246 used.

For a single-track structure the moments at C and D are as follows:

For hinged ends at A and B

$$M_c = P \cdot \frac{3\,a\,(b - a)\,I_2}{2\,h\,I_1 + 3\,b\,I_2}. \quad \ldots \ldots \quad (42)$$

For a strut, $A\,B$, of moment of inertia I_3,

$$M_c = \frac{P\,a\,(b - a)\,(2\,h\,I_3 + 3\,b\,I_2)\,I_2}{I_1\,I_3\,h^2 + 2\,b\,h\,(I_1 + I_3)\,I_2 + 3\,b^2\,I_2^2} \quad \ldots \quad (43)$$

For upper ends fixed,

$$M_c = P \cdot \frac{2\,a\,(b - a)\,I_2}{h\,I_1 + 2\,b\,I_2}. \quad \ldots \ldots \quad (44)$$

in which h is the distance from the centre of $C\,D$ to the lower edge of the overhead bracing.

For double-track structures it is sufficiently exact to consider each of the loads, P, as the total load on one track, and the distance, a, as the distance to the track centre.

To investigate the relative stresses in the floorbeam and post, consider eq. (42). Let f_1 and f_2 be the bending stresses at C in the beam and post, respectively, and c_1 and c_2 be the half-widths of these members. Then in general $f_1 = \dfrac{M\,c_1}{I_1}$ and $f_2 = \dfrac{M\,c_2}{I_2}$. From eq. (42), we have

$$f_2 = P\,\frac{3\,a\,(b - a)\,c_2}{2\,h\,I_1 + 3\,b\,I_2}.$$

Considering that I_2 is small as compared to I_1, and placing $a = \frac{1}{4}\,b$, we have approximately,

$$f_2 = P\,\frac{3\,a\,(b - a)\,c_2}{2\,h\,I_1} = \frac{9}{32}\,\frac{b^2\,c_2}{h\,I_1}\,P. \quad \ldots \ldots \quad (45)$$

If f_w is the working stress in the floorbeam used in its design, we have $f_w = \dfrac{P\,a\,c_1}{I_1} = \dfrac{P\,b\,c_1}{4\,I_1}$, whence

$$\frac{f_2}{f_w} = \frac{9\,b}{8\,h} \cdot \frac{c_2}{c_1}, \quad \ldots \ldots \ldots \quad (46)$$

that is, the ratio of stress in the post to the working stress in the floorbeam is equal to the ratio of the widths of the respective members, multiplied by $\frac{9\,b}{8\,h}$. For short-span trusses the ratio $\frac{9\,b}{8\,h}$ is from ½ to ¾, hence we have approximately $\frac{f_2}{f_w} = \frac{1}{2}\frac{c_2}{c_1}$ to $\frac{3}{4}\frac{c_2}{c_1}$.

Thus if $c_1 = 48$ in. and $c_2 = 16$ in., the stress f_2 will be from 12 per cent to 20 per cent of the working stress f_w.

For fixed ends, the bending stresses in the post will be relatively greater.

The secondary stresses in posts will be made a minimum by the use of deep beams, and posts whose width in the transverse plane is no greater than necessary to secure the desired rigidity as a strut. Where the floorbeams are attached to tension verticals, a narrow section can readily be used.

329. Secondary Stresses in Floor Members Due to the Action of Stringers and Laterals.—Fig. 56 represents the ordinary arrangement of floor members, chords, and laterals, in a single-track railroad bridge.

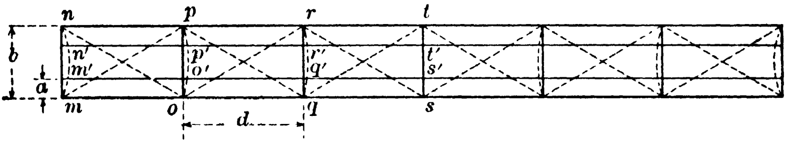


FIG. 56.

The stringers $m'\,o'$, $n'\,p'$, etc., are riveted between the floorbeams, $m\,n$, o, p, etc., and the latter are riveted to the posts at the joints m, n, o, p, etc. At these joints the beams are also more or less rigidly attached to the chord members, m–o–q, etc. The lateral diagonals are also generally fastened to the stringers, but such attachment is not usually a very rigid one.

Assuming a through bridge to be under consideration, the chords will be elongated under stress, and the lateral diagonals will also be stressed in tension in accordance with the analysis of Art. 232, Chapter VI. The total elongation of the entire chord will be equal to $\frac{n\,d\,s}{E}$ where n = number of panels, d = panel length, and s = average unit stress in the chord members. The stringers will be stressed principally in bending, and their axial length will remain nearly constant. The result of this action of chords and stringers is to cause a horizontal bending of the floorbeams in the direction indicated by the dotted lines.

Assuming the stringers to have no axial stress, the beam $q\,r$ will deflect at points q' and r' a distance $\frac{d\,s}{E}$, the beam $o\,p$, a distance $\frac{2\,d\,s}{E}$, and the end beam $m\,n$, a distance $\frac{3\,d\,s}{E}$. From Table No. 1, Art. 7, the deflection of a beam loaded in this manner and simply supported at the ends, is equal to

$$\Delta = \frac{P\,a^2}{6\,E\,I}\,(3\,l - 4\,a) = \frac{f\,a}{6\,E\,c}\,(3\,b - 4\,a), \quad . \quad . \quad (47)$$

in which f = fibre stress due to the deflection, and $2\,c$ = width of beam. From this, we have

$$f = \frac{6\,c\,E\,\Delta}{a\,(3\,b - 4\,a)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Then for beam $q\,r$, the first from the centre, we have $\Delta = \frac{d\,s}{E}$ and hence

$$f = \frac{6\,c\,d\,s}{a\,(3\,b - 4\,a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

as the secondary fibre stress in the beam under the given assumption.

Assuming ordinary values of d = 300 in., s = 10,000 lbs. per sq. in., b = 192 in., and a = 54 in., we get, for the first beam, $f = 920\,c$. For c = 6 in., f = 5,520 lbs. per sq. in. The second beam will be bent twice as much, and will therefore have a fibre stress of 11,000 lbs. per sq. in., etc.

In these calculations it has been assumed that the stringers are not elongated at all and that the connection to the beams permits of no deformation whatever. Practically the riveted joint is not entirely rigid, as the connection angles, the rivets, and the web of the beam all contribute some deformation. The stringers, also, receive some longitudinal stress, although the amount per square inch of section is small. On the whole, the deflections and stresses in the beams are not as great as deduced from the above calculations, but they are often large and of much importance. As the calculations show, they increase with the width of the beam and with the number of panels, and are greater as the distance *a* becomes smaller. In long-span structures, excessive stresses of this kind should be avoided by the use of expansion joints at intervals of a few panels. The use of relatively narrow flanges for the beams is also advisable.

In the above calculations it has been assumed that the beams are hinged, or simply supported at the ends. Where large joint plates are used between laterals, chords, and beams, the connection at the lower flange is quite rigid. The bending of the beam will therefore introduce some secondary stresses in the chords, and at the same time the stresses in the beams will be increased. If the beam and chord connection be considered as perfectly rigid, then the stresses may be found by the general method for secondary stresses, writing out an equation for each of the joints, *m*, *n*, *o*, and *p*. The changes of angles in the quadrilateral, $q\,q'\,s'\,s$, are readily determined from the known deflection of *q* relative to q'. This method of calculation is hardly warranted, however, on account of the imperfect rigidity of the various joints. Furthermore, the stresses are still further complicated by the attachment of the laterals to the stringers and beams. Since the chord deformations cannot be much reduced, and since the stringers are of relatively large section and cannot receive much direct stress, it follows that the more rigid the joints and connections of the system are made, the greater will be the secondary stresses due to the bending action which must take place. These conditions properly lead to the design of a floor system, rigid in itself with respect to its main duty, but independent so far as practicable of the action of the chord system of the main trusses. The laterals belong essentially to the main truss system and the stresses developed from chord deformations are readily taken care of.

That serious lateral bending of floorbeams takes place has been shown experimentally in many cases, some examples of which are reported in Bulletin No. 125 of the American Railway Engineering and Maintenance of Way Association, 1910, p. 44. In a 228-ft. span the lateral bending of the end beam added 100 per cent to the stresses due to vertical load in one flange and reduced to zero the stress in the other flange.

330. Deflection of Members Due to Secondary Stresses.—In the analysis of secondary stresses the effect of the deflection of the members themselves upon the stresses has been neglected. In Art. 331 the exact theory is briefly set forth, whereby this effect is taken into account from the beginning and the secondary stresses determined accordingly. That method is very laborious and not generally practicable. It is, however, often desirable to determine approximately the deflection of a member subjected to secondary stresses, especially in the case of compression members, in order that the influence of the secondary stresses upon the buckling action may be estimated.

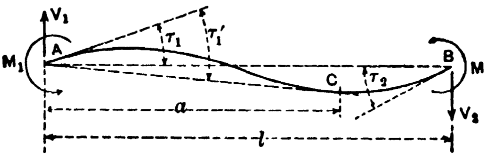


FIG. 57.

Having determined the secondary stresses by the methods already described, the deflection of a member at any point can readily be found as follows:

Consider the member *A B*, Fig. 57, subjected to end moments and shears as shown. The moment at any point *C*, distant *a* from *A*, is

$$M_c = M_2 - (M_1 + M_2)\frac{l-a}{l} = M_2\frac{a}{l} - M_1\left(\frac{l-a}{l}\right). \qquad (50)$$

Then, as in eq. (8), Art. 304, we have

$$\tau_1' = \frac{a}{6\,EI}(2\,M_1 - M_c), \quad . \quad . \quad . \quad . \quad . \quad (51)$$

also

$$\tau_1 = \frac{l}{6\,EI}(2\,M_1 - M_2). \quad . \quad . \quad . \quad . \quad . \quad (52)$$

The deflection of point *C* is

$$\Delta_c = (\tau_1' - \tau_1)\,a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Substituting from (50), (51), and (52), and reducing, we derive the value

$$\Delta_c = \frac{[M_2(l+a) - M_1(2\,l-a)](l-a)\,a}{6\,EIl}. \quad . \quad . \quad (54)$$

For a maximum value of Δ_c we get, by differentiating eq. (54), the value of $\frac{a}{l}$ is

$$\frac{a}{l} = \frac{M_1}{M_1 + M_2} \pm \sqrt{\left(\frac{M_1}{M_1 + M_2}\right)^2 - \frac{2\,M_1 - M_2}{3\,(M_1 + M_2)}}. \quad (55)$$

EXAMPLE.—(1) Let it be required to calculate the deflection of the vertical post 4–5 of the Pratt truss of Fig. 14, under a load of 75,000 lbs. at each of the joints 7, 9, and 11. From Table F we find the moments for this loading to be $M_{45} = -114{,}000$ in.-lbs., and $M_{54} = -121{,}000$ in.-lbs. By eq. (55) we find for the point of maximum deflection, $a/l = .485 \pm .288 = .773$ and .197. Substituting in (54), the deflection at these two points is found to be .007 in., and .010 in., respectively.

(2) Suppose, for example, the end moments in Ex. (1) to be − 120,000 and + 120,000, giving single instead of double curvature. The maximum deflection would be at the centre and, from (54), equal to 0.068 in. The direct stress in the member under the assumed load is 75,000 lbs., which stress, multiplied by the deflection of .068 in., gives a moment of 5,000 in.-lbs., which represents the additional effect due to deflection in the case assumed. The total moment at the centre would then be approximately 120,000 + 5,000 = 125,000 in.-lbs. The true deflection and moment would be somewhat greater than here calculated, as the effect of the moment of 5,000 in.-lbs. is to increase the deflection still more.

The examples here given indicate that in the ordinary truss the lateral deflection of compression members, due to secondary stresses, is not often of importance in increasing such stresses.

Effect of Transverse Loads.—In the above analysis no account has been taken of transverse loads, such as the weight of horizontal or inclined members. These loads can readily be taken account of in this approximate method of calculation, simply by adding to the deflections caused by the end moments, the deflection caused by the transverse loads, considering the beam as simply supported at the ends. The end moments are then obtained by duly considering the weight of the members in the secondary-stress calculation.

331. Exact Method of Calculating Secondary Stresses.—In the method of analysis explained in the foregoing articles, the bending moments due to the direct or primary stress, acting with an arm equal to the deflection of the member, have been neglected. Manderla's method of solution, published in 1880, was quite remarkable in the fact that while being the first adequate treatment of secondary stresses in general, it was at the same time an exact method, taking due account of the deflections. This exact method is considerably more laborious than the approximate method heretofore given, and as the results generally differ but little, it will rarely be expedient to resort to the exact process. However, as the method is a valuable one in dealing with certain special problems, it will be given here.

The beam $A B$, Fig. 58, is acted upon by the end moments M_1 and M_2, the shears V_1 and V_2, and the direct stress S, which may be either tension or compression. It will be necessary to consider the two cases separately.

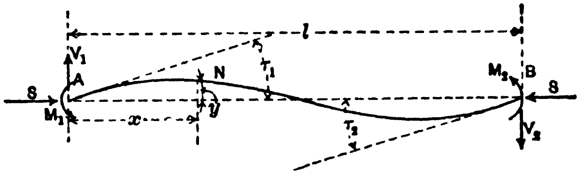


FIG. 58.

a. Compression Members.—The moment at any point N, distant x from A, is

$$M = M_1 + S y - V_1 x \quad . \quad . \quad . \quad . \quad . \quad (56)$$

in which M has the same sign as M_1.

The differential equation of the elastic line is

$$\frac{d^2 y}{d x^2} = -\frac{M}{EI} = -\frac{M_1}{EI} - \frac{S y}{EI} + \frac{V_1 x}{EI} \quad . \quad . \quad . \quad (57)$$

Placing $\frac{S}{EI} = q^2$ and integrating, we derive the general equation of the curve $A B$

$$y = C_1 \sin q x + C_2 \cos q x - \frac{M_1}{S} + \frac{V_1 x}{S} \quad . \quad . \quad (58)$$

in which C_1 and C_2 are integration constants to be determined by the conditions of the problem. In this case they are found from the condition that for $x = 0$, $y = 0$, and for $x = l$, $y = 0$. From the first we have

$$0 = C_2 - \frac{M_1}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

whence

$$C_2 = \frac{M_1}{S}. \tag{59}$$

and from the second

$$0 = C_1 \sin q\,l + C_2 \cos q\,l - \frac{M_1}{S} + \frac{V_1 l}{S}. \tag{b}$$

Substituting the value of C_2 we get

$$C_1 = \frac{M_1}{S} \cdot \frac{1 - \cos q\,l}{\sin q\,l} - \frac{V_1 l}{S \sin q\,l} = \frac{M_1}{S} \tan \frac{q\,l}{2} - \frac{V_1 l}{S \sin q\,l}. \tag{60}$$

The inclination of the elastic line is found by differentiating (58), giving

$$\frac{d\,y}{d\,x} = q\,C_1 \cos q\,x - q\,C_2 \sin q\,x + \frac{V_1}{S}. \tag{61}$$

For $x = 0$ the value of $\frac{d\,y}{d\,x}$ is τ_1, and for $x = l$, $\frac{d\,y}{d\,x} = \tau_2$, whence

$$\tau_1 = q\,C_1 + \frac{V_1}{S} \tag{c}$$

$$\tau_2 = q\,C_1 \cos q\,l + q\,C_2 \sin q\,l + \frac{V_1}{S}. \tag{d}$$

Substituting values of C_1 and C_2 we derive the formulas

$$\tau_1 = \frac{q\,M_1}{S} \tan \frac{q\,l}{2} - \frac{V_1}{S}\left(\frac{q\,l}{\sin q\,l} - 1\right) \tag{62}$$

$$\tau_2 = -\frac{q\,M_1}{S} \tan \frac{q\,l}{2} - \frac{V_1}{S}\left(\frac{q\,l \cos q\,l}{\sin q\,l} - 1\right). \tag{63}$$

Eliminating $\frac{V_1}{S}$ from these two equations we derive an expression for M_1 in terms of τ_1 and τ_2 and known quantities. It is

$$M_1 = \frac{S}{q}(A\,\tau_1 + B\,\tau_2) \tag{64}$$

in which

$$A = \tfrac{1}{2}\left(\frac{q\,l}{2 - q\,l \cot \frac{q\,l}{2}} + \cot \frac{q\,l}{2}\right) \tag{65}$$

$$B = \tfrac{1}{2}\left(\frac{q\,l}{2 - q\,l \cot \frac{q\,l}{2}} - \cot \frac{q\,l}{2}\right). \tag{66}$$

Eq. (64) may be written in the form

$$M_1 = \frac{2S}{q^2 l}\left(\frac{2Aql}{4}\tau_1 + \frac{Bql}{2}\tau_2\right) = \frac{2EI}{l}(2a_c\tau_1 + b_c\tau_2), \qquad (67)$$

in which a_c and b_c are coefficients respectively equal to $\frac{Aql}{4}$ and $\frac{Bql}{2}$. They are functions of ql only, which quantities involve l, I, E, and S for the member in question.

We may also write for M_2 the similar expression

$$M_2 = \frac{2EI}{l}(2a_c\tau_2 + b_c\tau_1). \quad . \quad . \quad . \quad . \quad (68)$$

The values of the coefficients a_c and b_c are conveniently expressed in the form of series. The value of cot θ in general is

$$\cot\theta = \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{45} - \frac{2\theta^5}{945} - \text{etc.} \quad . \quad . \quad . \quad (69)$$

From this are derived the values of A and B as follows:

$$A = \frac{4}{ql} - \frac{2ql}{15} - \frac{11(ql)^3}{6{,}300} - \text{etc.} \quad . \quad . \quad . \quad (70)$$

$$B = \frac{2}{ql} + \frac{ql}{30} + \frac{13(ql)^3}{12{,}600} + \text{etc.} \quad . \quad . \quad . \quad (71)$$

and, finally,

$$a_c = \frac{Aql}{4} = 1 - \frac{(ql)^2}{30} - \frac{11(ql)^4}{25{,}000} - \text{etc.} \quad . \quad . \quad (72)$$

$$b_c = \frac{Bql}{2} = 1 + \frac{(ql)^2}{60} + \frac{13(ql)^4}{25{,}000} + \text{etc.} \quad . \quad . \quad (73)$$

The value of ql is $l\sqrt{\frac{S}{EI}}$. If A = area of cross-section, s = unit stress = S/A, and r = radius of gyration, we have the convenient expression for compression members,

$$ql = \frac{l}{r}\sqrt{\frac{s}{E}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

b. Tension Members.—In case the stress S, Fig. 58, is tensile, the moment at N is

$$M = M_1 - Sy - V_1 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

and the differential equation of the elastic line is

$$\frac{d^2 y}{d x^2} = -\frac{M_1}{EI} + \frac{S y}{EI} + \frac{V_1 x}{EI}. \quad . \quad . \quad . \quad . \quad (76)$$

This integrates into the expression

$$y = C_1 e^{qx} + C_2 e^{-qx} + \frac{M_1}{S} - \frac{V_1 x}{S} \quad . \quad . \quad . \quad (77)$$

where $q = \sqrt{\frac{S}{EI}}$ as before, and C_1 and C_2 are integration constants.

For $x = 0$, $y = 0$, and for $x = l$, $y = 0$, whence

$$0 = C_1 + C_2 + \frac{M_1}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$0 = C_1 e^{ql} + C_2 e^{-ql} + \frac{M_1}{S} - \frac{V_1 l}{S}. \quad . \quad . \quad . \quad (b)$$

From these we get the values

$$C_1 = -\frac{M_1}{S}\left(\frac{1 - e^{-ql}}{e^{ql} - e^{-ql}}\right) + \frac{V_1 l}{S}\left(\frac{1}{e^{ql} - e^{-ql}}\right) \quad . \quad . \quad (78)$$

$$C_2 = \frac{M_1}{S}\left(\frac{1 - e^{ql}}{e^{ql} - e^{-ql}}\right) - \frac{V_1 l}{S}\left(\frac{1}{e^{ql} - e^{-ql}}\right). \quad . \quad . \quad (79)$$

The inclination of the elastic line is

$$\frac{dy}{dx} = q C_1 e^{qx} - q C_2 e^{-qx} - \frac{V_1}{S}. \quad . \quad . \quad . \quad . \quad (80)$$

Substituting $x = 0$ and $x = l$, we have

$$\tau_1 = q C_1 - q C_2 - \frac{V_1}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

$$\tau_2 = q C_1 e^{ql} - q C_2 e^{-ql} - \frac{V_1}{S}. \quad . \quad . \quad . \quad . \quad . \quad (82)$$

Substituting the values of C_1 and C_2 we have

$$\tau_1 = \frac{M_1}{S} q \left(\frac{e^{ql} - 1}{e^{ql} + 1}\right) + \frac{V_1}{S}\left(\frac{2 q l}{e^{ql} - e^{-ql}} - 1\right) \quad . \quad . \quad (83)$$

$$\tau_2 = -\frac{M_1}{S} q \left(\frac{e^{ql} - 1}{e^{ql} + 1}\right) + \frac{V_1}{S}\left(\frac{q l (e^{ql} + e^{-ql})}{e^{ql} - e^{-ql}} - 1\right). \quad . \quad (84)$$

In terms of hyperbolic functions, eqs. (83) and (84) transform into

$$\tau_1 = \frac{M_1}{S} q \tanh\frac{ql}{2} + \frac{V_1}{S}\left(\frac{ql}{\sinh ql} - 1\right) \quad . \quad . \quad . \quad (85)$$

$$\tau_2 = -\frac{M_1}{S} q \tanh\frac{ql}{2} + \frac{V_1}{S}\left(\frac{ql \cosh ql}{\sinh ql} - 1\right). \quad . \quad (86)$$

Eliminating V_1 from these equations we derive finally the value for M_1 in terms of τ_1 and τ_2

$$M_1 = \frac{S}{q}(A' \tau_1 + B' \tau_2) \quad . \quad . \quad . \quad . \quad (87)$$

in which

$$A' = \tfrac{1}{2}\left(\frac{ql}{ql \coth\frac{ql}{2} - 2} + \coth\frac{ql}{2}\right) \quad . \quad . \quad . \quad (88)$$

$$B' = \tfrac{1}{2}\left(\frac{ql}{ql \coth\frac{ql}{2} - 2} - \coth\frac{ql}{2}\right) \quad . \quad . \quad . \quad (89)$$

The value of coth θ in the form of a series is

$$\coth\theta = \frac{1}{\theta} + \frac{\theta}{3} - \frac{\theta^3}{45} + \frac{2\,\theta^5}{945} - \text{etc.} \quad . \quad . \quad . \quad (90)$$

By means of this expression eq. (87) may be transformed into

$$M_1 = \frac{2\,EI}{l}(2\,a_t \tau_1 + b_t \tau_2) \quad . \quad . \quad . \quad . \quad (91)$$

in which

$$a_t = \frac{A' ql}{4} = 1 + \frac{(ql)^2}{30} - \frac{11\,(ql)^4}{25{,}000} + \quad . \quad . \quad . \quad (92)$$

$$b_t = \frac{B' ql}{2} = 1 - \frac{(ql)^2}{60} + \frac{13\,(ql)^4}{25{,}000} - \quad . \quad . \quad . \quad (93)$$

These two equations differ from the similar equations (72) and (73) only in the signs of the second terms of the series.

Method of Application.—Equations (68) and (91) may be written in the general form

$$M_{nm} = \frac{2\,EI}{l}(2\,a\,\tau_{nm} + b\,\tau_{mn}) \quad . \quad . \quad . \quad . \quad (94)$$

corresponding to the general expression for M_{nm} of Art. 281. In the more exact expression here developed the quantities τ_{nm} and τ_{mn} are to be multiplied by the coefficients a and b, whose values differ from unity by the amounts represented by the second and following terms of a series. For compression members the value of $q\,l$ is such as to make the series rapidly convergent so that all terms beyond the second are negligible. For tension members the third term may need to be considered and in extreme cases it will be desirable to use the exact expressions of eqs. (88) and (89).

In applying this exact method to a problem, the sum of the moments at each joint, as given by eq. (94), is placed equal to zero as before. In formulating these equations, the values of the coefficients a and b must be determined for each member and the several values of τ multiplied by the respective coefficients. After this is done the multiplier, $K = I/l$, is used and the work is carried out as in the approximate method. The method of tabulating the values of $\Sigma\ \delta\ \angle$, and formulating the equations, explained in Art. 283, will be modified accordingly.

It is to be observed that in using the exact method of analysis it is necessary to consider the actual total load on the structure for which the stresses are desired in order that the values of the coefficients a and b may be determined. The secondary stresses are no longer proportional to the primary stresses.

The additional labor required by the exact method of solution is not very great, but the increased accuracy secured by the process is not generally of sufficient importance to warrant its use. Some notion of the errors involved in the usual approximate method may be gained by considering the relative size of the terms $\frac{(q\,l)^2}{30}$ and $\frac{(q\,l)^2}{60}$ in eqs. (72), (73), (92), and (93), as compared to unity. For compression members the value of l/r is not often greater than 100 and if $s = 15{,}000$ then $q\,l = \frac{l}{r}\sqrt{\frac{s}{E}} = 2.24$. Then $\frac{(q\,l)^2}{30} = 0.167$ and $\frac{(q\,l)^2}{60} = 0.083$. Hence $a_c = 0.83$ and $b_c = 1.08$. The equation for M_1 then becomes $M_1 = \frac{2\,E\,I}{l}(2 \times 0.83\,\tau_1 + 1.08\,\tau_2)$. Generally $q\,l$ will not exceed

1.0, in which case $M_1 = \frac{2EI}{l}(2 \times .965\,\tau_1 + 1.016\,\tau_2)$. For large tension members composed of eye-bars l/r is likely to be about 150, making $ql = 3.3$, $a_t = 1.36$, and $b_t = 0.82$. The error of the approximate method is thus greater for tension than for compression members. A matter of more importance, however, is the fact that in the case of compression members the effect of deflection is in general to increase the moments near the centre of the member, especially where the member is of single curvature. This is also fully shown in Art. 330.

332. Stresses in Riveted Expansion Suspenders.—A method sometimes used for providing for expansion is to support the end of the member or truss by means of a vertical plate, rigidly attached at top and bottom, and which permits the desired movement by bending. The plate is thus subjected to combined bending and tension. The conditions are shown in Fig. 59.

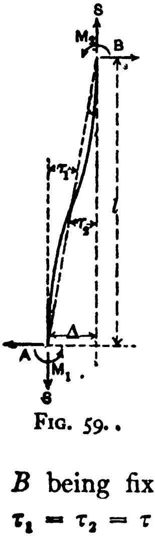


Fig. 59.

The stresses in such a plate are readily found by the application of the general formula $M_1 = \frac{2EI}{l}(2\,a_t\,\tau_1 + b_t\,\tau_2)$, given by eq. (91). In this case, since the angle τ is relatively large and l/r is also large, it will be necessary to use the exact values of the coefficients A' and B', given by eqs. (88) and (89). The angles τ_1 and τ_2 are known from the conditions of the problems.

Generally the problem is to determine the stresses in the plate due to a definite deflection Δ, the ends A and B being fixed in direction and the tangents parallel. In this case $\tau_1 = \tau_2 = \tau = \Delta/l$. Then

$$M_1 = \frac{2EI\tau}{l}(2\,a_t + b_t). \quad . \quad . \quad . \quad . \quad . \quad (95)$$

Substituting the values of a_t and b_t from (92), (93), (88), and (89), we have

$$M_1 = \frac{2EI\tau}{l} \cdot \frac{ql}{4}\left(\frac{1}{ql \coth \frac{ql}{2} - 2}\right) = S\Delta \cdot \frac{1}{ql \coth \frac{ql}{2} - 2} \quad (96)$$

Example.—Suppose an expansion joint be made by means of a plate $\frac{3}{8}$ in. thick and 6 ft. long to provide for a movement of $\frac{1}{2}$ in. The plate is also subjected to a direct tension of 8,000 lbs. per sq. in. What will be the additional stress due to bending?

Under the given conditions the width of the plate does not affect the result. Consider a width of 1 in. Then $S = 6{,}000$ lbs.; $l = 72$ in.; $I = .0352$ in^4.

$$q^2 = \frac{S}{E\,I} = \frac{1}{176};\quad q = \frac{1}{13.25};\quad q\,l = 5.43;\quad \frac{q\,l}{2} = 2.715.\quad \coth\frac{q\,l}{2} = 1.01\,.$$

$$\frac{1}{q\,l\coth\dfrac{q\,l}{2} - 2} = \frac{1}{5.48 - 2} = \frac{1}{3.48}.\quad \text{Whence}\quad M_1 = 6{,}000 \times \frac{1}{2} \times \frac{1}{3.48} =$$

862 in.-lbs. The fibre stress $= f = \dfrac{862 \times 3/8}{.0352} = 9{,}200$ lbs. per sq. in.

The bending stress which would exist if there were no direct stress is found from the formula for deflection of a cantilever beam of length of 36 in. The deflection in terms of fibre stress is $y = \frac{2}{3} \cdot \dfrac{f\,l^2}{2\,E\,c} = \dfrac{f\,l^2}{3\,E\,c}$. Placing $y = \Delta/2$ and $l = 36$ in., we derive the value $f = \dfrac{6\,\Delta\,E\,c}{l^2}$. In the above example this would be 6,530 lbs. per sq. in., a value 30 per cent less than the true stress. This comparison shows the importance of considering the effect of the direct stress in such problems.

333. Bending Moments in Members Subjected to Transverse Loads and Direct Compression or Tension.—In Art. 331, the effect of transverse loads was not considered. To take account of such loading in an exact manner, the value of M in the general expressions of eqs. (56) and (75) would have to be modified by the moment M' due to the transverse loads, considering the beam as a simply supported one, as in Art. 330. It would be possible then, for any given case, to integrate the resulting expression and finally to derive a value for M_1. Such an expression would, however, be very cumbersome and hardly of practical value. If the members are relatively short and the secondary stresses, or end moments, large, then the method explained in Art. 330 may be used; if the members are long, then the ends may be assumed either as fixed, with horizontal end tangents, or as hinged, as the case may be, and the bending stresses determined as explained below.

a. Compression Members.—Fixed Ends.—The beam $A\ B$, Fig. 60 is subjected to the transverse load of p per unit length, and also to the direct stress S, and the moments and shears as shown. The end tangents are assumed to be horizontal. As in Art. 331 the value of M at any point is

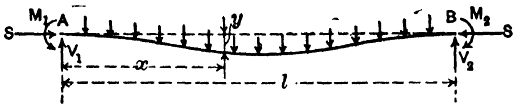


FIG. 60.

$$M = -M_1 + S\,y + V_1\,x - \frac{p\,x^2}{2} \quad . \quad . \quad . \quad (97)$$

and

$$\frac{d^2 y}{d x^2} = -\frac{M}{EI} = \frac{M_1}{EI} - \frac{S\,y}{EI} - \frac{V_1\,x}{EI} + \frac{p\,x^2}{2\,EI}. \quad . \quad . \quad (98)$$

Placing $q^2 = \frac{S}{EI}$, we have, by integration

$$y = C_1 \sin q\,x + C_2 \cos q\,x + \frac{M_1}{S} - \frac{V_1\,x}{S} + \frac{p\,x^2}{2\,S} - \frac{p}{q^2 S}. \quad . \quad (99)$$

For $x = 0$, $y = 0$, and $x = l$, $y = 0$, whence we derive the values

$$C_1 = \frac{V_1}{q\,S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

$$C_2 = \frac{p}{q^2 S} - \frac{M_1}{S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

Also for $x = l$, $\frac{d\,y}{d\,x} = 0$, whence

$$0 = q\,C_1 \cos q\,l - q\,C_2 \sin q\,l - \frac{V_1}{S} + \frac{p\,l}{S}. \quad . \quad . \quad (102)$$

Substituting and reducing $\left(\text{noting that } V_1 = \frac{p\,l}{2}\right)$, we have

$$M_1 = \frac{p}{q^2}\left[1 - \frac{q\,l}{2} \cot \frac{q\,l}{2}\right]. \quad . \quad . \quad . \quad . \quad (103)$$

For small values of $q\,l$, such as occur for compression members, we may express $\cot \frac{q\,l}{2}$ in the form of a series, as in Art. 330, deriving the expression

$$M_1 = \frac{p l^2}{12}\left[1 + \frac{(q l)^2}{60} + \ldots\right]. \quad . \quad . \quad . \quad (104)$$

But $\frac{p l^2}{12}$ is the end moment for a beam with fixed ends and subjected to the transverse load only. Call this moment M'_1. Omitting all terms of the series beyond the second we have then

$$M_1 = M'_1\left(1 + \frac{(q l)^2}{60}\right). \quad . \quad . \quad . \quad . \quad (105)$$

The moment at the centre is found from the value of $- E I \frac{d^2 y}{d x^2}$ for $x = \frac{l}{2}$. From (99) we have

$$\frac{d^2 y}{d x^2} = - q^2 (C_1 \sin q x + C_2 \cos q x) + \frac{p}{S}. \quad . \quad . \quad (106)$$

Placing $x = \frac{l}{2}$ and inserting values of C_1 and C_2 we have for the centre moment

$$M_c = \frac{p}{q^2}\left(\frac{q l}{2} \operatorname{cosec} \frac{q l}{2} - 1\right). \quad . \quad . \quad . \quad . \quad (107)$$

Writing $\operatorname{cosec} \frac{q l}{2} = \frac{2}{q l} + \frac{q l}{12} + \frac{7 (q l)^3}{2,880} + \ldots$ we derive

$$M_c = \frac{p l^2}{24}\left(1 + \frac{7 (q l)^2}{240}\right). \quad . \quad . \quad . \quad . \quad (108)$$

Or, since $\frac{p l^2}{24}$ is the centre moment for a beam fixed at the ends, we may call such moment M'_c and we have, finally,

$$M_c = M'_c\left(1 + \frac{7 (q l)^2}{240}\right). \quad . \quad . \quad . \quad . \quad (109)$$

Hinged Ends.—For hinged ends $M_1 = 0$ and the same process as employed in the preceding analysis results in the expression for centre moment

$$M_c = \frac{p}{q^2}\left(\sec \frac{q l}{2} - 1\right), \quad . \quad . \quad . \quad . \quad . \quad (110)$$

or approximately

$$M_c = \frac{p l^2}{8}\left(1 + \frac{5 (q l^2)}{48}\right). \quad . \quad . \quad . \quad . \quad . \quad (111)$$

Placing $\frac{pl^2}{8} = M'_c$ = centre moment for a simply supported beam, we have

$$M_c = M'\left(1 + \frac{5\,(q\,l)^2}{48}\right). \quad . \quad . \quad . \quad . \quad . \qquad (112)$$

6. Tension Members.—Fixed Ends.—In the case of tension members the term $S\,y$ of eq. (98) is negative, giving

$$\frac{d^2 y}{d x^2} = \frac{M_1}{EI} + \frac{S\,y}{EI} - \frac{V_1 x}{EI} + \frac{p\,x^2}{2\,EI}. \quad . \quad . \quad . \quad . \qquad (113)$$

This integrates into

$$y = C_1 e^{qx} + C_2 e^{-qx} - \frac{M_1}{S} + \frac{V_1 x}{S} - \frac{p\,x^2}{2\,S} - \frac{p}{q^2 S} \quad . \qquad (114)$$

in which $q^2 = \frac{S}{E\,I}$ as before.

From (114), we have

$$\frac{d\,y}{d\,x} = q\,(C_1 e^{qx} - C_2 e^{-qx}) + \frac{V_1}{S} - \frac{p\,x}{S} \quad . \quad . \qquad (115)$$

and

$$\frac{d^2 y}{d\,x^2} = q^2\,(C_1 e^{qx} + C_2 e^{-qx}) - \frac{p}{S}. \quad . \quad . \quad . \quad . \quad . \qquad (116)$$

To determine the constants C_1 and C_2 for a beam fixed at the ends, we use the conditions that for $x = 0$, $\frac{d\,y}{d\,x} = 0$ and for $x = \frac{l}{2}$, $\frac{d\,y}{d\,x} = 0$, giving

$$0 = q\,(C_1 - C_2) + \frac{V_1}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

and

$$0 = q\left(C_1 e^{\frac{ql}{2}} - C_2 e^{-\frac{ql}{2}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

from which

$$C_2 = C_1 e^{ql} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (117)$$

$$C_1 = \frac{V_1}{q\,S\,(e^{ql} - 1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (118)$$

Then from the condition that for $x = 0$, $y = 0$, we have

$$0 = C_1 + C_2 - \frac{M_1}{S} - \frac{p}{q^2 S}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (c)$$

Substituting the values of C_1 and C_2 we get

$$M_1 = -\frac{p}{q^2} + \frac{p\,l}{2\,q}\left[\frac{e^{ql}+1}{e^{ql}-1}\right] \quad . \quad . \quad . \quad . \quad . \qquad (119)$$

or

$$M_1 = \frac{p}{q^2}\left[\frac{q\,l}{2}\coth\frac{q\,l}{2} - 1\right]. \quad . \quad . \quad . \quad . \qquad (120)$$

For small values of $q\,l$ we may write $\coth\frac{q\,l}{2}$ as a series, deriving

$$M_1 = \frac{p\,l^2}{12}\left(1 - \frac{(q\,l)^2}{60}\right) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (121)$$

$$= M'_1\left(1 - \frac{(q\,l)^2}{60}\right), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (122)$$

in which M'_1 has the same meaning as in eq. (105). Frequently in the case of tension members the value of $q\,l$ will be too large to permit the form given in eq. (122) to be used.

The centre moment is found in the same manner as for compression members. It is

$$M_c = \frac{p}{q^2}\left[1 - \frac{q\,l}{2}\operatorname{cosech}\frac{q\,l}{2}\right] \quad . \quad . \quad . \quad . \quad . \qquad (123)$$

and for small values of $q\,l$

$$M_c = \frac{p\,l^2}{24}\left(1 - \frac{7\,(q\,l)^2}{240}\right) \quad . \quad . \quad . \quad . \quad . \qquad (124)$$

$$= M'_c\left(1 - \frac{7\,(q\,l)^2}{240}\right). \quad . \quad . \quad . \quad . \quad . \qquad (125)$$

Hinged Ends.—In this case M_1 of eq. (*c*) is zero, hence

$$0 = C_1 + C_2 - \frac{p}{q^2 S}.$$

From this equation, and eq. (*b*), we derive

$$C_2 = C_1\,e^{ql} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (126)$$

$$C_1 = \frac{p}{q^2 S\,(1 + e^{ql})}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (127)$$

Then from (116)

$$M_c = \frac{p}{q^2}\left(1 - \operatorname{sech}\frac{q\,l}{2}\right). \quad . \quad . \quad . \quad . \quad . \qquad (128)$$

and for small values of $q\,l$

$$M_c = \frac{p\,l^2}{8}\left(1 - \frac{5\,(q\,l)^2}{48}\right) \quad . \quad . \quad . \quad . \quad . \quad (129)$$

$$= M'_c\left(1 - \frac{5\,(q\,l)^2}{48}\right). \quad . \quad . \quad . \quad . \quad . \quad (130)$$

334. *Approximate General Formulas for Tension and Compression Members.*—A sufficiently exact formula for practical use may be derived on the assumption that the form of the deflection curve is the same as that of an ordinary beam uniformly loaded.

a. Hinged Ends.—From Table 1, Art. 7, the deflection of a uniformly loaded beam, simply supported, in terms of fibre stress, is $y = \frac{5}{48}\frac{f\,l^2}{E\,c}$, or very closely, $y = \frac{f\,l^2}{10\,E\,c}$. The bending moment in all cases $= M = \frac{f\,I}{c}$, hence the deflection in terms of the actual bending moment is

$$y = \frac{M\,l^2}{10\,E\,I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

If M' is the centre moment in an ordinary beam, we have, as in Art. 333,

$$M = M' \pm S\,y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

in which the plus sign is to be used for compression members and the minus sign for tension members. Substituting the value of y from (131) and solving for M, we have

$$M = \frac{M'}{1 \mp \frac{S\,l^2}{10\,E\,I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

in which the plus sign is to be used for tension and the minus sign for compression members.

b. Fixed Ends.—The centre deflection of an ordinary beam with fixed ends, uniformly loaded, is, in terms of the centre moment,

$$y = \frac{M\,l^2}{16\,E\,I}.$$

The effect of the direct stress, S, upon the moments at both end and centre may be placed equal to $S\,y/2$. Hence, for the centre moment,

$$M_c = M' \pm \frac{S\,y}{2} = M' \pm \frac{S\,M_c\,l^2}{32\,E\,I},$$

whence

$$M_c = \frac{M'}{1 \mp \dfrac{S\,l^2}{32\,E\,I}}. \qquad (134)$$

In terms of the end moment, M_1, the deflection is

$$y = \frac{M_1\,l^2}{64\,E\,I},$$

and we derive as before

$$M_1 = \frac{M'}{1 \mp \dfrac{S\,l^2}{64\,E\,I}}. \qquad (135)$$

In (134) and (135), M' is the moment at the centre and end, respectively $\left(\frac{1}{12}\,p\,l^2 \text{ and } \frac{1}{24}\,p\,l^2\right)$, in an ordinary beam with fixed ends.

EXAMPLES.—(1) An eye-bar 2 in. wide by 12 in. deep and 40 ft. long is subjected to a direct stress of 12,000 lbs. per sq. in., or a total stress of 288,000 lbs. What is the fibre stress due to bending from its own weight, the bar being assumed as pin-ended?

In this case $p = 80$ lbs. per ft., $l = 480$ in., $I = 288$ in.^{4} Using the approximate formula, eq. (133), we have $\frac{S\,l^2}{E\,I} = \frac{288{,}000 \times 480^2}{30{,}000{,}000 \times 288} = 7.7$, and $M = M'/1.77 = \frac{1}{8}\,p\,l^2/1.77 = 192{,}000/1.77 = 108{,}000$ in.-lbs. The fibre stress $= f = \frac{108{,}000 \times 6}{288} = 2{,}250$ lbs. per sq. in.

By the exact formula, eq. (128), we have $M = \frac{p}{q^2}\left(1 - \operatorname{sech}\frac{q\,l}{2}\right)$ $= \frac{p\,l^2}{8}\left[\frac{8}{(q\,l)^2}\left(1 - \operatorname{sech}\frac{q\,l}{2}\right)\right]$. The value of $(q\,l)^2 = \frac{S\,l^2}{E\,I} = 7.7$. Then $\frac{q\,l}{2} = 1.385$ and $\operatorname{sech} 1.385 = \frac{1}{2.12} = .473$. $M = \frac{p\,l^2}{8}\left[\frac{8}{7.7} \times (1 - .473)\right]$ $= \frac{p\,l^2}{8} \times .547 = 192{,}000 \times .547 = 105{,}000$ in.-lbs., a value about 3 per cent less than obtained by the approximate method.

(2) A compression member has the following dimensions: $l = 360$ in., sectional area $= A = 60$ sq. in., $I = 3{,}000$ in.^{4}, depth $= 18$ in. Direct

stress = S = 600,000 lbs. What is the fibre stress due to its own weight, assuming pin ends?

Here $\frac{S\,l^2}{E\,I} = \frac{600{,}000 \times 360^2}{30{,}000{,}000 \times 3{,}000} = 0.865$. Hence the approximate formula gives $M = \frac{M'}{1 - .0865} = \frac{p\,l^2}{8} \times 1.095 = 270{,}000 \times 1.095 = 295{,}600$ in.-lbs. The effect of the deflection is indicated by the term .0865 in the denominator, as compared to unity.

The more exact formula, eq. (112), gives $M = M'\left(1 + \frac{5\,(q\,l^2)}{48}\right) = M'\left(1 + \frac{5}{48} \times .865\right) = M' \times 1.09 = 294{,}000$ in.-lbs., a value agreeing closely with the one found above.

335. Impact Stresses.—*Static and Dynamic Stresses.*—In the analysis of stresses which has preceded, the loads considered have, in all cases, been assumed to be due to the action of gravity alone. In the case of moving loads, the position of the loads has been varied, but the resulting stresses have been calculated on the assumption that at any instant the loads have been stationary. Stated in other words, the dynamic effect of moving loads (the effect due to acceleration or retardation) has not been considered. The stresses thus obtained, whether from dead load or from live load, may properly be called static stresses. In their calculation no account is taken of dynamic effect.

In the design of structures it is necessary, however, to know not only the static stresses involved, but also, as nearly as practicable, the dynamic stresses, so that proper provision may be made for the combined effect. In some structures this dynamic effect is relatively small and unimportant, as, for example, where the fixed load is very large and the moving or variable load is small, or is applied gradually and steadily. Bridges carrying aqueducts, roof trusses, and many types of buildings, are examples of such structures. In other structures the dynamic effect of the live load is very great, as, for example, highway bridges subjected to the action of rapidly moving vehicles or animals, and railroad bridges under fast moving trains. Whatever may be the cause of such dynamic effect, the combined result is generally spoken of as *impact stress.*

In the design of bridges the impact stress is provided for either by increasing the live-load stress in accordance with some empirical rule supposed to represent the impact, or by a reduction in the live-load working stress below what would be used in case the stress were wholly static. The former method appears to be the more rational one to follow and is the one generally used.

A great variety of impact formulas have been used, most of which express the impact ratio or percentage in terms of span length or loaded length for maximum static stress. One of the earliest of these formulas, used for many years in the specifications of the American Railway Engineering and Maintenance of Way Association, is

$$I = S \frac{300}{300 + L}$$

in which S is the static stress and L the length of span loaded to produce the maximum live-load stress.

To secure more definite information on this subject the Maintenance of Way Association, through a sub-committee, has conducted a large series of experiments, the results of which are published in "Bulletin No. 125," 1910. Experiments were made on about 50 plate girder and truss spans, in which the effect of special test trains, moving at various rates of speed, was determined by means of a deflectometer and several extensometers attached to various members of the structure. Some of the results and the general conclusions contained in this report are here given.

336. *Causes of Impact.*—The chief factors in causing impact in railroad structures are: (1) Unbalanced locomotive drivers; (2) rough and uneven track; (3) flat or irregular wheels; (4) eccentric wheels; (5) rapidity of application of load; (6) deflection of beams and stringers, giving rise to variations in the action of the vertical load.

Where caused by open joints or rough or flat wheels, the impact is more or less in the nature of a "blow" upon the structure; but where caused by the other factors mentioned, it amounts in effect to a varying load, or a series of impulses, acting upon the structure. In either case the result is to produce deflections and stresses in excess of the static values, such excess of stress being commonly called *impact stress* or simply *impact.*

Of all the various causes of impact the fifth one above mentioned, namely, the rapidity of application of the load, is the one which has received the greatest amount of theoretical discussion. The other causes cannot be so readily traced from a theoretical standpoint; experimental determination is the only practical method of determining the results produced. The discussion here given will therefore be divided into, (*a*) effect of speed of application of load, and (*b*) effect of other causes.

337. (*a*) *Effect of Speed of Application of Load.*—Assuming the track perfectly smooth and all of the rotating parts perfectly balanced, the effect of a load moving over a structure at a high rate of speed depends wholly upon the vertical curvature of the track and the effect which this curvature has upon the path over which the centre of gravity of the load travels. If the load causes the structure to deflect so that the curvature of the track is concave upward, the pressure of the load upon the bridge will be in excess of its weight by reason of the centrifugal force caused by such curvature. If the track has an initial camber so that, when the load passes over the structure, the deflection produced is just sufficient to bring the track into a straight line, then there will be no centrifugal force developed and the pressure of the load upon the track will be constant and exactly equal to the static load. The impact in that case will be zero.

If we assume the track originally straight and absolutely rigid, the amount of impact or centrifugal force resulting from the deflection of the structure can be approximately determined on theoretical grounds. Such an analysis has been made by Dr. H. Zimmerman * for the case of a single rolling load, and a formula which is very closely approximate to his exact formula is as follows:

$$F = P \frac{1}{g l^2/16 v^2 d - 3}$$

in which F = centrifugal force, P = weight of rolling load, v = velocity in feet per second, d = deflection of structure, and l = span length. If, for example, $d = 1/2{,}400$ of span length and $v = 90$ ft. per second (about 60 miles per hour), we have

---

* "Die Schwingungen eines Trägers mit bewegter Last." Berlin.

$$F = P \frac{1}{0.595\,l - 3}$$

a formula which is practically exact for spans greater than 15 feet. For a 25-foot span this gives 8.7 per cent impact, and for a 50-foot span, 3.7 per cent. For a 100-foot span the value would be 1.7 per cent.

Considering these results and the fact that for spans of any considerable length the track is cambered and the abutments not rigid, we may conclude on theoretical grounds that the impact due to speed of application of load, for spans greater than 50 feet, is of no consequence.

The experimental data with reference to this point are very difficult to obtain by reason of other elements which are always present. The results obtained in these tests from balanced compound locomotives are, however, very significant, and indicate that under very favorable conditions as to track and rolling load the impact is of very little consequence even for short spans.

338. (*b*) *Effect of other Causes of Impact.*—The experiments obtained in this series of tests, indicate that with track and rolling stock in good condition the main cause of impact is the unbalanced condition of the drivers of the ordinary locomotive. Various rules are in use for the balancing of locomotives, but, excepting in the case of the balanced four-cylinder locomotive, or the electric locomotive, it is impracticable to balance an engine in all respects. The result aimed at by the mechanical departments is to so counterweight the wheels as to secure a reasonable compromise between the effect of the rotating parts and the reciprocating parts. This requires the use of counterweights considerably in excess of the amount necessary to balance the rotating parts. So far as the vertical effect on the track is concerned the reciprocating parts are of little influence, but by whatever amount the rotating parts are overbalanced, just so far will there be a variation in pressure upon the rail due to the centrifugal force of such overbalance. The amount of such overbalance is such that at high speeds the centrifugal force of this weight becomes a large percentage of the vertical load. In many locomotives, at 80 miles per hour, the excess pressure amounts to more than 100 per cent.

In consequence of the action of these centrifugal forces during the passage of a train, the load acting upon the bridge is a varying one. It varies with each rotation of the driver, and thus acts as a series of impulses tending to set the structure into vibration. In the case of short-span bridges these impulses will be repeated only two or three times during the passage of the locomotive, but in the case of long-span bridges they will be repeated many times. If now these impulses correspond in period with the normal rate of vibration of the loaded structure, the effect will be cumulative and the vibrations will be greatly increased. Such cumulative effect cannot occur for bridges of very short span length, as the normal rate of vibration of such structures is higher than the rate of rotation of the drivers at the highest practicable speeds. It may and frequently does occur in structures of span lengths as low as 75 feet and sometimes less. The speed at which the impulses here considered show a cumulative effect may be termed the *critical* speed.

In the report here referred to the following formula is derived as expressing the rate of vibration of a loaded bridge:

$$T = \sqrt{\frac{w + p}{p} \times d}$$

in which T = time of vibration of loaded structure in seconds;
w = dead load per foot, assumed as uniform;
p = live load per foot, assumed as uniform;
d = static deflection in feet due to load p, as determined by calculation or direct measurement.

The speed of the train which will produce cumulative vibration, as described above, depends upon the natural rate of vibration of the loaded structure and the diameter of the locomotive drivers. During the passage of a train the total weight on the structure varies to a considerable extent and hence the normal rate of vibration of the structure also varies. Exact agreement, therefore, between driver rotation and vibration period of the structure will exist only for a short time, and hence the cumulative effect will not continue during the passage of the locomotive entirely across a long-span structure. The results of the tests show clearly the importance of the critical speed, and the maximum vibrations were found to occur generally through a narrow range of speed corresponding to this critical speed.

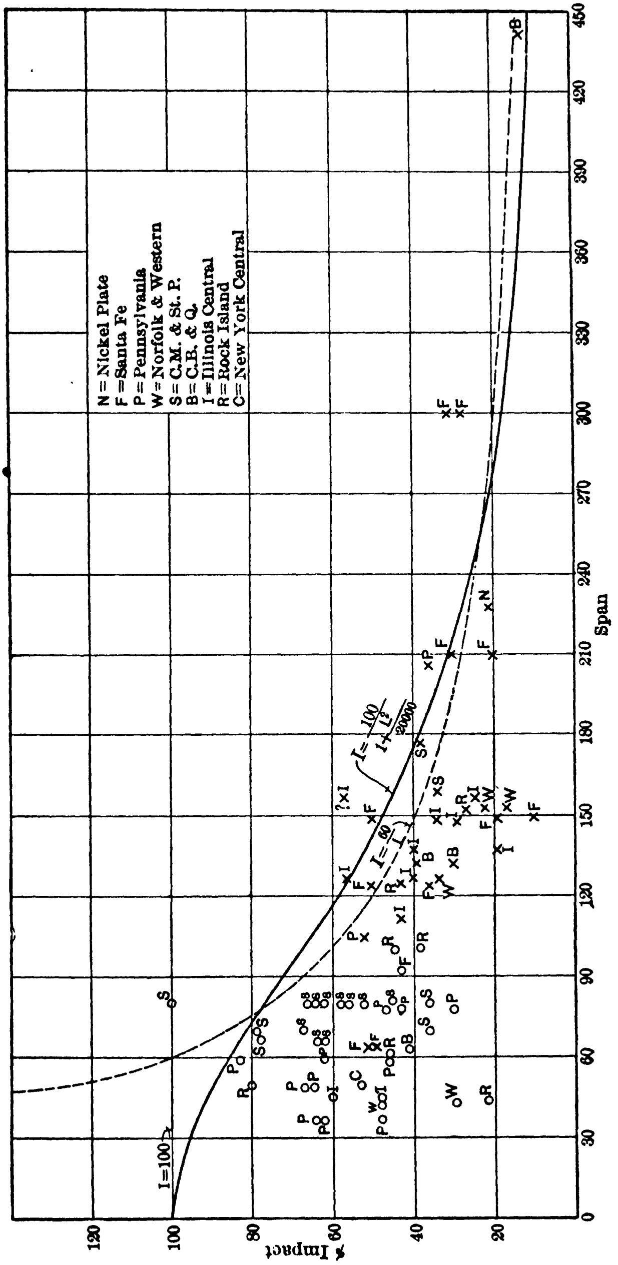


Fig. 61.—Results of Impact Tests. (Bul. 135 Am. Ry. Eng. and M. of W. Assn., 1910.)

339. *General Results for all Spans.*—The most important results obtained are shown in Fig. 61, taken from Plate IX of the "Bulletin." In this diagram are plotted the maximum impact percentages for all of the structures tested, determined by a comparison of the deflection and stresses in flanges and main truss members at high speeds with the same stresses at low speeds. Each point represents a separate structure or series of tests. The curve shown by the full line on the diagram is drawn to represent the maximum percentages as determined by the experiments. So far as our present knowledge goes it may be taken as a reasonable expression of maximum probable impact for ordinary bridge structures.

340. *Impact Percentages for Various Truss Members.*—The percentages referred to in Fig. 61 relate principally to flanges of girders and chords of trusses, all of which are directly affected by vibrations of the structure as a whole. The relative impact in various web members, as compared to chord members, was also investigated and, so far as these tests went, it was found that there was comparatively little difference in the impact ratios for the two classes of members.

341. *Summary of Results of Tests.*—The following summary is taken from the Bulletin.

(1) With track in good condition the chief cause of impact was found to be the unbalanced drivers of the locomotive. Such inequalities of track as existed on the structures tested were of little influence on impact on girder flanges and main truss members of spans exceeding 60 to 75 feet in length.

(2) When the rate of rotation of the locomotive drivers corresponds to the rate of vibration of the loaded structure, cumulative vibration is caused, which is the principal factor in producing impact in long spans. The speed of the train which produces this cumulative vibration is called the "critical speed." A speed in excess of the critical speed, as well as a speed below the critical speed, will cause vibrations of less amplitude than those caused at or near the critical speed.

(3) The longer the span length the slower is the critical speed, and therefore, the maximum impact on long spans will occur at slower speeds than on short spans.

(4) For short spans, such that the critical speed is not reached by the moving train, the impact percentage tends to be constant so far

as the effect of the counterbalance is concerned, but the effect of rough track and wheels becomes of greater importance for such spans.

(5) The impact as determined by extensometer measurements on flanges and chord members of trusses is somewhat greater than the percentages determined from measurements of deflection, but both values follow the same general law.

(6) The maximum impact on web members (excepting hip verticals) occurs under the same conditions which cause maximum impact on chord members, and the percentages of impact for the two classes of members are practically the same.

(7) The impact on stringers is about the same as on plate girder spans of the same length, and the impact on floor beams and hip verticals is about the same as on plate girders of a span length equal to two panels.

(8) The maximum impact percentages as determined by these tests is closely given by the formula:

$$I = \frac{100}{1 + \frac{L^2}{20,000}}$$

in which I = impact percentage and L = length of span in feet.

(9) The effect of differences of design was most noticeable with respect to differences in the bridge floors. An elastic floor, such as furnished by long ties supported on widely spaced stringers, or by a ballasted floor, gave smoother curves than were obtained with more rigid floors. The results clearly indicate a cushioning effect with respect to impact due to open joints, rough wheels, and similar causes. This cushioning effect was noticed on stringers, floor beams, hip verticals, and short-span girders.

(10) The effect of design upon impact percentages for main truss members was not sufficiently marked to enable conclusions to be drawn. The impact percentage here considered refers to variations in the axial stresses in the members, and does not relate to vibrations of members themselves.

(11) The impact due to the rapid application of a load, assuming smooth track and balanced loads, is found to be, from both theoretical and experimental grounds, of no practical importance.

(12) The impact caused by balanced compound and electric locomotives was very small and the vibrations caused under the loads were not cumulative.

(13) The effect of rough and flat wheels was distinctly noticeable on floor beams, but not on truss members. Large impact was, however, caused in several cases by heavily loaded freight cars moving at high speeds.

CHAPTER VIII

ANALYSIS OF QUADRANGULAR FRAMES AND SECONDARY STRESSES BY THE METHOD OF SLOPE AND DEFLECTION

SEC. I.—GENERAL THEORY

342. Introduction.—The so-called method of Slope and Deflection developed by Professor Otto Mohr * in Germany, and later developed independently by Professor G. A. Maney,† has certain advantages in the solution of problems involving quadrangular frames (frames composed of beams rigidly connected at joints but without triangular bracing), and may also be preferred by some in the determination of secondary stresses in ordinary trusses. In the method of solution of quadrangular frames employed in Chapter VI, the unknowns were expressed in terms of moment, shear, and direct stress at some conveniently cut section of the frame. In the method here explained, the unknowns are expressed in general as the changes in inclination of the axes of the beams and the angles of twist of the several joints from their original positions. The method might well be called the method of twist and deflection angles, and these terms will generally be used herein. The significance of these functions will now be explained.

In Fig. 1 (*a*), suppose A B to represent a member in an unstrained state. Now suppose end B is moved through a vertical distance Δ, the tangents at A and B remaining parallel to the original position of the member, thus assuming complete restraint of the end connections. The beam axis A B has been rotated through an angle α (Fig. 1 (*b*)), and moments M_A and M_B now exist at the ends of the beam. The angle α is here called the "deflection" angle of the beam and the moments M_A and M_B can readily be expressed in terms of the angle α and the dimensions of the beam.

---

* Die Berechnung der Fachwerke mit starren Knotenverbindungen. Professor O. Mohr. Der Civil Ingenieur; 1892, page 577; 1893; page 67.

† Studies in Engineering No. 1, The University of Minnesota, March, 1915, by G. A. Maney. Bulletins No. 80, 1915, and 108, 1918, University of Illinois Engineering Experiment Station, by Wilson, Maney, Richart, and Weiss.

If the ends of the member are only partially restrained, there will be a rotation of the end joints through angles θ_A and θ_B, respectively, these angles being inversely proportional to the restraint at the ends of the beam. Fig. 1 (*c*) shows the existing conditions. Angles θ_A and θ_B are the angles of "twist" at the joints, and the moments M_A and M_B are now functions of these angles as well as of the deflection angle α. Formulas for moments are developed in the following article.

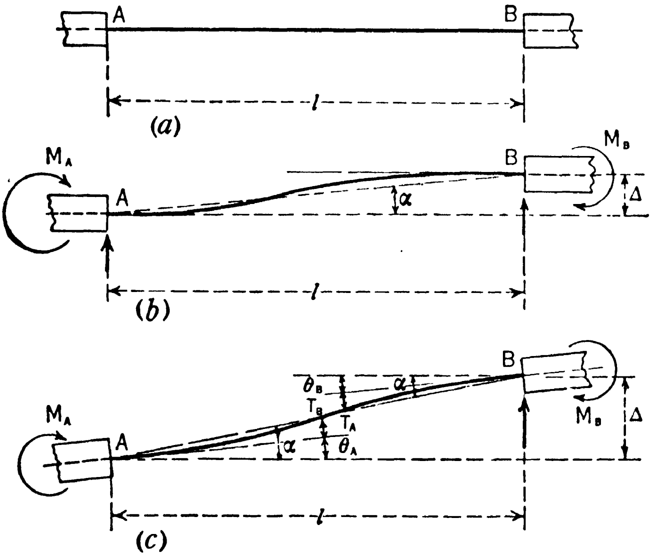


Fig. 1.

In a structure consisting of several members rigidly connected at the joints, the twist angles for all members at any joint are the same. There will thus be one unknown twist angle for each joint in a structure. As conditions of static equilibrium exist at each joint, which may be expressed as a zero summation of moments, it is possible to write moment equations for each joint of the structure which will result in a set of simultaneous equations equal in number to the number of unknown twist angles. The resulting equations contain also the deflection angles for the several members. These deflection angles may in certain cases be determined by calculation or by direct measurement and inserted in the equations as known quantities. Again, they may also be unknown quantities, in which case additional equations based on known conditions of static equilibrium are required for their determination. These special methods will be fully explained in the articles which follow.

343. Bending Moments at the Ends of a Member in Terms of Angles of Twist and Deflection.—It will be assumed, as in Chapter VII, that angles of twist and deflection are positive when the rotation is in a counter-clockwise direction. Also, counter-clockwise moments will be assumed as positive. The effect of the direct stresses in the members on the moments will be neglected, for the reasons given in Article 277.

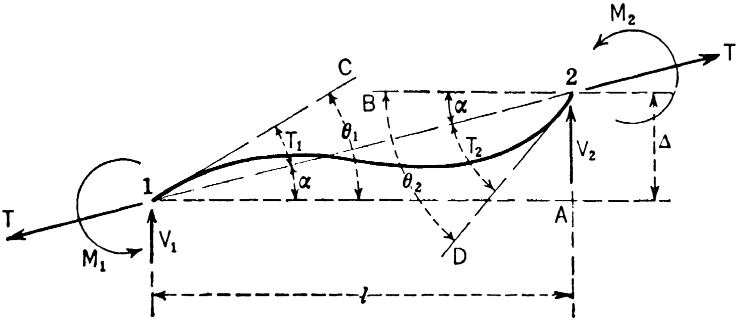


FIG. 2.

In Fig. 2, 1-*A* shows the original position of a member. Due to the distortion of the structure, point 1-*A* has been moved vertically a distance Δ to a new position at 2, the axis 1–2 of the member being rotated through an angle α in a positive direction. Then, also, due to the twist at the joints, the tangent 1–*C* at joint 1 has been rotated through a positive angle θ_1, and at joint 2, the tangent 2–*D* has been rotated through the angle θ_2, measured from a line 2–*B* parallel to the original position of the member. Angles θ_1 and θ_2 are the twist angles at joints 1 and 2, and these angles are common to all members entering these joints from other parts of the structure. Note that all angles shown in Fig. 2 are positive angles. Moments M_1 and M_2 at joints 1 and 2 are also positive moments, according to the assumed notation. Since Δ is very small compared to l we may write

$$\alpha = \Delta/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Values of M_1 and M_2, in terms of θ_1, θ_2 and α may be determined by the same methods as used in Art. 277. However, we note from Fig. 5, Art. 277, that angles C–1–2 and D–2–1 of Fig. 2 are the same as τ_1 and τ_2 of Fig. 5. Hence the desired values of M_1 and M_2 may be obtained from eqs. (10) and (11), Art. 277, by writing $\tau_1 = \theta_1 - \alpha$ and $\tau_2 = \theta_2 - \alpha$. Substituting these values in eq. (10), we have

$$M_1 = \frac{2EI}{l}[2(\theta_1 - \alpha) + (\theta_2 - \alpha)],$$

from which

$$M_1 = \frac{2EI}{l}(2\theta_1 + \theta_2 - 3\alpha). \quad . \quad . \quad . \quad . \quad (2)$$

In a similar manner, we find from eq. (11)

$$M_2 = \frac{2EI}{l}(2\theta_2 + \theta_1 - 3\alpha). \quad . \quad . \quad . \quad . \quad (3)$$

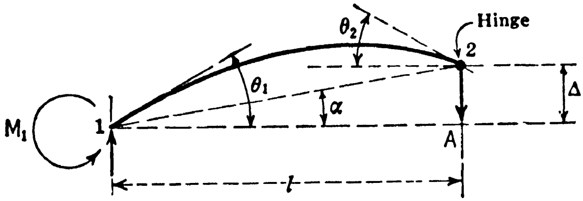


FIG. 3.

In eqs. (2) and (3), the twist angles θ_1 and θ_2 and the deflection angle α replace angles τ_1 and τ_2 of eqs. (10) and (11).

If member 1–2 is hinged at 2, as shown in Fig. 3, we have $M_2 = 0$. Then from eq. (3)

$$2\frac{EI}{l}(2\theta_2 + \theta_1 - 3\alpha) = 0,$$

from which

$$\theta_2 = -\tfrac{1}{2}(\theta_1 - 3\alpha). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Placing this value of θ_2 in eq. (2), we have

$$M_1 = 2\frac{EI}{l}[\tfrac{3}{2}(\theta_1 - \alpha)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When joint 1 is hinged, we have

$$\theta_1 = -\tfrac{1}{2}(\theta_2 - 3\alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$M_2 = \frac{2EI}{l}[\tfrac{3}{2}(\theta_2 - \alpha)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Fixed end conditions at any joint may be accounted for in eqs. (2) and (3) by placing θ at the fixed end equal to zero.

When a member supports applied loads between the joints, as shown in Fig. 4, the resulting end moments may be found by methods similar to those used in Art. 305, or, as before, we may substitute

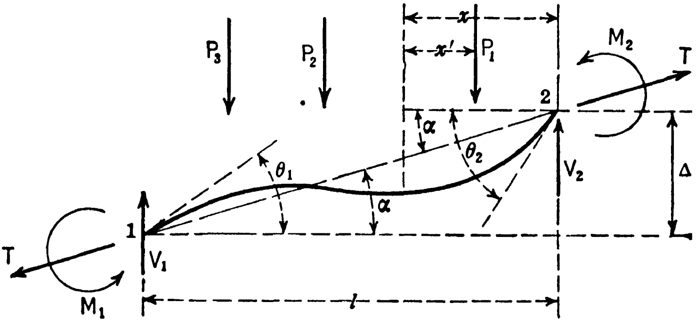


Fig. 4.

$\tau_1 = \theta_1 - \alpha$ and $\tau_2 = \theta_2 - \alpha$ in eqs. (26) and (27) of Art. 305. We then have

$$M_1 = \frac{2\,E\,I}{l}(2\,\theta_1 + \theta_2 - 3\,\alpha) - \frac{2}{l^2}\int_0^l M'\,(l - 3\,x)\,d\,x. \quad . \quad (8)$$

and

$$M_2 = \frac{2\,E\,I}{l}(2\,\theta_2 + \theta_1 - 3\,\alpha) + \frac{2}{l^2}\int_0^l M'\,(3\,x - 2\,l)\,d\,x. \quad . \quad (9)$$

As explained in Art. 305, M' is the moment in a beam assumed as freely supported at 1 and 2.

The terms $\frac{2}{l^2}\int_0^l M'\,(l - 3\,x)\,d\,x$ and $\frac{2}{l^2}\int_0 M'\,(3\,x - 2\,l)\,d\,x$ of eqs. (8) and (9) are equal to the end moments at the left and right ends respectively of a beam fixed at the ends and supporting the applied loads. Let C_1 and C_2 respectively denote these terms for the left and right ends of the beam. Equations (8) and (9) may then be written in the form

$$M_1 = \frac{2\,E\,I}{l}(2\,\theta_1 + \theta_2 - 3\,\alpha) + C_1, \quad . \quad . \quad . \quad . \quad (10)$$

and

$$M_2 = \frac{2\,E\,I}{l}(2\,\theta_2 + \theta_1 - 3\,\alpha) - C_2. \quad . \quad . \quad . \quad . \quad (11)$$

Note that proper provision has been made in eqs. (10) and (11) for the signs of C_1 and C_2. Values of C_1 and C_2 for various forms of loading are given in Table II, where they are shown as positive quantities.

When member 1–2 is hinged at 2 but restrained at 1, $M_2 = 0$. Then from eq. (9)

$$\theta_2 = -\tfrac{1}{2}\left[(\theta_1 - 3\,\alpha) + \frac{1}{E\,I\,l}\int_0^l M'\,(3\,x - 2\,l)\,d\,x\right],$$

and

$$\theta_2 = -\tfrac{1}{2}\left[(\theta_1 - 3\,\alpha) - \frac{l}{2\,E\,I}C_2\right]. \quad \ldots\ldots \quad (12)$$

Also

$$M_1 = \frac{3\,E\,I}{l}(\theta_1 - \alpha) + \frac{3}{l^2}\int_0^l M'\,x\,d\,x. \quad \ldots\ldots \quad (13)$$

When member 1–2 is hinged at 1 but restrained at 2, $M_1 = 0$, and

$$\theta_1 = -\tfrac{1}{2}\left[(\theta_2 - 3\,\alpha) - \frac{1}{E\,I\,l}\int_0^l M'\,(l - 3\,x)\,d\,x\right],$$

or

$$\theta_1 = -\tfrac{1}{2}\left[(\theta_2 - 3\,\alpha) + \frac{l}{2\,E\,I}C_1\right], \quad \ldots\ldots \quad (14)$$

and

$$M_2 = \frac{3\,E\,I}{l}(\theta_2 - \alpha) - \frac{3}{l^2}\int_0^l M'\,(l - x)\,d\,x. \quad \ldots\ldots \quad (15)$$

Terms $\int_0^l M'\,x\,d\,x$ and $\int_0^l M'\,(l - x)\,d\,x$ of eqs. (13) and (15) are the statical moments about the ends of the beam of the area of the moment diagram for M'. If these terms, multiplied by $\frac{3}{l^2}$ be denoted by D_1 and D_2, respectively, eqs. (13) and (15) may be written in the form

$$M_1 = \frac{3\,E\,I}{l}(\theta_1 - \alpha) + D_1, \quad \ldots\ldots \quad (16)$$

and

$$M_2 = \frac{3\,E\,I}{l}(\theta_2 - \alpha) - D_2. \quad \ldots\ldots \quad (17)$$

Values of D_1 and D_2 are given in Table II. Table I contains a summary of the foregoing equations arranged in convenient form for reference.

TABLE I

SUMMARY OF GENERAL EQUATIONS

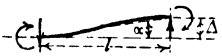
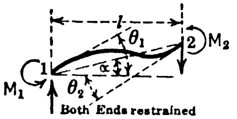
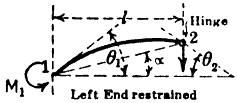
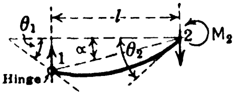
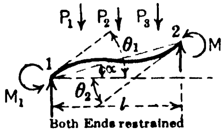
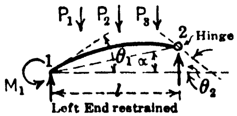
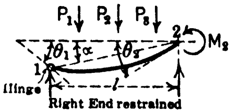
Loading Condition	Values of M_1, M_2, θ, and α
	$\alpha = \frac{\Delta}{l}$. . . . . (1)
 Both Ends restrained	$M_1 = \frac{2EI}{l}(2\theta_1 + \theta_2 - 3\alpha)$. . . . . . (2) $M_2 = \frac{2EI}{l}(2\theta_2 + \theta_1 - 3\alpha)$. . . . . . (3)
 Left End restrained Right End hinged	$M_2 = 0$ $\theta_2 = -\frac{1}{2}(\theta_1 - 3\alpha)$. . . . . (4) $M_1 = \frac{3EI}{l}(\theta_1 - \alpha)$. . . . . (5)
 Right End restrained Left End hinged	$M_1 = 0$ $\theta_1 = -\frac{1}{2}(\theta_2 - 3\alpha)$. . . . . (6) $M_2 = \frac{3EI}{l}(\theta_2 - \alpha)$. . . . . (7)
 Both Ends restrained Loads between Joints	$M_1 = \frac{2EI}{l}(2\theta_1 + \theta_2 - 3\alpha) + C_1$. . . . (10) $M_2 = \frac{2EI}{l}(2\theta_2 + \theta_1 - 3\alpha) - C_2$. . . . (11) For values of C_1 and C_2, see Table II
 Left End restrained Right End hinged Loads between Joints	$M_2 = 0$ $\theta_2 = -\frac{1}{2}\left[(\theta_1 - 3\alpha) - \frac{l}{2EI}C_2\right]$. . . . (12) $M_1 = \frac{3EI}{l}(\theta_1 - \alpha) + D_1$. . . . . . (13) For values of D_1 and C_2, see Table II
 Right End restrained Left End hinged Loads between Joints	$M_1 = 0$ $\theta_1 = -\frac{1}{2}\left[(\theta_2 - 3\alpha) + \frac{l}{2EI}C_1\right]$. . . . (14) $M_2 = \frac{3EI}{l}(\theta_2 - \alpha) - D_2$. . . . . . (15) For values of D_2 and C_1, see Table II

TABLE II

VALUES OF C_1, C_2, D_1, AND D_2 OF TABLE I

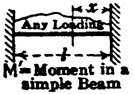
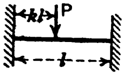
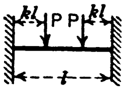
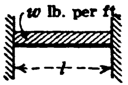
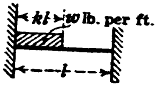
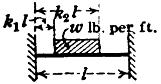
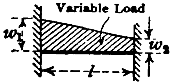
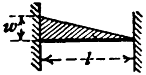
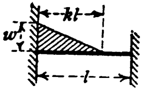
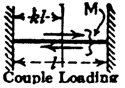
Loading Condition	C_1	C_2	D_1 Right End Hinged	D_2 Left End Hinged
	$-\frac{2}{l^2}\int_0^l M'(l-3x)dx$	$-\frac{2}{l^2}\int_0^l M'(3x-2l)dx$	$\frac{3}{l^2}\int_0^l M'x dx$	$\frac{3}{l^2}\int_0^l M'(l-x)dx$
	$Pk(1-k)^2l$	$Pk^2(1-k)l$	$\frac{1}{2}P(1-k)(2-k)kl$	$\frac{1}{2}P(1-k^2)kl$
	$Pk(1-k)l$	$Pk(1-k)l$	$\frac{3}{2}Pk(1-k)l$	$\frac{3}{2}Pk(1-k)l$
	$\frac{1}{12}wl^2$	$\frac{1}{12}wl^2$	$\frac{1}{8}wl^2$	$\frac{1}{8}wl^2$
	$\frac{wl^2k^2}{12}(3k^2-8k+6)$	$\frac{wl^2k^3}{12}(4-3k)$	$\frac{1}{8}wl^2k^2(2-k)^2$	$\frac{1}{8}wl^2k^2(2-k^2)$

TABLE II—*Continued*

VALUES OF C_1, C_2, D_1, AND D_2 OF TABLE I

Loading Condition	C_1	C_2	D_1 Right End Hinged	D_2 Left End Hinged
	$\frac{wl^2}{12}\left[k^2(3k^2-8k+6)\right]_{k_1}^{k_2}$	$\frac{wl^2}{12}\left[k^3(4-3k)\right]_{k_1}^{k_2}$	$\frac{1}{8}wl^2\left[k^2(2-k)^2\right]_{k_1}^{k_2}$	$\frac{1}{8}wl^2\left[k^2(2-k^2)\right]_{k_1}^{k_2}$
	$\frac{l^2}{60}(3w_1+2w_2)$	$\frac{l^2}{60}(2w_1+3w_2)$	$\frac{l^2}{120}(8w_1+7w_2)$	$\frac{l^2}{120}(7w_1+8w_2)$
	$\frac{wl^2}{20}$	$\frac{wl^2}{30}$	$\frac{wl^2}{15}$	$\frac{7wl^2}{120}$
	$\frac{wk^2l^2}{60}(10-10k+3k^2)$	$\frac{wk^3l^2}{60}(5-3k)$	$\frac{wk^2l^2}{120}(20-15k+3k^2)$	$\frac{wk^2l^2}{120}(10-3k^2)$
	$M(1-k)(1-3k)$	$Mk(2-3k)$	$\frac{M}{2}[3(1-k)^2-1]$	$\frac{M}{2}(1-3k^2)$

344. Special Cases of End Restraint.—*Restraint Factor.*—When the deflection angle α is zero, and there are no applied loads on the member, and when there is a known or assumed relation between the twist angles at the ends of the member, the equations of Art. 343 may be stated in a simple and useful form. Thus in the case where $\theta_1 = -\theta_2$, the general equation $M_2 = \frac{2EI}{l}(2\theta_2 + \theta_1)$ becomes

$$M_2 = \frac{2EI}{l}\theta_2 = 2EK\theta_2, \quad \ldots\ldots \quad (18)$$

where

$$K = \frac{I}{l}.$$

In general, let $\theta_1 = r\theta_2$. Then the general expression for M_2 becomes

$$M_2 = 2(2 + r)EK\theta_2, \quad \ldots\ldots \quad (19)$$

or

$$M_2 = REK\theta_2, \quad \ldots\ldots\ldots\ldots \quad (20)$$

in which

$$R = 2(2 + r). \quad \ldots\ldots\ldots\ldots \quad (21)$$

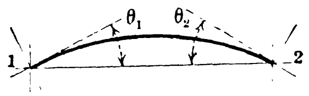
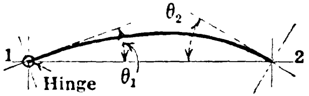
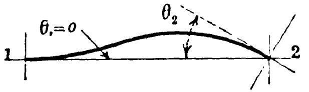
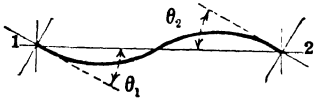
The quantity R (eq. 21), indicates the effect of the relative values of θ_1 and θ_2 upon the moment M_2. This quantity R may be called the *restraint* factor; it is a function of the ratio of θ_1 and θ_2.

Table III gives the values of R and M_2 for several simple and frequently assumed relations between the end twist angles θ_1 and θ_2. For conditions of end restraint lying between these particular values, R varies between the given values in proportion to the variation in the ratios of the end twist angles, and can be interpolated between the values given in the table. Thus if, for a given case, it can be assumed that $\theta_1 = \frac{1}{2}\theta_2$, then $r = \frac{1}{2}$ and $R = 5$, a value midway between cases (*c*) and (*d*). For $\theta_1 = -\frac{1}{4}\theta_2$, $r = -\frac{1}{4}$ and $R = 3\frac{1}{2}$, midway between (*b*) and (*c*). These relations will be found useful in approximate analysis of building frames where it is possible to estimate with reasonable accuracy the relative values of θ_1 and θ_2 (see Sect. V).

345. Moments, Shears, and Fiber Stresses in Members.—After the end moments have been determined, the shear and bending moment at any point in the member may be determined by the same methods

TABLE III

Restraint Factors for Particular Cases

Case	Relation Between θ Angles	Deformation Diagrams	Moment Equation in Terms of θ_2 $K = \frac{I}{l}$	Restraint Factor for Member 2 – 1
a	$\theta_1 = -\theta_2$		$M_2 = 2EK\theta_2$	$R = 2$
b	Hinged at 1 $\theta_1 = -\frac{1}{2}\theta_2$		$M_2 = 3EK\theta_2$	$R = 3$
c	Fixed at 1 $\theta_1 = 0$		$M_2 = 4EK\theta_2$	$R = 4$
d	$\theta_1 = +\theta_2$		$M_2 = 6EK\theta_2$	$R = 6$

as in Art. 10 for Continuous Girders. Thus from Fig. 4, the end shears, assumed as positive in direction in the figure, are

$$V_1 = \frac{1}{l}(M_1 + M_2 + \Sigma P x'), \quad . \quad . \quad . \quad . \quad (22)$$

and

$$V_2 = -\frac{1}{l}[M_1 + M_2 - \Sigma P(l - x')]. \quad . \quad (23)$$

Considering forces to the right of a section distance x from the right end of the member, the moment is

$$M_x = V_2 x + M_2 - m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

in which $m = \Sigma P x'$ = moment of loads to right of the section. Eq. (24) may also be written in the form of eq. (22), Art. 305, thus

$$M_x = M_2 - (M_1 + M_2)\frac{x}{l} + M', \quad . \quad . \quad . \quad (25)$$

in which M' has the value previously defined.

Fiber stresses in members are determined as in Chapter VII by means of eq. (18*a*), Art. 282, or by eq. (30), Art. 305. The character of fiber stress is determined as explained in Art. 296. Thus, if the moment at the end of any member is positive (plus sign), the fiber stress in the first fiber met with in passing around the joint in a clockwise direction will have a plus sign (tension). Opposite signs indicate compression.

346. General Equation for Equilibrium of a Joint in Terms of θ and α. Fig. 5 represents a joint of a structure where all members are concentric and subjected to moments at the ends. It will be noted that the state of strain of each member is similar to that shown in Fig. 4. Let θ_n = angle of twist of all members at joint n; θ_1, θ_2, etc. = angles of twist at joints 1, 2, etc.; and α_1, α_2, etc. = changes of inclination or the deflection angles of the several members from their original positions. The moments at the ends of the several members may be expressed by means of the formulas derived in the preceding article.

If it be assumed that joint n is in a state of static equilibrium, ΣM for the ends of all members entering joint n must equal zero. We may then write

$$\Sigma M_n = 0 = M_{n-1} + M_{n-2} + \quad . \quad . \quad .$$

Substituting values of the several moments, as given by eq. (2), we have

$$\frac{2\,EI_{n-1}}{l_{n-1}}(2\,\theta_n + \theta_1 - 3\,\alpha_1) + \frac{2\,E\,I_{n-2}}{l_{n-2}}(2\,\theta_n + \theta_2 - 3\,\alpha_2) + \ldots = 0$$

Let $\frac{I_{n-1}}{l_{n-1}}, \frac{I_{n-2}}{l_{n-2}}$, etc. $= K_{n-1}, K_{n-2}$, etc., and omit $2\,E$. Substituting these values in the above equation and collecting terms, we have

$$2\,(K_{n-1} + K_{n-2} + \ldots)\,\theta_n + K_{n-1}\,\theta_1 + K_{n-2}\,\theta_2 + \ldots$$
$$- 3\,(K_{n-1}\,\alpha_1 + K_{n-2}\,\alpha_2 + \ldots) = 0 \ldots\ldots \quad (26)$$

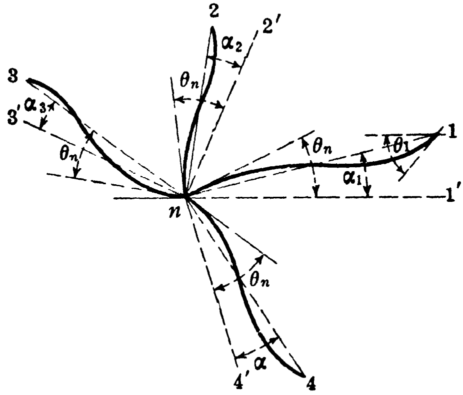


FIG. 5.

Equation (26) is a general equation for equilibrium at a joint in terms of θ and α. On comparing eq. (26) with eq. (15), Art. 281, it will be noted that the τ's of eq. (15) have been replaced by θ's in eq. (26), and that the coefficients of τ's and θ's are exactly the same in the two equations. However, the terms $\delta\angle$ of eq. (15) have been replaced by terms containing α in eq. (26).

In case either end of a member is hinged, or if applied loads are supported between joints, use the equation given in Table I, which covers the loading conditions when formulating eq. (26). If any members at a joint are eccentric, the method of procedure is the same as given in Art. 281.

Equations similar to eq. (26) may be written for each joint of the structure. When the relative movements of the joints of a structure can be determined in any manner, numerical values may be substituted for α and the resulting equations solved for θ. When the relative joint movements are not known, the values of α also enter eq. (26) as unknowns. In this case, values of α may be determined by writing additional equations, one for each unknown α, based on known conditions of static equilibrium existing in properly chosen portions of the structure. On solving the system of equations like (26) and the additional equations based on statics, values of θ and α may be determined.

After the values of θ and α are known, the moments at the ends of the members may be determined from the moment equations given in Table I.

347. Applications.—The general method of analysis outlined in the preceding articles may be applied to the solution of many problems in statically indeterminate structures. It will be found that the solutions obtained by this method are in some cases much shorter than those obtained by the general method given in the preceding chapters. This is due to the manner in which the unknowns are selected in the method under discussion. As stated in Art. 346, all members entering any joint have a common twist angle. There will then be one unknown for each joint of the frame. Also, there will be an unknown deflection angle α for each member of the frame. However, in many cases the α values may be calculated from known movements of the joints of the frame. In other cases, the α's remain unknown, but may be determined by means of additional equations based on known conditions of statical equilibrium existing in properly chosen portions of the frame.

In most cases it will be found best to solve for particular solutions of the problem under consideration by substituting, as far as possible, numerical values for terms involved in the equations. In this manner, each solution will result in a system of simultaneous equations in which the unknown θ's and α's have numerical coefficients. This procedure will, in general, lead to simpler solutions than if general values are substituted for the purpose of deriving general formulas.

In the determination of secondary stresses, the angles α are determined by the same general process used in finding the values of $\delta\alpha$ in Art. 276, but the several joint equations like eq. (26) are somewhat more readily set up than the form of eq. (15), Art. 281. The solution of the several equations is precisely similar, but in subsequent determination of moments, eq. (18), Art. 282, is a little briefer than eq. (2).

In the articles which follow, complete solutions will be given for some of the more common statically indeterminate structures encountered in practice.

SECTION II.—RESTRAINED, CONTINUOUS AND PARTIALLY CONTINUOUS BEAMS

348. Restrained Beams.—If the ends of a restrained beam are fixed, values of θ at the ends of the beam are zero. When the ends of the beam are partially restrained, values of θ may be determined by direct measurement and become known quantities. If the supports have not been subjected to settlement, α is also zero. When a known settlement of the supports has taken place, the value of α is readily calculated from eq. (1).

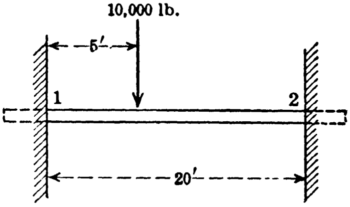


FIG. 6.

Example 1.—Given a beam fixed at the ends with supports at the same level, and loaded as shown in Fig. 6. Find the end moments. Since the ends of the beam are fixed, the twist angles θ_1 and θ_2 at the ends of the beam are equal to zero, and since the supports are at the same level, $\alpha = 0$. Then from eqs. (10) and (11), the end moments are

$$M_1 = + C_1 \quad \text{and} \quad M_2 = - C_2.$$

From Table II, $C_1 = P\,k\,(l - k)^2$ and $C_2 = P\,k^2\,(l - k)$. With $P = 10{,}000$ lb., $l = 20$ ft., and $k = \frac{5}{20} = \frac{1}{4}$

$$M_1 = + (10{,}000)(0.25)(0.75)(20) = + 28{,}125 \text{ ft.-lb.}$$

$$M_2 = - (10{,}000)(0.25)(0.75)(20) = - 9{,}375 \text{ ft.-lb.}$$

As explained in Art. 345, the fiber stress in the top fiber at the left end of the beam is tension and at the right end, the bottom fiber is in compression.

Example 2.—Given a beam hinged at the right end, partially restrained at the left end, and loaded as shown in Fig. 7. Measurements made on the beam show that the right end has settled 2 inches, that the left end has settled 1 inch, and that the left end joint has been subjected to a twist of 0.006 radian in a negative (clockwise) direction. The moment of inertia of the beam section is 100 in.^{4}, and $E = 30{,}000{,}000$ lb. per sq. in. Find the moment at joint 1.

Since joint 2 has settled 2 in. and joint 1 has settled 1 in., the beam has been subjected to a negative rotation of $\frac{2-1}{240} = 0.00417$ radian. Hence $\alpha = -0.00417$. As given above, $\theta_1 = -0.006$. From eq. (13)

$$M_1 = \frac{3EI}{l}(\theta_1 - \alpha) + D_1 = \frac{3EI}{20}[-0.006 - (-0.00417)] + D_1$$

$$= \frac{3EI}{20}(-0.00183) + D_1.$$

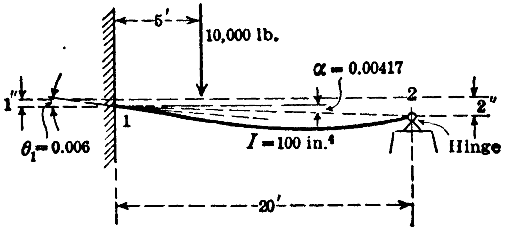


FIG. 7.

From Table II, $D_1 = \frac{1}{2}P(1-k)(2-k)kl$. With $P = 10{,}000$ lb., $k = \frac{1}{4}$ and $l = 20$ ft., $D_1 = 32{,}800$ ft.-lb. In foot and pound units,

$$EI = [(30{,}000{,}000)(12)^2](100)/(12)^4 = 20{,}830{,}000 \text{ ft.-lb.}$$

Then

$$M_1 = \tfrac{3}{20}(20{,}830{,}000)(-0.00183) + 32{,}800$$

$$M_1 = -5{,}720 + 32{,}800 = +27{,}080 \text{ ft.-lb.}$$

The fiber stress in the top fiber at joint 1 is tension.

349. Continuous Beams.—The general method of this chapter is particularly well adapted to the solution of problems in continuous girders involving settlement of supports and unusual conditions of end restraint. In such cases, solutions by this method are usually more direct and require less time than solutions by the theorem of Three Moments, as given in Chapter I.

It will generally be found that the number of unknowns to be determined will be least if the twist angles θ at the several supports be selected as the unknowns. If fixed conditions exist at a support, the θ for that support will be zero, and if a hinge is present at any support, the θ at this point is dependent upon the θ at the next joint. Where a known settlement exists at a support, the deflection angles α for the adjoining spans may be calculated. Thus the α's are known quantities, whose values may be substituted in the general joint equations.

Example.—A continuous beam is subjected to the loads shown in Fig. 8. Assume that the left end is hinged; that the right end is fixed; and that support 2 has settled 0.01 ft. If $I = 100$ in.^{4}, and $E = 30{,}000{,}000$ lb. per sq. in., find the moment at support 2.

Since support 2 has settled 0.01 ft., span No. 1 is subjected to a negative rotation of $0.01/30 = 0.000333$ radian, and span No. 2 is subjected to a positive rotation of $0.01/10 = 0.001$ radian. Hence,

$$\alpha_{12} = \alpha_{21} = -0.000333;\ \alpha_{23} = \alpha_{32} = +0.001;\ \alpha_{34} = \alpha_{43} = 0.$$

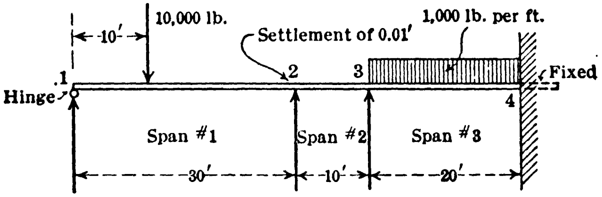


FIG. 8.

Since the right end is fixed, $\theta_4 = 0$, and since the left end is hinged, $M_{12} = 0$.

The unknowns to be determined are the twist angles θ_2 and θ_3. Equilibrium equations written for joints 2 and 3 will give a pair of simultaneous equations from which the unknowns may be determined.

Joint 2. $\Sigma M = 0 = M_{21} + M_{23}$.

From eqs. (17) and (2)

$$\frac{3\,E\,I}{30}(\theta_2 + 0.000333) - D_{21} + \frac{2\,E\,I}{10}(2\,\theta_2 + \theta_3 - 0.00300) = 0.$$

Collecting terms

$$0.5\,\theta_2 + 0.2\,\theta_3 = +\,0.000567 + \frac{D_{21}}{E\,I}.$$

From Table II, $D_{21} = \frac{1}{2}P(1 - k^2)\,k\,l$. With $P = 10{,}000$ lb., $k = 1/3$, $l = 30$ ft.; $D_{21} = 44{,}440$ ft.-lb.; $E\,I = 20{,}830{,}000$ lb. sq. ft. (see Example 2, Art. 348); we have $\frac{D_{21}}{E\,I} = \frac{44{,}440}{20{,}830{,}000} = 0.002133$. Inserting this value in above equation

$$0.5\,\theta_2 + 0.2\,\theta_3 = +\,0.00270. \qquad \text{(a)}$$

Joint 3. $\Sigma M = 0 = M_{32} + M_{34}$.

From eqs. (2) and (10), noting that $\theta_4 = 0$

$$\frac{2\,E\,I}{10}(2\,\theta_3 + \theta_2 - 0.00300) + \frac{2\,E\,I}{30}(2\,\theta_3) + C_{34} = 0.$$

From Table II, $C_{34} = \frac{1}{12} wl^2 = (1/12)(1{,}000)(20)^2 = 33{,}330$ ft.-lb.

Collecting terms and reducing

$$0.2\,\theta_2 + 0.6\,\theta_3 = -\,0.00100. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

On solving eqs. (*a*) and (*b*), we find $\theta_2 = +\,0.00700$ and $\theta_3 = -\,0.00400$. From eq. (17)

$$M_{21} = \frac{3\,E\,I}{l}(\theta_2 - \alpha) - D_2$$

$$= \frac{(3)(20{,}830{,}000)}{30}(0.00700 + 0.000333) - 44{,}440$$

$$M_{21} = -\,29{,}170 \text{ ft.-lb.}$$

350. Partially Continuous Beams.—Structures of the general type shown in Fig. 16, Art. 75, are readily analyzed by the method under discussion in this Chapter. An analysis by this method is shorter and more direct than the one given in Art. 75. As an illustration we will determine general values for the moments at supports 2 and 3 of Fig. 9. It is assumed, as in Art. 75, that the shear in panel 2–3 is zero and that moments at 2 and 3 are equal.

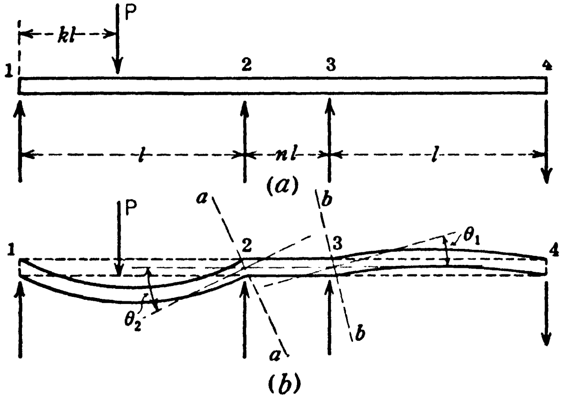


FIG. 9.

Writing summations of moments at supports 2 and 3 equal zero, we have $M_{21} + M_{23} = 0$ and $M_{32} + M_{34} = 0$. Since the moments at 2 and 3 are equal, $M_{23} = -\,M_{32}$, and therefore

$$M_{21} = -\,M_{34}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

From Table I, Art. 343, noting that hinges, or their equivalent, are provided at supports 1 and 4

$$M_{21} = \frac{3\,E\,I}{l}\,\theta_2 - D_2 \text{ and } M_{34} = \frac{3\,E\,I}{l}\,\theta_3. \quad . \quad . \quad . \quad (b)$$

From Table II, Art. 343, $D_2 = \frac{1}{2}\,P\,(1 - k^2)\,k\,l$. Substituting these values in eq. (*a*), and solving in terms of the twist angles, we have

$$\theta_2 + \theta_3 = \frac{D_2 l}{3\,E\,I} = \frac{P\,l^2}{6\,E\,I}\,(1 - k^2)\,k. \quad . \quad . \quad . \quad (c)$$

Due to the presence of little or no web bracing in the center panel (see Fig. 18, Art. 76), the continuity of the structure is broken at supports 2 and 3 and cusps in the elastic curve are formed at these supports. Tangents to the elastic curves for spans 1–2 and 3–4 at 2 and 3 are not tangent to the elastic curve for span 2–3. In fact, the span 2–3 does not bend as a beam but is distorted only by the lengthening of the top chord or flange and the shortening of the lower chord. The conditions are as represented in Fig. 9 (*b*). If lines *a–a* and *b–b* are drawn normal to the axes of the beams 2–1 and 3–4, respectively, the angle between these normals = $\theta_2 - \theta_3$. But this angle can also be expressed in terms of the distortion of the portion 2–3 and is given by the general formula of Art. 2

$$\int_0^{nl} \frac{M\,d\,x}{E\,I} = \frac{M_{3\text{-}4}\,n\,l}{E\,I}$$

Hence

$$\theta_2 - \theta_3 = M_{3\text{-}4}\,\frac{n\,l}{E\,I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

On solving eqs. (*c*) and (*d*), we have

$$\theta_3 = -\,M_{3\text{-}4}\,\frac{n\,l}{E\,I} + \frac{P\,l^2}{12\,E\,I}\,(1 - k^2)\,k.$$

Substituting this value of θ_3 in eq. (*b*), we derive

$$M_{3\,4} = +\,P\,\frac{(1 - k^2)\,k\,l}{4 + 6\,n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This moment is positive, indicating that tension exists in the top fibers of the beam to the right of support 3. Hence the reaction at support is downward, and its value is

$$R_4 = P\,\frac{(1 - k^2)\,k}{4 + 6\,n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

On reducing the values given by eq. (21), Art. 75 to corresponding dimensions, we note that the result checks eq. (2).

Example.—Calculate the reactions for the partially continuous structure shown in Fig. 10 under a single concentrated load. Assume the moment of inertia I to be constant for all spans.

From eqs. (1) and (2), with $P = 1{,}000$ lb.; $l = 50$ ft.; $k = 10/50 = 0.2$, and $n = 10/50 = 0.2$, we have

$$R_4 = 1{,}000 \frac{(1 - 0.04)0.2}{4 + 1.2} = 36.9 \text{ lb.}$$

and

$$M_{3\text{-}4} = 36.9 \times 50 = 1{,}845 \text{ ft.-lb.}$$

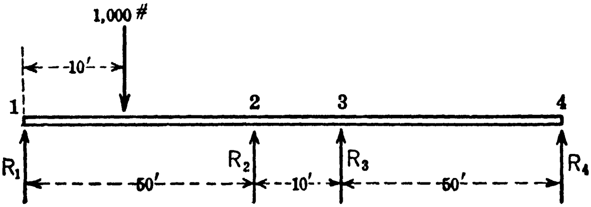


Fig. 10.

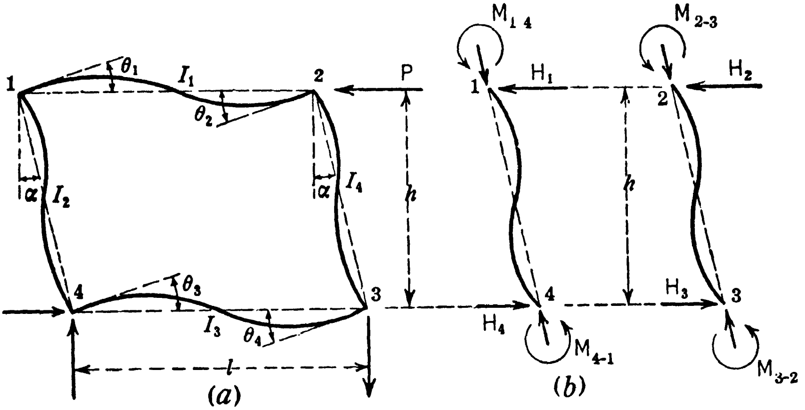
To determine R_3, cut a section just to the left of support 3. Noting that the shear is zero in panel 3–4 and that R_4 acts downward, $R_3 = - R_4 = 36.9$ lb., acting upward.

To determine R_1, cut a section to the left of 2, and consider the portion of the structure to the left of the section. Noting that $M_{21} = - M_{34}$, we have $M_{21} = - 1{,}845$ ft.-lb., a moment acting in a clockwise direction at 2. On taking moments about 2, we have $R_1 \times 50 - 1{,}000 \times 40 + 1{,}845 = 0$, from which $R_1 = 763.1$ lb., acting upward. To determine R_2, we note that since the shear in the center panel is zero, $R_1 + R_2 = P$, or $R_2 = 1{,}000 - 763.1 = 236.9$ lb., acting upward. The value of R_2 may also be determined by cutting a section just to the left of support 3. The moment at this section is $M_{32} = - M_{34} = - 1{,}845$ (clockwise). On taking moments about the section, using the known value of R_1 as determined above, we have, $R_2 \times 10 + 763.1 \times 60 - 1{,}000 \times 50 + 1{,}845 = 0$, from which $R_2 = 236.9$ lb.

SECTION III.—RECTANGULAR FRAMES

351. General Methods of Analysis for Rectangular Frames.—In the application of this method of analysis to quadrangular frames, the axial deformations of the members may be neglected, as is done in Chapter VI, since the important deformations are those due to the bending of the members of the open frame. The common case is where horizontal shearing forces produce a skew in the frame, as shown in Fig. 11. Neglecting the axial deformations of the horizontal members, all vertical members are deflected through a common angle α, the horizontal members being without change of slope.

352. The General Case.—Referring to Fig. 11, which shows the general case of an open frame with all members of unequal cross-section, there will be five unknowns to be determined. These unknowns will be four twist angles, one for each corner of the frame, and the common deflection angle α for the vertical members.



(a) (b)

Fig. 11.

To determine these unknowns, we first write equilibrium equations for each joint after the manner of eq. (26), Art. 346. Letting $\frac{I_1}{l} = K_1$; $\frac{I_2}{h} = K_2$; $\frac{I_3}{l} = K_3$; and $\frac{I_4}{h} = K_4$, the resulting equations are as follows:

Joint 1:

$$2\,(K_1 + K_2)\,\theta_1 + K_1\,\theta_2 + K_2\,\theta_4 - 3\,K_2\,\alpha = 0 \quad . \quad . \quad . \quad (1)$$

Joint 2:

$$2\,(K_1 + K_3)\,\theta_2 + K_1\,\theta_1 + K_4\,\theta_3 - 3\,K_4\,\alpha = 0 \quad . \quad . \quad . \quad (2)$$

Joint 3:

$$2\,(K_3 + K_4)\,\theta_3 + K_4\,\theta_2 + K_3\,\theta_4 - 3\,K_4\,\alpha = 0 \quad . \quad . \quad . \quad (3)$$

Joint 4:

$$2\,(K_2 + K_3)\,\theta_4 + K_2\,\theta_1 + K_3\,\theta_3 - 3\,K_2\,\alpha = 0 \quad . \; . \; . \quad (4)$$

The skew angle α adds an extra unknown in this case, and an additional condition equation is needed for the solution of the five unknowns. This additional equation may be obtained from the conditions of statical equilibrium existing in a portion of the frame formed by cutting the vertical members just below the top joints and just above the lower joints. Fig. 11 (*b*) shows this portion of the frame with all forces in position. Moments at the ends of the members shown on Fig. 11 (*b*) have been assumed as positive. $H_1 + H_2 =$ sum of the shearing forces at top of members $= P$. On taking moments about 3 or 4, we have

$$M_{14} + M_{41} + M_{23} + M_{32} + P\,h = 0. \quad . \; . \; . \quad (5)$$

Writing out the values of the several moments in terms of twist angles and the deflection angle α after eqs. (2) and (3), Table I, we have

$$\Sigma M = 2\,E\,[K_2\,(2\,\theta_1 + \theta_4 - 3\,\alpha) + K_2\,(2\,\theta_4 + \theta_1 - 3\,\alpha) + K_4\,(2\,\theta_2 + \theta_3 - 3\,\alpha) + K_4\,(2\,\theta_3 + \theta_2 - 3\,\alpha)] = -\,P\,h.$$

On collecting terms, we have

$$6\,E\,[K_2\,(\theta_1 + \theta_4 - 2\,\alpha) + K_4\,(\theta_2 + \theta_3 - 2\,\alpha)] = -\,P\,h. \quad . \quad (6)$$

Equation (6) is the general form of the necessary additional equation. Substituting numerical values of K and $P\,h$, in eqs. (1), (2), (3), (4) and (6), enables the equations to be readily solved for values of θ and α. On substituting these values of θ and α in eqs. (2) and (3) of Table I, the moments at the ends of the members are determined.

353. Modified Solution.—Since all values of θ and α in the above equation are in this case proportional to $P\,h$, the equations may readily be solved by assuming a value of $\alpha = 1$ in eqs. (1) to (4), and solving the resulting equations for θ. Then insert these values of θ in eq. (6) with $\alpha = 1$, and solve for the corresponding value of $P\,h$. This gives a value of $P\,h$ which will produce a skew α equal to unity. This value of $P\,h$ is then divided into the actual $P\,h$ value and the resulting factor used to multiply all values of θ and α to obtain the true results.

If it is desired to obtain directly numerical values for the moments at joints of the frame of Fig. 11, or if it is desired to express these moments in the form of general equations, the method of solution given below may be followed.

354. Derivation of General Formulas.—Since there are but two members entering each joint of the frame of Fig. 12, it is evident that the moments at the ends of these two members are equal in magnitude but opposite in sense. Hence the unknown moments at any joint may be represented as the moment at the end of one member entering the joint. In a general case there will be four unknown moments to be determined. For Fig. 12, these moments will be taken as M_{1-2}, M_{2-3}, M_{3-4}, and M_{4-1}. Writing values of these moments in terms of twist and deflection angles after the equations given in Table I, Art. 343, we may obtain two independent equations for each of the selected unknowns. Thus at joint 1, two values of M_{1-2} may be written, one for member 1–2 in terms of the twist angles at 1 and 2, and the other for member 1–4 in terms of the twist angles at 1 and 4 and the deflection angle α. The resulting eight equations and a ninth equation like eq. (5), Art. 352, may then be combined so as to eliminate all

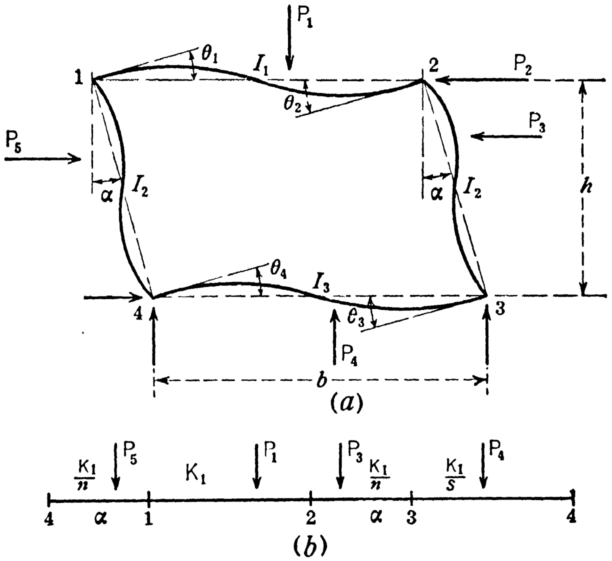


Fig. 12.

values of θ and α, giving four independent equations containing the unknown moments.

In most rectangular frames of the type shown in Fig. 12, the loads carried by the vertical members are interchangeable. Hence, it is probable that in a general case these members will be made alike, as shown in Fig. 12. Assuming the two vertical members to have the same cross-section, and letting

$$\frac{I_1}{b} = K_1; \ \frac{I_2}{h} = K_2 = \frac{K_1}{n} \ \text{and} \ \frac{I_3}{b} = K_3 = \frac{K_1}{s}, \quad . \quad . \quad (5)$$

the moments at the ends of each member may readily be written from the equations given in Table I. To facilitate the writing of these equations, the frame may be straightened out as shown in Fig. 12 (*b*). Beginning at point 4 of Fig. (*b*), the equations are as follows:

$$M_{4\text{-}1} = 2\,E\frac{K_1}{n}\,(2\,\theta_4 + \theta_1 - 3\,\alpha) + C_{4\text{-}1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$M_{1\text{-}2} = -\,M_{1\text{-}4} = -\,2\,E\frac{K_1}{n}\,(2\,\theta_1 + \theta_4 - 3\,\alpha) + C_{1\text{-}4} \quad . \quad (8)$$

$$M_{1\text{-}2} = 2\,E\,K_1\,(2\,\theta_1 + \theta_2) + C_{1\text{-}2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$M_{2\text{-}3} = -\,M_{2\text{-}1} = -\,2\,E\,K_1\,(2\,\theta_2 + \theta_1) + C_{2\text{-}1} \quad . \quad . \quad . \quad (10)$$

$$M_{2\text{-}3} = 2\,E\frac{K_1}{n}\,(2\,\theta_2 + \theta_3 - 3\,\alpha) + C_{2\text{-}3} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$M_{3\text{-}4} = -\,M_{3\text{-}2} = -\,2\,E\frac{K_1}{n}\,(2\,\theta_3 + \theta_2 - 3\,\alpha) + C_{3\text{-}2} \quad . \quad (12)$$

$$M_{3\text{-}4} = 2\,E\frac{K_1}{s}\,(2\,\theta_3 + \theta_4) + C_{3\text{-}4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$M_{4\text{-}1} = -\,M_{4\text{-}3} = -\,2\,E\frac{K_1}{s}\,(2\,\theta_4 + \theta_3) + C_{4\text{-}3} \quad . \quad . \quad . \quad (14)$$

In these equations, the terms $C_{1\text{-}2}$, etc., are to be written for the loading conditions existing for the several members.

Rewriting eq. (5), Art. 352, substituting $M_{1\text{-}4} = -\,M_{1\text{-}2}$, and $M_{3\text{-}2} = -\,M_{3\text{-}4}$, and letting $\Sigma\,M$ = moment of horizontal forces about the axis 3–4, Fig. 12 (*a*), we have

$$-\,M_{1\text{-}2} + M_{2\text{-}3} - M_{3\text{-}4} + M_{4\text{-}1} = -\,\Sigma\,M \quad . \quad . \quad . \quad (15)$$

When eqs. (7) to (15) are combined in the manner indicated in the following table,* the final condition equations are as follows:

Equation	Coefficients of M				Absolute Term	Combinations of Equations (7) to (15)
	M_{1-2}	M_{2-3}	M_{3-4}	M_{4-1}		
15	-1	$+1$	-1	$+1$	A	15
16	$+n$	$+n$	$+(2n+3s)$	$+(2n+3s)$	B	(8+11)+2(7+12)+3(13+14)
17	$-n$	$+(2n+1)$	$+(2n+s)$	$-n$	C	(10+13)+2(11+12)−(7+8)
18	$+(n+1)$	$+(n+1)$	$+(n+s)$	$+(n+s)$	D	7+8+9+10+11+12+13+14

In these equations the absolute terms have the following values:

$$\left.\begin{aligned} A &= -\Sigma M \\ B &= nC_{1\text{-}4} + n\,C_{2\text{-}3} + 2\,n\,C_{3\text{-}2} + 3\,s\,C_{3\text{-}4} \\ &\qquad + 2\,n\,C_{4\text{-}1} + 3\,s\,C_{4\text{-}3} \\ C &= -\,n\,C_{1\text{-}4} + C_{2\text{-}1} + 2\,n\,C_{2\text{-}3} + 2\,n\,C_{3\text{-}2} \\ &\qquad + s\,C_{3\text{-}4} - n\,C_{4\text{-}1} \\ D &= C_{1\text{-}2} + n\,C_{1\text{-}4} + C_{2\text{-}1} + n\,C_{2\text{-}3} + n\,C_{3\text{-}2} \\ &\qquad + s\,C_{3\text{-}4} + s\,C_{4\text{-}3} + n\,C_{4\text{-}1} \end{aligned}\right\} \quad . \; . \; . \qquad (19)$$

On solving these equations we have

$$\left.\begin{aligned} M_{4-1} &= +\frac{1}{2}\left\{\left[\frac{B\,(n+1)-Dn}{X}\right]+\left[\frac{A\,(3\,n+1)+D-2\,C}{Y}\right]\right\} \\ M_{3-4} &= +\frac{1}{2}\left\{\left[\frac{B\,(n+1)-Dn}{X}\right]-\left[\frac{A\,(3\,n+1)+D-2\,C}{Y}\right]\right\} \\ M_{2-3} &= -\frac{1}{2}\left\{\left[\frac{B\,(n+1)-Dn}{X}\right]Z+\left[\frac{A\,(3\,n+1)+D-2\,C}{Y}\right]\right\} \\ &\qquad +\left(\frac{B+An}{2\,n}\right) \\ M_{1-2} &= +\frac{1}{2}\left\{\left[\frac{B\,(n+1)-Dn}{X}\right](-Z)+\left[\frac{A\,(3\,n+1)+D-2\,C}{Y}\right]\right\} \\ &\qquad +\left(\frac{B-An}{2\,n}\right) \end{aligned}\right\} \qquad (20)$$

In these equations,

$$\left.-X=n^2+2\,ns+2\,n+3\,s; \quad Y=6\,n+s+1; \quad Z=\left(\frac{2\,n+3\,s}{n}\right)\right\} \quad . \qquad (21)$$

---

* After Bull. 108, Engineering Experiment Station, University of Illinois.

For any given loading conditions, the values of A, B, C, D may be obtained from eq. (19). Then from eqs. (20) and (21) the desired moments may be determined.

355. Special Loading Conditions.—When the rectangular frame supports special loadings of the type shown in Fig. 13, the desired moments may be determined by substituting values of the constants C_1 and C_2 of Table II, Art. 343, in the general equations of the preceding article. A direct solution may also be made after the method suggested in Art. 352. An application of this method is given in Art. 356. For Fig. (*a*), the right hand term of eq. (5) Art. 352 will be $P\,h$ as before; for Fig. (*b*) it will be zero, and for balanced forces, as in Fig. (*c*), it will also be zero. If the frame is symmetrical about a central vertical axis and the loads are also symmetrical about this axis, then α will be zero and eq. (5) need not be written. When, as shown in Figs. 13 (*d*) and 13 (*e*), there is an applied moment at one corner of the frame or on the side of a member due to a side walk bracket, the moments for the members are written after eqs. (10) and (11) of Table I, using values of C_1 and C_2 as given in Table II for couple loading.

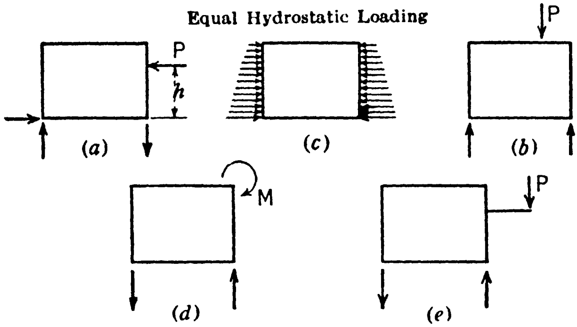


FIG. 13.

356. General Formulas for a Single Lateral Force.—For the case shown in Fig. 14, the formulas for moments are comparatively simple. They may be written out by substitution in the formulas of Art. 354, or may be readily derived directly by the method outlined in Art. 354 for a general case. Noting that the frame of Fig. 14 is symmetrical about a vertical central axis, the moments in members entering joints

2 and 3 are equal to those entering joints 1 and 4, respectively. Hence the problem may be solved by considering the equilibrium of joints 1 and 4.

From eqs. (9) and (8), Art. 354, for joint 1, and (7) and (14) for joint 4, noting that, due to symmetrical conditions $\theta_2 = \theta_1$ and $\theta_4 = \theta_3$, and letting

$$K_1 = \frac{I_1}{b}, \; K_2 = \frac{I_2}{h}, \; K_3 = \frac{I_3}{b},$$

we have

$$\textit{Joint } 1\text{:} \quad M_{12} = 2\,E\,K_1\,(3\,\theta_1) \quad \ldots\ldots\ldots \quad (a)$$

$$n\,M_{12} = -\,n\,M_{14} = -\,2\,E\,K_1(2\,\theta_1 + \theta_4 - 3\,\alpha)\,. \quad (b)$$

$$\textit{Joint } 4\text{:}\, n\,M_{41} = 2\,E\,K_1\,(2\,\theta_4 + \theta_1 - 3\,\alpha) \quad \ldots\ldots \quad (c)$$

$$s\,M_{41} = -\,s\,M_{43} = -\,2\,E\,K_1\,(3\,\theta_4). \quad \ldots\ldots \quad (d)$$

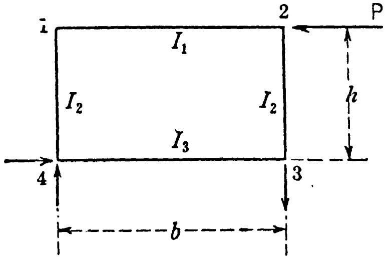


FIG. 14.

In these equations, $n = \frac{K_1}{K_2} = \frac{I_1}{b}\frac{h}{I_2}$ and $s = \frac{K_1}{K_3} = \frac{I_1}{I_3}$.

On combining these equations by adding three times [(b) + (c)] to [(a) + (d)], we have

$$(3\,n + 1)\,M_{12} + (3\,n + s)M_{41} = 0. \quad \ldots\ldots \quad (e)$$

A second condition equation may be derived from eq. (5), Art. 352, noting that $M_{14} = -\,M_{12}$; $M_{23} = M_{14} = -\,M_{12}$; and $M_{32} = M_{41}$, from which we have

$$-\,M_{12} + M_{41} = -\,\frac{P\,h}{2}. \quad \ldots\ldots\ldots \quad (f)$$

On solving equations (*e*) and (*f*) and substituting values of n and s, we have

$$M_{12} = \frac{(3n+s)}{(s+6n+1)} \cdot \frac{Ph}{2} = \frac{(3K_3+K_2)K_1}{(K_1K_2+6K_1K_3+K_2K_3)} \cdot \frac{Ph}{2}. \quad (22)$$

and

$$M_{41} = -\frac{(3n+1)}{(s+6n+1)} \cdot \frac{Ph}{2} = -\frac{(3K_1+K_2)K_3}{(K_1K_2+6K_1K_3+K_2K_3)} \cdot \frac{Ph}{2}. \quad (23)$$

These equations may also be written in the form

$$M_{12} = \frac{Ph}{2} \frac{\left(\frac{3h}{I_2} + \frac{b}{I_3}\right)}{\left(\frac{b}{I_1} + \frac{6h}{I_2} + \frac{b}{I_3}\right)}, \quad . \; . \; . \; . \quad (24)$$

and

$$M_{41} = -\frac{Ph}{2} \frac{\left(\frac{3h}{I_2} + \frac{b}{I_1}\right)}{\left(\frac{b}{I_1} + \frac{6h}{I_2} + \frac{b}{I_3}\right)}. \quad . \; . \; . \quad (25)$$

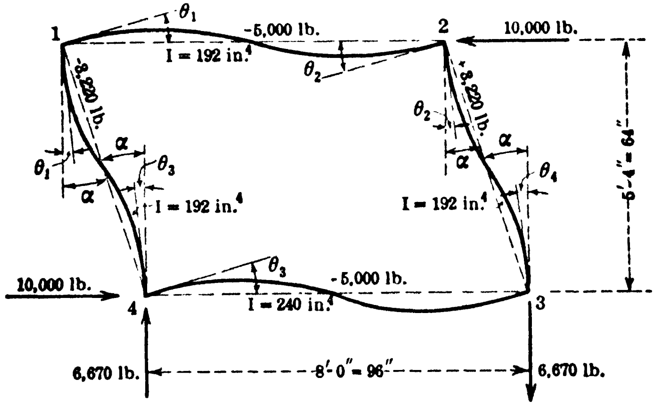


FIG. 15.

A solution of this problem by the general method of Chapter VI is given in Art. 249, where it is shown that the points of inflection of the horizontal members are at their midpoints and formulas are given for the shears at these points. The desired end moments are found by multiplying the shears $b/2$, resulting in values as given in eqs. (24) and (25).

Example.—Given the rectangular frame shown in Fig. 15. Determine the moments at the joints and the direct stresses in the members.

From eqs. (24) and (25), we have

$$M_{12} = \frac{10{,}000 \times 64}{2} \; \frac{\dfrac{3 \times 64}{192} + \dfrac{96}{240}}{\dfrac{96}{192} + \dfrac{6 \times 64}{192} + \dfrac{96}{240}}$$

$$= 154{,}500 \text{ in.-lb.}$$

$$M_{14} = -M_{12} = -154{,}500 \text{ in.-lb.}$$

$$M_{41} = -320{,}000 \times \frac{\dfrac{3 \times 64}{192} + \dfrac{96}{192}}{\dfrac{96}{192} + \dfrac{6 \times 64}{192} + \dfrac{96}{240}}$$

$$= -165{,}500 \text{ in.-lb.}$$

$$M_{4\text{-}3} = +165{,}500 \text{ in.-lb.}$$

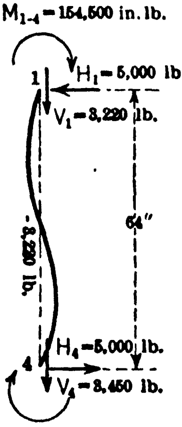


Fig. 16.

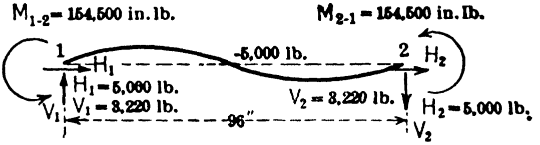


Fig. 17.

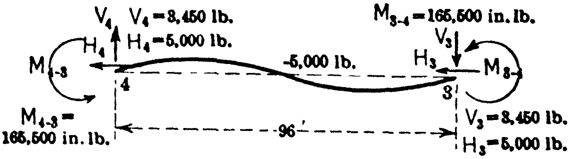


Fig. 18.

The moments at 2 and 3 are equal and opposite to 1 and 4

Figure 15 shows the true form of the frame as distorted by the moments calculated above.

The direct stresses in the frame of Fig. 15 may be determined by separating the frame into its parts, as shown in Figs. 16, 17, and 18, and considering the conditions of statical equilibrium of the several parts.

From Fig. 16, the shear H_1 at Joint 1 of member 1–4 is

$$H_1 = \frac{154{,}520 + 165{,}500}{64} = 5{,}000 \text{ lb.}$$

Also H_4, the shear at joint 4 of member 1–4 = 5,000 lb. This shear acts on member 1–2 as shown in Fig. 17, causing a compression of 5,000 lb. in member 1–2. This stress, combined with the shear of 5,000 lb. brought by member 2–3 to joint 2 of member 2–3, balances the applied load of 10,000 lb. at joint 2.

From Fig. 17, the shear V_1 at joint 1 of member 1–2 is

$$V_1 = \frac{154{,}500 + 154{,}500}{96} = 3{,}220 \text{ lb.}$$

As shown in Fig. 16, this shear causes a compression of 3,220 lb. in member 1–4. From Fig. 18, the shear V_4 at joint 4 of member 4–3 is:

$$V_3 = \frac{165{,}500 + 165{,}500}{96} = 3{,}450 \text{ lb.}$$

This shear, plus the direct stress in member 4–1 is 3,450 + 3,220 = 6,670 lb., which balances the external reaction at joint 4. All stresses are as indicated on Fig. 15.

Applying the general method of Art. 354, we note from eq. (19) and Fig. 15 that $A = -\Sigma M = -(10{,}000)(64) = -640{,}000$ in.-lb. and that $B = 0$; $C = 0$, and $D = 0$. Then from eq. (20),

$$M_{4\text{-}1} = +\frac{1}{2}\frac{A\,(3\,n + 1)}{b}.$$

From Figs. 12 and 15

$$n = \frac{K_1}{K_2} = \frac{2}{3} \quad \text{and} \quad s = \frac{K_1}{K_3} = \frac{2}{2{,}5} = \frac{4}{5}.$$

Then from eq. (21), $b = 6\,n + s + 1 = 5.8$. Hence $M_{41} = +\frac{1}{2}\,(-640{,}000)\,\frac{(2+1)}{(5{,}8)} = -165{,}500$ in.-lb., which checks the value given above.

SECTION IV.—PORTAL FRAMES

357. General Methods for the Analysis of Portal Frames.—Portal frames and viaduct bents of the plate girder type are readily analyzed by the method developed in this chapter. Trussed or partially trussed portal frames of the type shown in Art. 250 are best analyzed by the method of Chap. VI.

In applying the method of this Chapter to the analysis of viaduct bents or plate girder portals, it is generally best to consider the twist and deflection angles as unknowns. Since deformations due to direct stress may be neglected, there will in general be one unknown twist angle for the joint at the top of each vertical member and one unknown deflection angle which is common to all of the vertical members. If the frames are fixed at the bases, the twist angles at these points are zero. If the bases are hinged, there will be unknown twist angles at these points, but as shown in Art. 343, these twist angles are dependent upon the twist angles at the far end of the same member. The value of the twist angle at the base of the member is given by eqs. (12) or (14) of Table I.

358. Applications.—

Example 1.—Fig. 19 represents an elevated railway with dimensions and loading as shown. Let $I_1 = 50{,}000$ in.^{4} and $I_2 = 640$ in.^{4} Determine the moments at joints A and D. This example is also solved in Art. 247 by the method of Chap. VI.

(*a*) *Fixed Ends at D and E.*—Since the posts are fixed at the bases θ_D and $\theta_E = 0$. Also, since the structure and its loading are symmetrical about a vertical center line, $\theta_A = -\theta_B$. There being no external horizontal applied loads, the deflection angles for the vertical members are zero. Hence θ_A, the twist angle at joint A is the only unknown. Applying eq. (26), Art. 346, to joint A, with

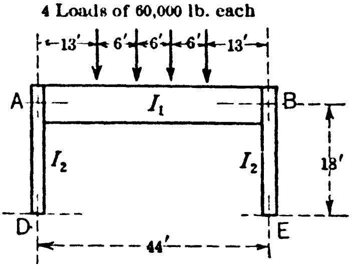


FIG. 19.

$$K_1 = \frac{I_1}{(44)(12)} = \frac{50{,}000}{(44)(12)} = 94.7;$$

and

$$K_2 = \frac{I_2}{(18)(12)} = \frac{640}{(18)(12)} = 2.96,$$

we have

$$2\,E\,[2\,\theta_A\,(2.96 + 94.7) + 94.7\,\theta_B] + C_{A\text{-}B} = 0.$$

As noted above

$$\theta_B = -\,\theta_A.$$

Hence

$$\theta_A = -\,\frac{C_{A\text{-}B}}{201.24\,E}.$$

From eq. (2), Table I, noting that $\theta_D = 0$ and $\alpha = 0$, we have

$$M_{AB} = -\,M_{AD} = -\,2\,E\,K_2\,(2\,\theta_A).$$

Substituting values of θ_A and K_2, as given above

$$M_{AB} = \frac{(4)(2.96)}{201.24}\,C_{A\text{-}B}.$$

From Table II, $C_{AB} = P\,l\,\Sigma\,k\,(1 - k)^2$. With $P = 60{,}000$ lb., $l = 44$ ft. and k for the several loads equal to 0.296, 0.432, 0.568 and 0.704 respectively, we have $C_{AB}. = 14{,}389{,}000$ in.-lb.

Then

$$M_{AB} = \frac{(4)(2.96)}{(201.24)}\,(14{,}389{,}000) = 846{,}600 \text{ in.-lb.}$$

From eq. (2), Table I

$$M_{DA} = 2\,E\,K_2\,\theta_A = -\frac{(2)(2.96)}{(201.24)}\,(14{,}389{,}000)$$

$$M_{DA} = -\,423{,}300 \text{ in.-lb.}$$

(*b*) *Hinged Ends at D and E.*—Writing the equilibrium equation for joint A, we have $M_{AB} + M_{AD} = 0$. From Table *I*, Art. 343

$$2\,E\,K_1\,(2\,\theta_A + \theta_B) + C_{A\text{-}B} + 3\,E\,K_2\,\theta_A = 0.$$

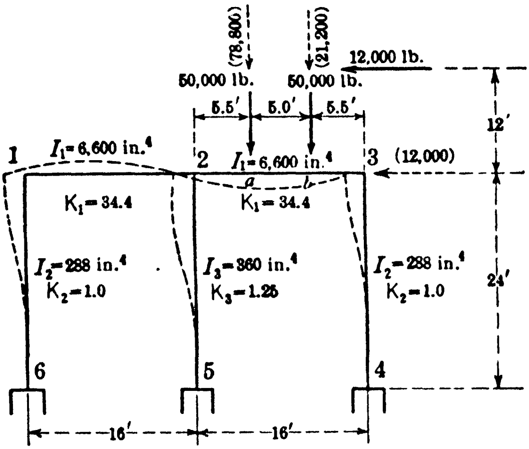


FIG. 20.

As before $\theta_A = -\,\theta_B$. With $K_1 = 94.7$, $K_2 = 2.96$, and $\alpha = 0$.

$$\theta_A = -\frac{C_{AB}}{198.28\,E}$$

in which $C_{AB} = 14{,}389{,}000$ in.-lb., as calculated above. From eq. (5), Table I, noting that $\alpha = 0$, we have

$$M_{AB} = -\,M_{AD} = -\,2\,E\,K_2\,(\tfrac{3}{2}\,\theta_A)$$

$$M_{AB} = \frac{(3)(2.96)}{198.28}\,(14{,}389{,}000) = +\,644{,}400 \text{ in.-lb.}$$

These particular problems are more readily solved by the use of the general methods and formulas of Arts. 246 and 247.

Example 2.—A railroad viaduct of the dimensions shown in Fig. 20 supports two vertical loads of 50,000 lb. due to train loading and a horizontal load of 12,000 lb. due to wind and lateral forces. Determine the moment in member 2–3 at joint 2. Assume that the columns are fixed at the bases.

In Fig. 20, the 12,000 lb. horizontal load acts on the side of a train. The effect of this load on the frame may be represented as a single horizontal force of 12,000 lb., shown by the dotted arrow at joint 3, and a couple applied to member 2–3 at points *a* and *b*. The forces comprising this couple are equal to 12,000 × 12/5 = 28,800 lb., acting downward at *a* and upward at *b*. Combining these loads with the 50,000 lb. vertical loads, the resultant loads are 78,800 lb. at *a* and 21,200 lb. at *b*. These loads are shown by the dotted arrows. Moments in the frame are to be determined for the forces represented by the dotted arrows.

Since the columns are fixed at the bases, the twist angles at joints 4, 5, and 6 are zero. There will then be four unknowns to be determined. These unknowns are the twist angles θ_1, θ_2, and θ_3 at joints 1, 2, and 3, and the deflection angle which is common to the three vertical members.

Writing the equilibrium equations for the several joints, we have

Joint 1:

$$2\,(1.0 + 34.4)\,\theta_1 + 34.4\,\theta_2 - 3\,(1)\,\alpha = 0,$$

from which

$$70.8\,\theta_1 + 34.4\,\theta_2 - 3\,\alpha = 0. \quad \ldots\ldots\ldots \quad (a)$$

Joint 2:

$$2\,(34.4 + 1.25 + 34.4)\,\theta_2 + 34.4\,\theta_1 + 34.4\,\theta_3 - (3)(1.25)\,\alpha + \frac{C_{3\text{-}2}}{2\,E} = 0,$$

from which

$$34.4\,\theta_1 + 140.1\,\theta_2 + 34.4\,\theta_3 - 3.75\,\alpha = -\frac{C_{2\text{-}3}}{2\,E}. \quad \ldots \quad (b)$$

Joint 3:

$$2\,(34.4 + 1.0)\,\theta_3 + 34.4\,\theta_2 - 3\,\alpha - \frac{C_{3\text{-}2}}{2\,E} = 0,$$

from which

$$34.4\,\theta_2 + 70.8\,\theta_3 - 3\,\alpha = +\frac{C_{3\text{-}2}}{2\,E}. \quad \ldots\ldots \quad (c)$$

Additional Equation.

$$M_{1\text{-}6} + M_{2\text{-}5} + M_{3\text{-}4} + M_{6\text{-}1} + M_{5\text{-}2} + M_{4\text{-}3} + 12{,}000(24) = 0.$$

Substituting values of the several moments, we have

$$(1)(2\,\theta_1 - 3\,\alpha) + (1.25)(2\,\theta_2 - 3\,\alpha) + (1.0)(2\,\theta_3 - 3\,\alpha)$$
$$+ (1)(\theta_1 - 3\,\alpha) + 1.25(\theta_2 - 3\,\alpha) + (1.0)(\theta_3 - 3\,\alpha) + \frac{288{,}000}{2\,E} = 0.$$

from which

$$3\,\theta_1 + 3.75\,\theta_2 + 3\,\theta_3 - 10.5\,\alpha = -\frac{144{,}000}{E}. \quad \ldots\ldots \quad (d)$$

Values of $C_{2\text{-}3}$ and $C_{3\text{-}2}$ in eqs. (*b*) and (*c*) are given in Table II, Art. 343, from which

$$C_{2\text{-}3} = \Sigma P k (1 - k)^2 l \text{ and } C_{3\text{-}2} = \Sigma P k^2 (1 - k) l.$$

With $k = 5.5/16 = 0.344$ and $10.5/16 = 0.657$ and with $P = 78{,}800$ lb. and 21,200 lb. we have,

$$C_{2\text{-}3} = [(78{,}800)(0.344)(0.656)^2 + (21{,}200)(0.657)(0.343)^2]16 = 196{,}000 \text{ ft.-lb.}$$

and

$$C_{3\text{-}2} = [(78{,}800)(0.344)^2(0.656) + (21{,}200)(0.657)^2(0.343)]16 = 148{,}000 \text{ ft.-lb.}$$

On substituting these values in eqs. (*b*) and (*c*) and solving eqs. (*a*) to (*d*), we have

$$\theta_1 = +\frac{903.8}{E};\ \theta_2 = -\frac{1{,}198.7}{E};\ \theta_3 = +\frac{1{,}949.1}{E};\ \text{and } \alpha = +\frac{7{,}587.3}{E}.$$

From eq. (10), Table I, noting that $\alpha = 0$ for horizontal members, we have

$$M_{2\text{-}3} = 2 E K_1 (2\,\theta_2 + \theta_3) + C_{2\text{-}3}.$$

With $K_1 = 34.4$; $\theta_2 = -\frac{1{,}198.7}{E}$; $\theta_3 = +\frac{1{,}949.1}{E}$ and $C_{2\text{-}3} = 196{,}500$, we have

$$M_{2\text{-}3} = (2)(34.4)[2(-1198.7) + 1{,}949.1] + 196{,}500$$

$$M_{2\text{-}3} = -30{,}843 + 196{,}500 = +165{,}657 \text{ ft.-lb.}$$

If the section of the horizontal girder is 36 inches deep, $\frac{I}{c} = 6{,}600/18 = 366$ in.^{3} and $f = \frac{(165{,}657)(12)}{366} = +5{,}420$ lb. per sq. in. Since the sign of this result is positive, the stress in the top fiber of girder 2–3 is tensile. The dotted lines of Fig. 20 show the form of the distorted frame.

Example 3.—Fig. 21 shows an *A* Frame viaduct bent supporting two vertical loads of 50,000 lb. due to train loading and a horizontal force of 12,000 lb. due to wind and lateral forces. Assuming fixed bases, find the moment in member 2–3 at joint 2. All dimensions are shown on Fig. 21. As in Example 2, the frame will be analyzed for the system of forces shown by the dotted arrows.

Since the bases are fixed, twist angles θ_1 and θ_4 are zero. The twist angles θ_2 and θ_3 at joints 2 and 3 are unknown. Also, the common deflection angle α_1 for the two posts and the deflection angle α_2 for the horizontal member 2–3 are unknown. However, since the deformations due to direct stress are neglected, α_2 may be expressed in terms of α_1. Assuming positive values of α_1, it can be seen from Fig. 21 that joint 2 moves upward a distance $6\,\alpha_1$ and joint 3 moves downward a distance $6\,\alpha_1$. Hence joint 2 moves upward a distance $12\,\alpha_1$ with respect to joint 3, and $\alpha_2 = -12\,\alpha_1/16 = -3/4\,\alpha_1$.

The equilibrium equations are as follows:

Joint 2:

$$2\,(1.0 + 34.4)\,\theta_2 + 34.4\,\theta_3 - 3\,[1.0\,\alpha_1 + 34.4\,(-3/4\,\alpha_1)] + \frac{C_{2-3}}{2\,E} = 0$$

Joint 3:

$$2\,(1.0 + 34.4)\,\theta_3 + 34.4\,\theta_2 - 3\,[1.0\,\alpha_1 + 34.4\,(-3/4\,\alpha_1)] - \frac{C_{3-2}}{2\,E} = 0$$

Noting that the loading conditions are the same as for example 2, these equations become

$$70.8\,\theta_2 + 34.4\,\theta_3 + 74.4\,\alpha_1 = -\frac{98{,}250}{E}, \quad . \; . \; . \; . \quad (a)$$

$$34.4\,\theta_2 + 70.8\,\theta_3 + 74.4\,\alpha_1 = +\frac{74{,}000}{E}. \quad . \; . \; . \; . \quad (b)$$

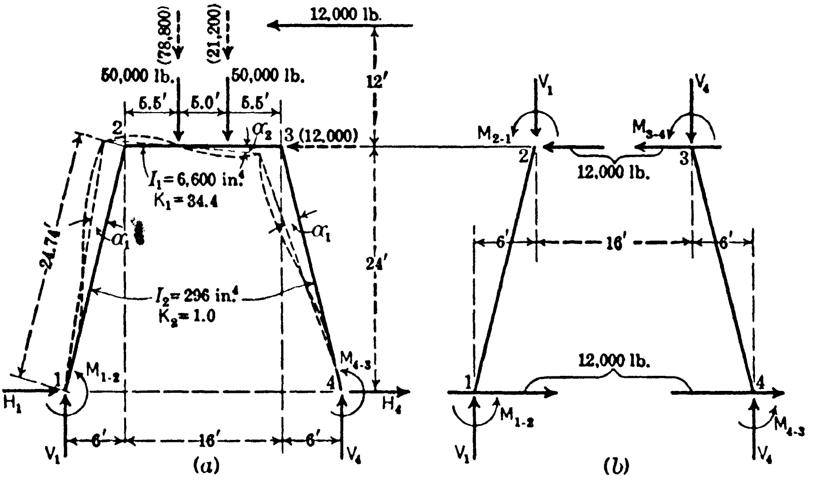


FIG. 21

A third independent equation can be written for the inclined posts considered together as a free body, as shown in Fig. 21(*b*), thus

$$M_{2\text{-}1} + M_{3\text{-}4} + M_{1\text{-}2} + M_{4\text{-}3} - 6\,V_1 + 6\,V_4 + (12{,}000)\,(24) = 0. \quad (c)$$

From Fig. 21(*a*) for $\Sigma M_4 = 0$:

$$28\,V_1 - 50{,}000\,(11.5 + 16.5) - M_{1\text{-}2} - M_{4\text{-}3} - (12{,}000)\,(36) = 0. \quad (d)$$

Also $$V_1 + V_4 = 100{,}000. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (e)$$

Substituting values of V_1 and V_4, as determined from eqs. (*d*) and (*e*), in eq. (*c*), and writing values of the several moments in terms of θ and α, eq. (*c*) becomes

$$\theta_2 + \theta_3 - 3.667\,\alpha_1 = -\frac{19{,}999}{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

On solving eqs. (*a*), (*b*), and (*f*),

$$\alpha_1 = +\frac{3{,}873}{E}; \quad \theta_2 = -\frac{5{,}374}{E}; \quad \text{and} \quad \theta_3 = -\frac{413.2}{E}$$

Then

$$M_{2\text{-}3} = 2\,EK\,(2\,\theta_2 + \theta_3 - 3\,\alpha_2) + \Sigma C_{2\text{-}3}$$

$$= 2\,(34.4)\,[2\,(-5{,}374) - 413.2 + 3\,(\tfrac{3}{4})\,(3{,}873)] + 213{,}000.$$

$$M_{2\text{-}3} = +\,44{,}650 \text{ ft. lb.}$$

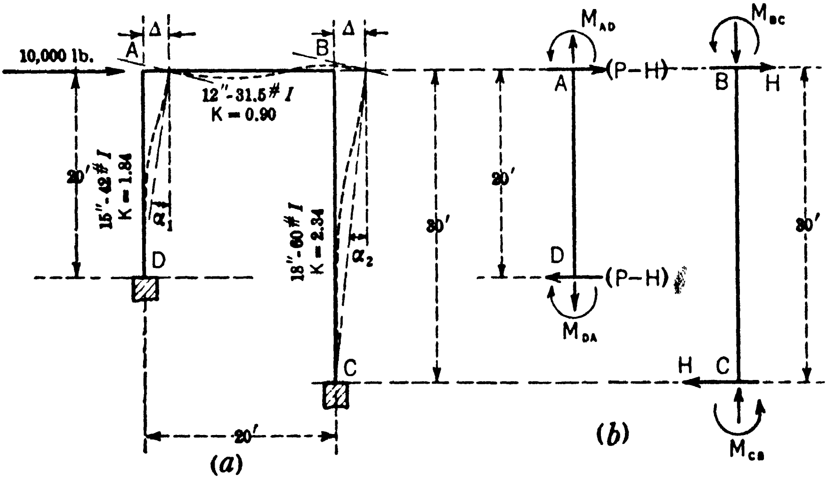


FIG. 22.

Example 4.—Fig. 22 shows a plate girder portal frame with vertical legs of unequal length. Assume fixed bases and find the moment in member *AB* at joint *A*. The unknowns to be determined are the twist angles θ_A and θ_B at joints *A* and *B* and the deflection angles α_1 and α_2 for the vertical posts. However, the two unknown α's may be expressed in terms of a single unknown by noting that (neglecting deformations due to direct stress) the horizontal movements of *A* and *B* must be equal. Let Δ, as shown in Fig. 22, represent this movement.

Then

$$\alpha_1 = \frac{\Delta}{(20)(12)} = 0.004166\,\Delta$$

and

$$\alpha_2 = \frac{\Delta}{(30)(12)} = 0.002777\,\Delta.$$

Then the three unknowns to be determined are θ_A, θ_B, and Δ.

Writing the equilibrium equations for joints A and B, we have:

Joint A:

$$2\,(1.84 + 0.90)\,\theta_A + 0.90\,\theta_B - 3\,(1.84)\,\alpha_1 = 0,$$

from which

$$5.48\,\theta_A + 0.90\,\theta_B - 0.0230\,\Delta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Joint B:

$$2\,(0.90 + 2.34)\,\theta_B + 0.90\,\theta_A - 3\,(2.34\,\alpha_2) = 0,$$

from which

$$0.90\,\theta_A + 6.48\,\theta_B - 0.0195\,\Delta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

A third independent equation may be written by cutting the vertical posts just below the horizontal member and just above the bases. These members are shown in Fig. 22 (*b*) with all forces in position. Let H = horizontal force at the foot of post $B\,C$. Then, if P = horizontal applied load $(P - H)$ = force at foot of post $A\,D$. On taking moments about point C, considering the forces acting on members $A\,D$ and $B\,C$, we have

$$M_{AD} + M_{DA} + M_{BC} + M_{CB} - 30\,P + (P - H)\,10 = 0,$$

from which

$$M_{AD} + M_{DA} + M_{BC} + M_{CB} - 20\,P = 10\,H. \quad . \quad . \quad . \quad (c)$$

On taking moments about point B, considering forces acting on member $B\,C$, we have

$$M_{BC} + M_{CB} - 30\,H = 0,$$

from which

$$10\,H = \tfrac{1}{3}\,(M_{BC} + M_{CB}).$$

Substituting this value of $10\,H$ in eq. (*c*), we have

$$M_{AD} + M_{DA} + \tfrac{2}{3}\,(M_{BC} + M_{CB}) = 20\,P.$$

Substituting values of the several moments in terms of θ and α, as given in Table I, we have

$$2\,E\{1.84(2\,\theta_A - 3\,\alpha_1) + 1.84\,(\theta_A - 3\,\alpha_1) + \tfrac{2}{3}\,[2.34(2\,\theta_B - 3\,\alpha_2)$$

$$+ 2.34\,(\theta_B - 3\,\alpha_2)]\} = 20\,P.$$

Replacing α_1 and α_2 by values in terms of Δ, as given above, and with $P = 10{,}000$ lb., we have finally

$$5.52\,\theta_A + 4.68\,\theta_B - 0.0720\,\Delta = +\frac{100{,}000}{E}. \quad . \quad . \quad . \quad (d)$$

On solving eqs. (*a*), (*b*), and (*d*) the values of the several unknowns are found to be

$$\Delta = -\frac{2{,}535{,}000}{E};\ \theta_B = -\frac{6{,}294}{E};\ \text{and}\ \theta_A = -\frac{9{,}610}{E}.$$

From eq. (2), Table I

$$M_{AB} = 2\,E\,K\,(2\,\theta_A + \theta_B) = 2\,(0.90)[(2)(-\,9{,}610) - 6{,}294]$$

$$M_{AB} = -\ 550{,}000\ \text{in.-lb.}$$

Hence the fiber stress in the top fiber of member *A B* at joint *A* is compressive. The dotted lines in Fig. 22 (*a*) show the form of the distorted frame.

SECTION V.—BUILDING FRAMES UNDER VERTICAL LOADING

359. General Methods of Analysis.—Moments, shears, and direct stresses in the members of a frame such as shown in Fig. 23 may be determined by an extension of the methods used in Sec. IV for Portal Frames. Joint equations may be written after the manner of eq. (26) Art. 346. Thus for joint 24 of Fig. 23 (*a*), we have

$$\Sigma M = M_{24\text{-}23} + M_{24\text{-}17} + M_{24\text{-}25} + M_{24\text{-}31} = 0. \quad . \quad (a)$$

In substituting values of these moments in terms of θ and α, the proper equations must be selected from Table I with due regard to the loading conditions to which the member is subjected. Experience has shown, however, that although α values for vertical members under unsymmetrical vertical loading do exist, these angular rotations are so small, compared to the θ values, that they may be omitted from the calculations without causing material errors in the values of the resulting moments. Making this assumption and using eqs. (1) and (10) of Table I, eq. (*a*) becomes:

$$2EK_{24\text{-}23}(2\theta_{24} + \theta_{23}) + 2EK_{24\text{-}17}(2\theta_{24} + \theta_{17}) + 2EK_{24\text{-}25}(2\theta_{24} + \theta_{25}) + C_{24\text{-}25} + 2EK_{24\text{-}31}(2\theta_{24} + \theta_{31}) = 0$$

from which

$$2(K_{24\text{-}23} + K_{24\text{-}17} + K_{24\text{-}25} + K_{24\text{-}31})\theta_{24} + K_{24\text{-}23}\theta_{23} + K_{24\text{-}17}\theta_{17} + K_{24\text{-}25}\theta_{25} + K_{24\text{-}31}\theta_{31} = -\frac{C_{24\text{-}25}}{2E}. \quad . \quad (1)$$

Eq. (1) may be written directly from Fig. 23 by noting that the first term of the equation is θ_{24}, the twist angle at the joint under consideration, multiplied by a term which is equal to twice the sum of all the K's $\left(\text{values of } \frac{I}{l}\right)$ for all members entering the joint. Next come in order the twist angles at the far end of each member entering joint 24, each twist angle being multiplied by the K value for the member. After a little practice these equations may readily be written out from a diagram of the frame like Fig. 23 without reference to Eq. (1).

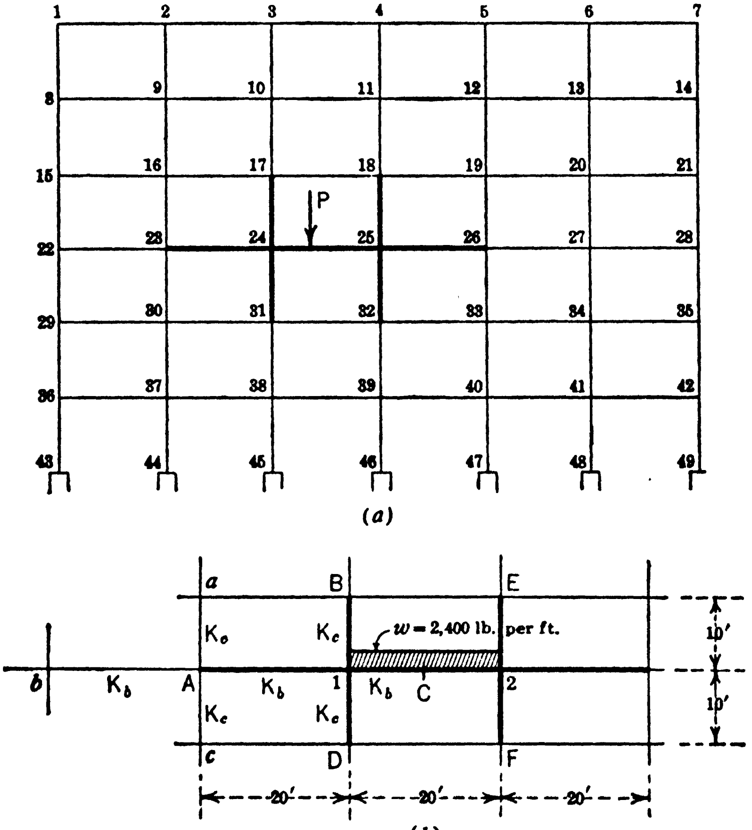
A similar equation must be written out for each joint of the frame except joints 43 to 49 at the bases of the vertical members. On solving the system of equations like eq. (1), all values of θ for the frame of Fig. 23 may be determined. Moments, shears and direct stresses may then be determined, as in the example given in Art. 356.

If member 23-24 be considered as fixed at joint 23, use eq. (3) for $M_{23\text{-}24}$ with $\theta_2 = \theta_{24}$ and $\theta_1 = \theta_{23} = 0$. If joint 23 is considered as hinged, use eq. (7). If the value of θ_{23} is desired, use eq. (6) noting that $\alpha = 0$, and hence $\theta_{23} = -\frac{1}{2}\theta_{24}$.

360. Approximate Methods.—An exact solution for a frame of many panels under vertical concentrated loads, as shown in Fig. 23, is very long and is unnecessary. Sufficiently precise results may be obtained by neglecting the effect of the frame beyond one or two panels in each direction, as was shown in the case of secondary stresses in Chap. VII.

When the maximum moment in a beam is required for loads in a single panel, as in Fig. 23, it will be shown in Art. 361 that a sufficiently accurate determination of the required moment may be had by considering only that portion of the frame shown by heavy lines. It may also be assumed that the terminal joints 17, 18, 26, 32, 31 and 23 are fixed. The resulting solution is thus relatively short.

When the maximum moment in a beam or column is desired for loads in several panels, these loads being placed so as to give the greatest moment at the point, it is also possible to obtain the desired result by considering only a portion of the frame. In such cases it is often possible to select the proper portion of the structure and to determine the boundary conditions from a study of deformation diagrams which show the probable shape of the distorted frame. Such diagrams are shown by the dotted lines in Figs. 24, 25, and 28. These diagrams were constructed by drawing first an approximate representation of the elastic line for the beams of the several floors considered as independent continuous beams. After the elastic lines for the several floors have been drawn, the elastic lines for the columns are sketched to conform to the twist angles at each floor as determined by the elastic curve for the horizontal members. On studying the



(a)

(b)

Fig. 23.

shape of the resulting deformation diagram it is possible to draw some conclusions regarding the probable relation between the unknown twist angles and the boundary twist angles. This method has been adopted in the solution of the problems given in Arts. 362 to 365.

Results obtained by this method give values which are the probable maximum moments for the case under consideration and assume that the structure is loaded in exactly the right manner to give an absolute maximum value. This condition is an improbable one but the calculations are greatly simplified by the boundary conditions thus assumed, and as loads on remote panels have little effect on the moments in question, the resulting values are not greatly in error.

Values of moments of inertia for beams and columns in a steel frame building are generally based wholly on the properties of the steel shapes used to form the members. In many buildings, the floor panels are formed by incasing the beams in concrete, pouring the slab and beams to form a monolithic structure. In such cases the moment of inertia for the floor beams should include all or a part of the concrete floor beams and slab. When this is done, it will usually be found that the moments of inertia of the beams are very large and that the moments of inertia of the columns are so small that they may be neglected. Under such conditions, the several beams forming any floor may be assumed to carry all of the moment due to vertical loads, no moment being taken by the columns. The floor may then be considered as a continuous girder and the moments determined by the methods given in Chap. I, neglecting the effect of the columns.

361. Moments Due to Loads in a Single Panel.—Fig. 23 (*b*) shows a portion of the frame of Fig. 23 (*a*). It will be assumed that panel 1–2 carries a uniform load, and it will further be assumed that in the panels surrounding the one in question the columns are all alike and the beams all alike. A study will be made as to the variation in values of moments in member 1–2 at joint 1 and at point *C* due to a variation in the restraint conditions at the terminal joints *A*, *B*, and *D*.

The terminal joints at *A*, *B*, and *D*, may be assumed as partially fixed or wholly fixed. It will be first assumed that these joints are partially fixed and that the remote joints *a*, *b*, *c*, etc., are fixed. These joints *A*, *B*, and *D* will later be assumed as fixed and results compared.

Let K_b and K_c denote the I/l values for the beams and columns respectively, then from Art. 344, the equilibrium equation for joint $A(\theta_a, \theta_b \text{ and } \theta_c = 0)$ is

$$4EK_c\theta_A + 4EK_b\theta_A + 4EK_c\theta_A + 2EK_b(2\theta_A + \theta_1) = 0$$

From this equation, the relation between θ_A and θ_1 is

$$\theta_A = -\frac{K_b}{4K_c + 4K_b}\theta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

A similar analysis, noting that $\theta_B = -\theta_E$ and $\theta_D = -\theta_F$, gives

$$\theta_B = \theta_D = -\frac{K_c}{4K_c + 3K_b}\theta_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

Eqs. (2) and (2*a*) show that the relative values of θ_A, θ_B, θ_D and θ_1 depend upon the relative rigidity of the beams and columns. Suppose $K_c = 2$ and $K_b = 1$. These values are based on beam and column sections with equal moments of inertia and a panel length of twice the height of a story, a case frequently encountered in practice. Then from eqs. (2) and (2*a*), $\theta_A = -\frac{1}{12}\theta_1$ and $\theta_B = \theta_D = -\frac{2}{11}\theta_1$. Again suppose $K_c = 1$ and $K_b = 2$. This case assumes a floor which is relatively rigid compared to the columns. From eqs. (2) and (2*a*), $\theta_A = -\frac{1}{6}\theta_1$ and $\theta_B = \theta_D = -\frac{1}{10}\theta_1$.

For the two assumed conditions, r of Art. 344 varies from $-\frac{1}{10}$ to $-\frac{2}{11}$ and the restraint factors for $1 - A$, $1 - B$ and $1 - D$ vary from $R = 2(2 + r) = 3.80$ for $r = -\frac{1}{10}$, to $R = 3.64$ for $r = -\frac{2}{11}$. For $K_b = K_c$, $R = 3.75$ for $1 - A$ and 3.71 for $1 - B$ and $1 - D$.

Following up the analysis into panel 1–2 we will first assume $K_c = K_b$. Then the moment equation for joint 1, noting that $\theta_2 = -\theta_1$, will be

$$3.75EK_b\theta_1 + 2 \times 3.71K_b\theta_1 + 2EK_b\theta_1 + C_1 = 0$$

from which

$$\theta_1 = -\frac{C_1}{13.17EK_b},$$

and

$$M_{1\text{-}2} = 2EK_b\theta_1 + C_1$$

$$= C_1(1 - 0.152) = 0.848C_1.$$

If it now be assumed that joints A, B, and D are fixed, the restraint factor for $1 - A$, $1 - B$, and $1 - D$ is 4 and the moment equation for joint 1 is

$$3 \times 4EK_b\theta_1 + 2EK_b\theta_1 + C_1 = 0$$

whence

$$\theta_1 = -\frac{C_1}{14EK_b},$$

and

$$M_{1\text{-}2} = C_1(1 - 0.143) = 0.857C_1.$$

Comparing the two above values of $M_{1\text{-}2}$ it is seen that they differ by only one per cent, showing the small effect of the differences in assumptions in the two cases. At the center of the beam the bending moment $M_c = \frac{1}{8}wl^2 - M_{1\text{-}2} = \frac{3}{2}C_1 - M_{1\text{-}2}$. The values of M_c corresponding to the two values of $M_{1\text{-}2}$ given above will be

$$(1.5 - 0.848)C_1 = 0.652C_1 \quad \text{and} \quad (1.5 - 0.857)C_1 = 0.643C_1,$$

a difference of about 1.4 per cent.

A similar analysis for two other assumed proportions of K_c and K_b gives the following results:

For $K_c = 2K_b$

$$\text{Joints } a, b, c, \text{ etc., fixed} \quad \begin{cases} M_{1\text{-}2} = 0.902C_1 \\ M_c \quad = 0.598C_1 \end{cases}$$

$$\text{Joints } A, B, D, \text{ etc., fixed} \quad \begin{cases} M_{1\text{-}2} = 0.909C_1 \\ M_c \quad = 0.591C_1 \end{cases}$$

For $K_c = \frac{1}{2}K_b$

$$\text{Joints } a, b, c, \text{ etc., fixed} \quad \begin{cases} M_{1\text{-}2} = 0.789C_1 \\ M_c \quad = 0.711C_1 \end{cases}$$

$$\text{Joints } A, B, D, \text{ etc., fixed} \quad \begin{cases} M_{1\text{-}2} = 0.800C_1 \\ M_c \quad = 0.700C_1 \end{cases}$$

Comparing results in general it is seen that whether the adjacent joints A, B, D, etc., be assumed as fixed, or that more remote joints be so assumed, makes a difference of only from 1.0 to 1.5 per cent in the values of the center moment M_c. We may therefore conclude that *the bending moments due to loads in one panel only may be found with sufficient precision by assuming that the far ends of all beams and columns adjacent to the panel in question are fixed.*

For the case of uniform load, symmetrical arrangement of beams and columns about the panel, and fixed joints at adjacent panel points, the general expressions for the moments are

$$M_{1\text{-}2} = C_1\left(1 - \frac{K_b}{3K_b + 4K_c}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$M_c = \tfrac{3}{2}C_1 - M_{1\text{-}2} = C_1\left(0.5 + \frac{K_b}{3K_b + 4K_c}\right). \quad . \quad (4)$$

in which $C_1 = \frac{1}{12}wl^2$, K_b and K_c are the I/l values for the beams and columns respectively.

For unsymmetrical loading, or variations in size of beams and columns, the full moment equations would need to be written out for joints 1 and 2, from which θ_1 and θ_2 would be found and thence $M_{1\text{-}2}$, $M_{2\text{-}1}$, and any desired moments in the beam.

The foregoing analysis shows that while fixed conditions may properly be assumed at A, B, D, etc., variations in the values of K for the different members have a considerable influence on the results and hence their values should be estimated fairly closely.

Example.—Assume the frame to have the dimensions shown in Fig. 23 (*b*). The moments of inertia of column and beam sections are each 240 in.^{4} Then $K_b = 1$ and $K_c = 2$. If the load in panel is 2,400 lb. per ft., find M_c.

From Table II, Art. 343, $C_1 = \frac{1}{12}wl^2$. With $l = 20$ ft., and $w = 2{,}400$ lb. per ft., $C_1 = 80{,}000$ ft.-lb. Then substituting in eq. (4), we have

$$M_c = C_1(0.5 + \tfrac{1}{11}) = 0.591C_1 = 80{,}000 \times 0.591 = 47{,}280 \text{ ft.-lb.}$$

The term $\frac{1}{11}C_1 = 7{,}280$ ft.-lb. represents the difference between this case and that of a fixed end beam.

362. Maximum Moment at the Center of a Beam.—Preceding, as suggested in Art. 360, it is found that the maximum moment at the center, C, of the beam 1–2 of Fig. 24 will be produced by loading the structure as shown. The form of the loaded structure is shown by the dotted lines. It is evident from the distortion of the frame, that the loaded panels contribute to increased positive values of θ_2 and negative values of θ_1. To show that this condition tends to produce maximum M_c, we note from Mechanics that, due to symmetrical conditions,

$$M_c = \tfrac{1}{8}wl^2 - M_{1\text{-}2}.$$

Therefore, to make M_c a maximum, $M_{1\text{-}2}$ must be as small as possible. From Tables I and II,

$$M_{1\text{-}2} = \frac{2EI}{l}(2\theta_1 + \theta_2) + \tfrac{1}{12}wl^2.$$

Since the loading on the frame of Fig. 24 is symmetrical about the panel in question it is evident that $\theta_2 = -\theta_1$. Now, as shown in Fig. 24, θ_1 is negative. Hence the term $\frac{2EI}{l}(2\theta_1 + \theta_2)$ is negative, and $M_{1\text{-}2}$, as required above, will have its smallest value when θ_1 and θ_2 are as large as possible, which condition is provided by the loading shown in Fig. 24.

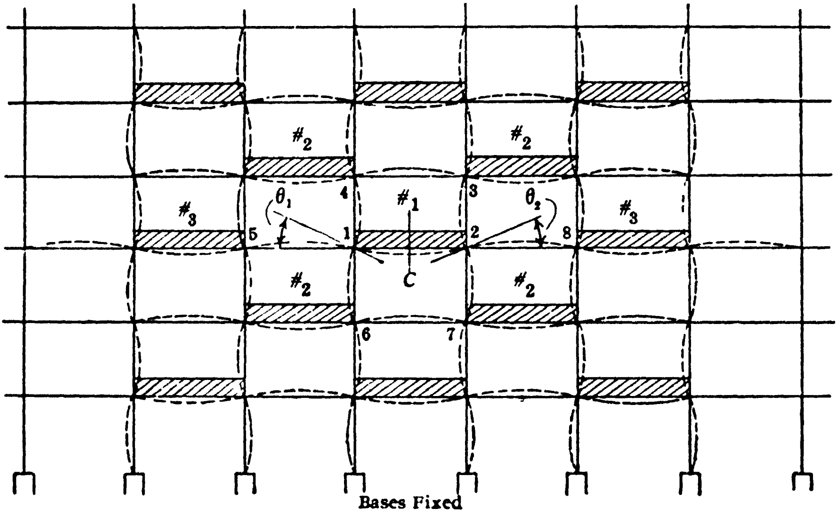


FIG. 24.

As suggested in Art. 360, the entire frame need not be considered in calculating the value of M_c. We will, therefore, consider the portion of the frame bounded by joints 3, 4, 5, 6, 7, and 8. Since the frame is symmetrical and the portion under consideration is surrounded by other panels it may reasonably be assumed from the dotted lines of Fig. 24, that the twist angles at the boundary joints are equal to θ_1 and θ_2 but opposite in sign, as indicated by the shape of the deformed frame. Then we have $\theta_2 = -\theta_1$; $\theta_4 = -\theta_1$; $\theta_5 = -\theta_1$ and $\theta_6 = -\theta_1$.*

---

* This relation would be exact for a frame of indefinite extent.

The restraint factors for members 1–4, 1–5, and 1–6 will therefore be 2 (see Table III) and hence the moment equation for joint 1 is

$$2EK_b\theta_1 + 2EK_c\theta_1 + 2EK_c\theta_1 + 2EK_b\theta_1 + C_1 = 0$$

whence

$$\theta_1 = -\frac{C_1}{2E(2K_b + 2K_c)}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

and

$$M_c = C_1\left(0.5 + \frac{K_b}{2K_b + 2K_c}\right). \quad . \; . \; . \; . \; . \; . \quad (6)$$

Example.—Assuming the same values of K_c, K_b and load as in the example of Art. 361, we have

$$\begin{aligned} M_c &= C_1(0.5 + \tfrac{1}{6}) \\ &= 80{,}000 \times 0.667 = 53{,}330 \text{ ft.-lb.} \end{aligned}$$

Comparing this case with the one given in Art. 361 where only a single panel is loaded, it is seen that the center moment is increased from 47,280 to 53,330 by making the extreme assumption that all panels are loaded which may contribute to this moment.

363. Maximum Moment at the End of a Girder.—Fig. 25 shows a frame loaded for maximum moment at joint 2 of girder 2–1. From Tables I and II, the moment at joint 2 of girder 1–2 is

$$M_{2\text{-}1} = \frac{2EI}{l}(2\theta_2 + \theta_1) + \tfrac{1}{12}wl^2.$$

To make $M_{2\text{-}1}$ a maximum, θ_2 and θ_1 should be positive angles. The loading conditions should then be such that all panel loads will either contribute positive values to these angles or will at least increase the value of the function $2\theta_2 + \theta_1$. It is quite evident that loads in panels 1, 2, 4, and 10 contribute to the value of $M_{2\text{-}1}$. A load in panel 6 causes a positive θ_2 and an approximately equal negative θ_1 but adds to the function $2\theta_2 + \theta_1$. A load in panel 5 causes a positive θ_1 with little effect on θ_2. In the same way it can be seen that panels 13 and 14 should both be loaded. Loads in panels 8 and 11 tend to cause a negative θ_3, but this negative twist at joint 3 tends to cause a positive twist at joint 2, thus increasing the positive value of θ_2.

The unknowns to be determined will be taken as θ_1, θ_2, and θ_3, the twist angles at joints 1, 2, and 3. Considering a portion of the frame bounded by joints 4, 5, 6, 7, 8, 9, 10, 11 as shown in Fig. 26, and based on the probable shape of the deformed frame, as shown by the dotted lines in Fig. 25, it may be assumed that the relations between the boundary twist angles and the unknown twist angles θ_1, θ_2, and θ_3 are as follows:

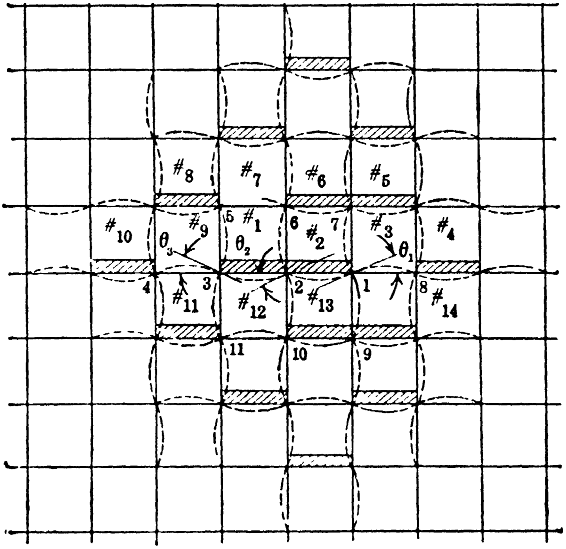


FIG. 25.

$$\theta_4 = -\theta_3;\ \theta_5 = -\theta_3;\ \theta_6 = \theta_3;\ \theta_7 = \theta_2;$$

$$\theta_8 = -\theta_1;\ \theta_9 = \theta_2;\ \theta_{10} = \theta_3;\ \text{and}\ \theta_{11} = -\theta_3.$$

From these relations three moment equations can be written for joints 1, 2, and 3 and solved for the twist angles at these joints. In this case a solution will be given in the form of a numerical example.

Assuming as in the other examples that $K_b = 1$, $K_c = 2$, and that the dimensions and uniform load are as before, whence

$$C_1 = C_2 = \tfrac{1}{12}wl^2 = 80{,}000 \text{ ft.-lb.},$$

the several joint equations are as follows:

Joint 1

$$2(1 + 2 + 1 + 2)\theta_1 + \theta_2 + 2\theta_7 + \theta_8 + 2\theta_9 = +\frac{40{,}000}{E}.$$

On substituting the assumed values of the twist angles, we have finally:

$$11\theta_1 + 5\theta_2 = +\frac{40{,}000}{E}. \quad . \; . \; . \; . \; . \; . \; . \quad (a)$$

Joint 2

$$2(1 + 2 + 1 + 2)\theta_2 + \theta_3 + 2\theta_6 + \theta_1 + 2\theta_{10} = 0,$$

from which

$$\theta_1 + 12\theta_2 + 5\theta_3 = 0. \quad . \; . \; . \; . \; . \; . \; . \quad (b)$$

Joint 3

$$2(1 + 2 + 1 + 2)\theta_3 + \theta_4 + 2\theta_5 + \theta_2 + 2\theta_{11} = -\frac{40{,}000}{E},$$

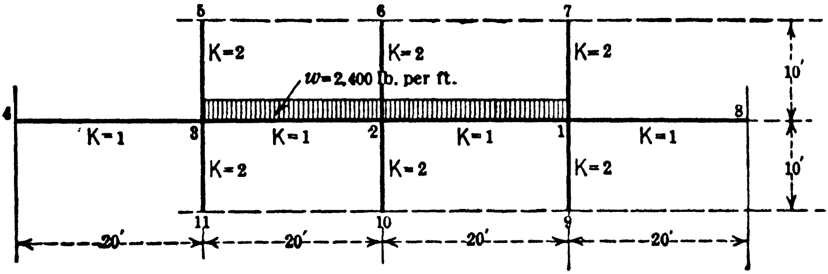


FIG. 26.

from which

$$\theta_2 + 7\theta_3 = -\frac{40{,}000}{E}. \quad . \; . \; . \; . \; . \; . \; . \quad (c)$$

On solving these equations we have

$$\theta_1 = +\frac{2{,}589.9}{E}; \; \theta_2 = +\frac{2{,}302.2}{E}; \quad \text{and} \quad \theta_3 = -\frac{6{,}043.3}{E}.$$

From Tables I and II

$$M_{2\text{-}1} = 2EK(2\theta_2 + \theta_1) + \tfrac{1}{12}wl^2.$$

With $K = 1$, θ_2 and θ_1 as given above, and $\frac{1}{12}wl^2 = 80{,}000$ we have

$$M_{2\text{-}1} = 2[2(+2{,}302.3) + 2.589.9] + 80{,}000 = +94{,}389 \text{ ft.-lb.}$$

Fig. 27 shows the moment and deformation diagrams for the frame.

By omitting loads in panels 6 and 13, symmetrical conditions will result with $\theta_2 = \theta_6 = 0$. Also, approximately,

$$\theta_1 = -\theta_7 = -\theta_8 = -\theta_9.$$

Then the moment equation for joint 1 is:

$$4EK_b\theta_1 - C_1 + 2EK_c\theta_1 + 2EK_b\theta_1 + 2EK_c\theta_1 = 0,$$

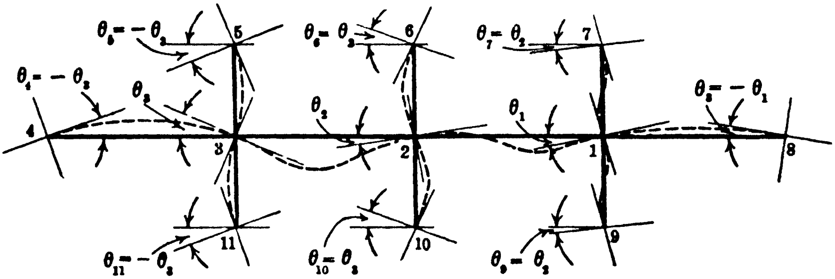
whence

$$\theta_1 = \frac{C_1}{2E(3K_b + 2K_c)},$$

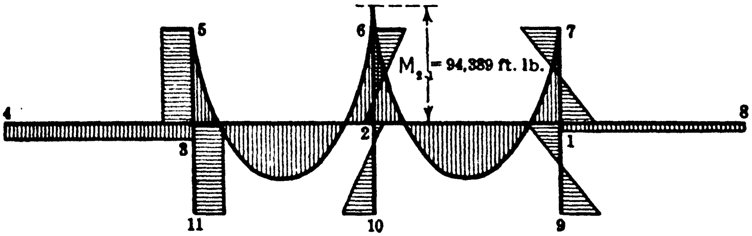
and

$$M_{2\text{-}1} = 2K_b\theta_1 + C_1$$

$$= C_1\left(1 + \frac{K_b}{3K_b + 2K_c}\right). \quad . \quad . \quad . \quad . \quad . \quad (7)$$



Deformation Diagram



Moment Diagram

FIG. 27.

For $K_c = 2K_b$, as in the foregoing example,

$$M_{2\text{-}1} = C_1(1 + \tfrac{1}{7}) = 80{,}000 \times 1.143 = 91{,}440 \text{ ft.-lb.}$$

This value is somewhat smaller than previously found but it would appear to be a satisfactory value for practical purposes.

If only loads in panels 1 and 2 be considered and joints 7, 8, and 9 be assumed as fixed, as in Art. 361, there results

$$\theta_1 = \frac{C_1}{2E(4K_b + 4K_c)}.$$

For $K_c = 2K_b$

$$M_{2-1} = C_1(1 + \tfrac{1}{12}) = 80{,}000\ (1.083) = 86{,}670\ \text{ft.-lb.}$$

Comparing the several results it is seen that the effect of loads in remote panels is not great. The last value is probably smaller than should be used in practice.

364. Maximum Moment in an Interior Column.—Fig. 28 shows the loading conditions for maximum moment in an interior column. From Table I, the moment at joint 1 of member 1–2 is:

$$M_{1-2} = \frac{2EI}{l}(2\theta_1 + \theta_2).$$

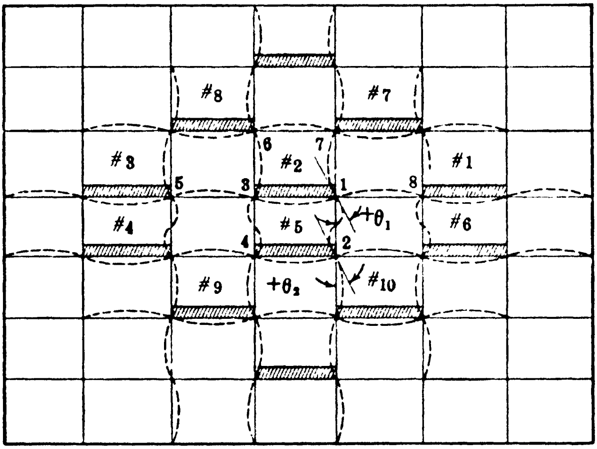


FIG. 28.

To make M_{1-2} a maximum θ_1 and θ_2 should be of the same sign. Member 1-2 must therefore be of reversed curvature, as shown by the dotted lines in Fig. 28. Loads in panels 1, 2, and 3 cause positive θ_1 and loads in panels 4, 5, and 6, cause positive θ_2. Loads in panels 7 and 10 contribute directly to positive values of θ_1 and θ_2 and loads in panels 8 and 9 contribute to negative twist angles at joints 3 and 4 and thus indirectly to positive twist angles at joints 1 and 2.

On studying the shape of the deformed frame, as shown by the dotted lines of Fig. 28 it seems reasonable to calculate M_{1-2} by considering the portion of the frame bounded by joints 2, 4, 5, 6, 7, and 8, as shown in Fig. 29. The following boundary conditions will be assumed, based on symmetry and the deformations shown by dotted lines in Fig. 28:

$$\theta_2 = \theta_1;\ \theta_3 = -\theta_1;\ \theta_4 = \theta_3 = -\theta_1;\ \theta_5 = -\theta_3 = +\theta_1;$$

$$\theta_6 = -\theta_3 = +\theta_1;\quad \theta_7 = -\theta_1;\ \text{and}\ \theta_8 = -\theta_1.$$

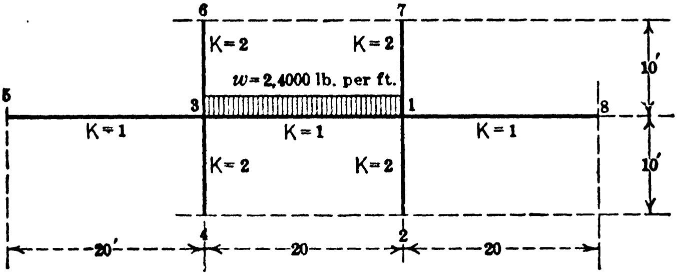


FIG. 29.

The moment equation for joint 1 is then

$$2EK_c\theta_1 + 2EK_b\theta_1 + 6EK_c\theta_1 + 2EK_b\theta_1 - C_{1\text{-}3} = 0,$$

whence

$$\theta_1 = \frac{C_{1\text{-}3}}{2E(2K_b + 4K_c)}, \quad \ldots\ldots\ldots \quad (8)$$

and

$$M_{1\text{-}3} = 6EK_c\theta_1$$

$$= C_{1\text{-}3}\frac{3K_c}{2K_b + 4K_c}. \quad \ldots\ldots\ldots \quad (9)$$

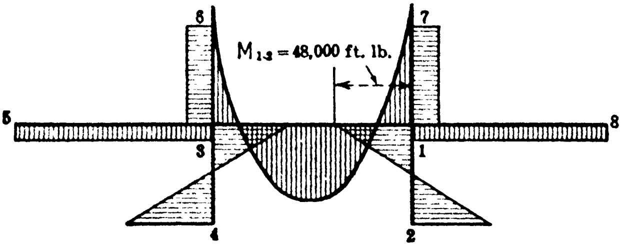


FIG. 30.

Assuming, as in the preceding examples, $K_c = 2$; $K_b = 1$ and $C_{1\text{-}3} = 80{,}000$ ft.-lbs. we have

$$M_{1\ 3} = \tfrac{6}{10} \times C_{1\text{-}3} = 48{,}000 \text{ ft. lbs.}$$

Fig. 30 shows the moment diagram for the various members.

If the loading assumed in the foregoing analysis is considered as too improbable, a more reasonable assumption would be to consider the loads on two floors only, that is, panels 1 to 6, inclusive. Under this condition we have again $\theta_1 = \theta_2$, $\theta_3 = \theta_8 = -\theta_1$, and θ_7 may be taken at some small value, say $-\frac{1}{4}\theta_1$, midway between hinged and fixed end conditions, making the restraint factor for 1–7 equal to 3.5 (see Table III). Then the moment equation for joint 1 is:

$$3.5EK_c\theta_1 + 2EK_b\theta_1 + 6E_c\theta_1 + 2EK_b\theta_1 - C_{1\text{-}3} = 0,$$

whence

$$\theta_1 = \frac{C_{1\ 3}}{4EK_b + 9.5K_c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$M_{1\text{-}3} = C_{1\text{-}3}\frac{6K_c}{4EK_b + 9.5K_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

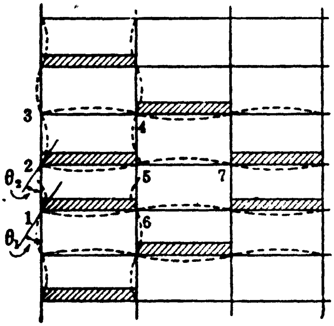


FIG. 31.

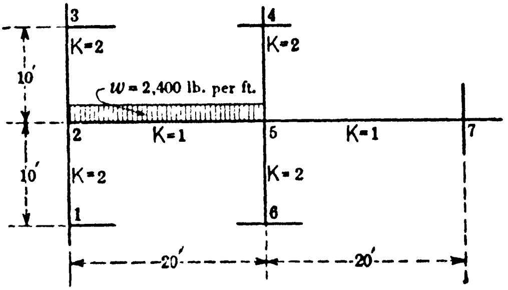


FIG. 32.

Applying this to the same example as before we have

$$M_{1\text{-}3} = \tfrac{12}{23} \times C_{1\text{-}3} = 41{,}700 \text{ ft.-lbs.}$$

This is about 13% less than the moment based on extreme assumptions. It would appear to be sufficiently high for any ordinary case.

365. Maximum Moment in an Exterior Column.—Fig. 31 shows panel loads in position for maximum moment at joint 2 in an exterior column 2–1. As in the case of an interior column, maximum moment occurs when the curvature of the member is reversed, as shown in Fig. 31. To determine the desired moment, it will be assumed that the portion of Fig. 31 bounded by 1, 3, 4, 7, and 6 may be removed, as shown in Fig. 32. Based on the deformation of the frame, as shown by the dotted lines of Fig. 31, it will be assumed that the boundary conditions in Fig. 32 are as follows:

$$\theta_3 = -\,\theta_2;\quad \theta_4 = -\,\theta_5;\quad \theta_7 = -\,\theta_5;\quad \theta_6 = \theta_5;\quad \text{and}\quad \theta_1 = \theta_2.$$

The unknowns to be determined are θ_2 and θ_5. Using the numerical data shown in Fig. 32, and writing joint equations for joints 2 and 5, we have:

Joint 2,

$$2(2 + 1 + 2)\theta_2 + 2\theta_3 + \theta_5 + 2\theta_1 = -\,\frac{C_{2-5}}{E}.$$

With $\theta_3 = -\,\theta_2$ and $\theta_1 = \theta_2$ and $C_{2-5} = \frac{1}{12}wl^2 = 80{,}000$ ft.-lbs., this equation becomes,

$$10\theta_2 + \theta_5 = -\,\frac{40{,}000}{E}. \qquad (a)$$

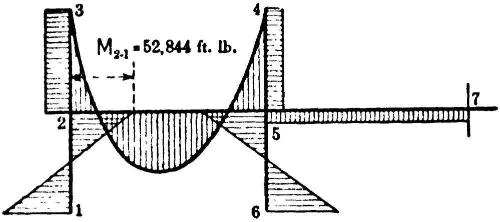


FIG. 33.

Joint 5

$$2(1 + 2 + 1 + 2)\theta_5 + \theta_2 + 2\theta_4 + \theta_7 + 2\theta_6 = +\,\frac{C_{5-2}}{E},$$

from which

$$\theta_2 + 11\theta_5 = +\,\frac{40{,}000}{E}. \qquad (b)$$

Solving eqs. (*a*) and (*b*),

$$\theta_2 = -\,\frac{4{,}403.7}{E} \quad \text{and} \quad \theta_5 = +\,\frac{4{,}036.7}{E}.$$

Then

$$M_{2-1} = 6EK_c\theta_2 = -\,12 \times 4{,}403.7 = -\,52{,}844 \text{ ft.-lbs.}$$

Fig. 33 shows the moment diagram.

If θ_5 be assumed equal to θ_2 the following values would result:

$$\theta_2 = -\,\frac{4{,}444}{E} \quad \text{and} \quad M_{2-1} = -\,53{,}333 \text{ ft.-lbs.}$$

SECTION VI.—BUILDING FRAMES UNDER HORIZONTAL LOADING

366. General Methods of Analysis of Building Frames Under Horizontal or Wind Loading.—Moments, shears, and direct stresses in the members of a frame such as shown in Fig. 34 under horizontal loading may be determined by an extension of the methods used in Sec. III for the analysis of quadrangular frames. In the frame of Fig. 34, the shearing strains due to the horizontal forces will be different for each story, giving rise to a different α for each. However, since the deformation of members due to direct stresses is neglected, the α's for the several vertical members in any one story will be equal.

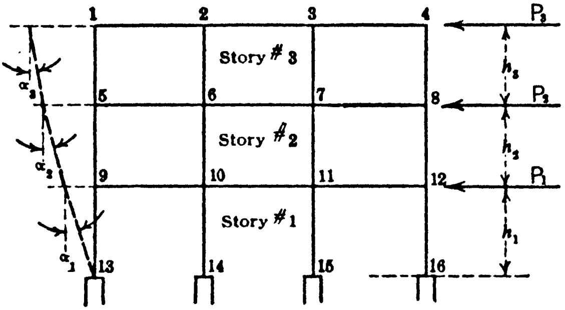


FIG. 34.

In the frame of Fig. 34, the unknowns to be determined are twelve twist angles for joints 1 to 12, and three deflection angles, one for each story. At joints 13 to 16, the twist angles depend upon the condition of the column base. If the bases are fixed, these θ's are zero; if the bases are hinged these θ's are also unknowns, but as explained in Art. 343, they are dependent upon the θ's at joints 9 to 12 respectively. To determine the unknown θ's and α's, independent equations equal in number to the unknowns must be provided. These equations may be obtained by writing an equilibrium equation after eq. (26) Art. 346 for each joint. There is also required an additional equation for each story in which there is an unknown α. This equation is obtained by equating the total shear in that story, times the story height, to the sum of the moments at the top and bottom of all verticals of that story.

Joint equations may be written out after the manner of eq. (26), Art. 346. Thus for joint 6 of Fig. 34, we have

$$\Sigma M = M_{6-2} + M_{6-7} + M_{6-10} + M_{6-5} = 0.$$

Substituting values of these moments, using eqs. (2) and (3) of Table I, this equation becomes

$$2EK_{6-2}(2\theta_6 + \theta_2 - 3\alpha_3) + 2EK_{6-7}(2\theta_6 + \theta_7)$$
$$+ 2EK_{6-10}(2\theta_6 + \theta_{10} - 3\alpha_2) + 2EK_{6-5}(2\theta_6 + \theta_5) = 0.$$

Collecting terms, and omitting $2E$, which is common to all terms, the equilibrium equation takes the form,

$$2(K_{6-2} + K_{6-7} + K_{6-10} + K_{6-5})\theta_6 + K_{6-2}\theta_2 + K_{6-7}\theta_7$$
$$+ K_{6-10}\theta_{10} + K_{6-5}\theta_5 - 3K_{6-2}\alpha_3 - 3K_{6-10}\alpha_2 = 0. \quad . \quad (1)$$

Eq. (1) may be written directly from Fig. 34 by noting that the first term of the equation is θ_6, the twist angle at the joint under consideration, multiplied by a term which is equal to twice the sum of all the K's $\left(\text{values of } \frac{I}{l}\right)$ for all members entering the joint. Next come in order the twist angles at the far end of each member entering joint 6, each twist angle being multiplied by the K value for the member. Finally, we have the values of α for vertical members entering joint 6, each α being multiplied by three times the K for the member. Note that these α terms have a minus sign, while the θ terms have plus signs. After a little practice, this equation may readily be written out from a diagram of the frame like Fig. 34 without reference to eq. (1).

In addition to joint equations like eq. (1), there are required additional independent equations for the determination of the α values, one equation being required for each α value to be determined. These additional equations may be written by considering the equilibrium of a portion of the frame formed by cutting sections across the frame just below the top and just above the bottom of all columns in any story. Thus, for the second story of Fig. 34, we have the system shown in Fig. 35.

On taking moments about a horizontal axis through the foot of the columns, noting that the sum of the shears V_{8-12}, V_{7-11}, etc., at the top of the columns is equal to the total horizontal shear above story No. 2, which from Fig. 34, is $P_3 + P_2$, we have from Fig. 35:

$$M_{5\text{-}9} + M_{6\text{-}10} + M_{7\text{-}11} + M_{8\text{-}12} + M_{9\text{-}5} + M_{10\text{-}6} + M_{11\text{-}7} + M_{12\text{-}8} + (P_3 + P_2)h_2 = 0.$$

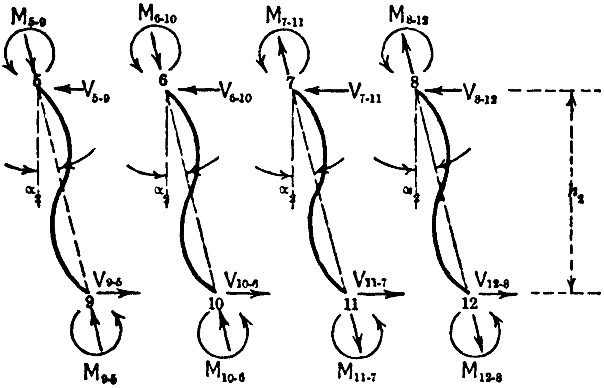


FIG. 35.

Writing out the values of these moments, using eqs. (2) and (3) of Table I, we have

$$\begin{aligned}2K_{5\text{-}9}(2\theta_5 + \theta_9 - 3\alpha_2) &+ 2K_{6\text{-}10}(2\theta_6 + \theta_{10} - 3\alpha_2)\\ &+ 2K_{7\text{-}11}(2\theta_{11} + \theta_7 - 3\alpha_2) + 2K_{8\text{-}12}(2\theta_{12} + \theta_8 - 3\alpha_2)\\ &+ 2K_{9\text{-}5}(2\theta_9 + \theta_5 - 3\alpha_2) + 2K_{10\text{-}6}(2\theta_{10} + \theta_6 - 3\alpha_2)\\ &+ 2K_{11\text{-}7}(2\theta_{11} + \theta_7 - 3\alpha_2) + 2K_{12\text{-}8}(2\theta_{12} + \theta_8 - 3\alpha_2)\\ &+ (P_3 + P_2)h_2 = 0.\end{aligned}$$

Collecting terms, noting that $K_{9\text{-}5} = K_{5\text{-}9}$, $K_{10\text{-}6} = K_{6\text{-}10}$, etc., we have finally

$$\begin{aligned}K_{5\text{-}9}(\theta_5 + \theta_9) &+ K_{6\text{-}10}(\theta_6 + \theta_{10}) + K_{7\text{-}11}(\theta_7 + \theta_{11}) + K_{8\text{-}12}(\theta_8 + \theta_{12})\\ &- 2(K_{5\text{-}9} + K_{6\text{-}10} + K_{7\text{-}11} + K_{8\text{-}12})\alpha_2 = -\frac{(P_3 + P_2)h_2}{6E}. \quad (2)\end{aligned}$$

This equation may readily be written out directly from Fig. 34 by noting that the left-hand member consists of the sum of the θ's at both ends of each member multiplied by the K for that member, minus the α for the story multiplied by twice the sum of the K's for all verticals in the story, and that the right-hand member of this equation is a negative term equal to the sum of the horizontal loads above the story in question multiplied by the story height and divided by $6E$. Thus for story No. 3, the right-hand term is $-\frac{P_3h_3}{6E}$, and for story No. 1 this term is $-\frac{(P_1 + P_2 + P_3)}{6E}h_1$.

Solving the system of equations like (1) and (2) will give all values of θ and α. Moments at the ends of the members may be determined by the formulas of Table I. Shears and direct stresses in the members may be determined by the methods explained for the example of Art. 356.

In determining values of θ and α for a frame under horizontal loading, all joints of the frame must be taken into consideration. If theoretically correct results are desired, based on the assumed conditions, it is not possible to limit the portion of the frame included in the calculations to members adjoining the one in which the stress is desired. However, the calculations are somewhat reduced in the case of symmetrical frames. Thus in Fig. 34, if the structure is symmetrical about a vertical center line with respect to dimensions of panels and cross sections of members, there will be two unknown θ's and one unknown α for each story. For the first story, θ_9 will equal θ_{12} and θ_{10} will equal θ_{11}. The unknown α will be α_1, the same for all verticals.

367. Calculations of Wind Stresses in a Five-story Building Frame.—In the problem which follows, it has been assumed that the entire resistance to wind forces is provided by the columns and beams, as represented by the combinations of rolled shapes which form these members. This is the usual assumption made in practice. No account is taken of the resistance of the floor slabs, which are usually rigidly fastened to the floor beams, or of the stiffening effect of partition and exterior walls. Under the usual assumptions, the calculated stresses in the frame are thus in many cases much greater than the actual stresses.

In view of the amount of work required in the solution of such problems, and the approximate nature of the results obtained, it would seem best to adopt one of the shorter methods suggested in Art. 368.

As will be noted in the problem which follows, the sectional areas of all members must be known before the wind stresses can be determined. A preliminary design must be made, based on vertical loading conditions. The adopted sections must be based on these preliminary calculations, modified to provide for the final stresses. These modifications can be made correctly only by an experienced calculator.

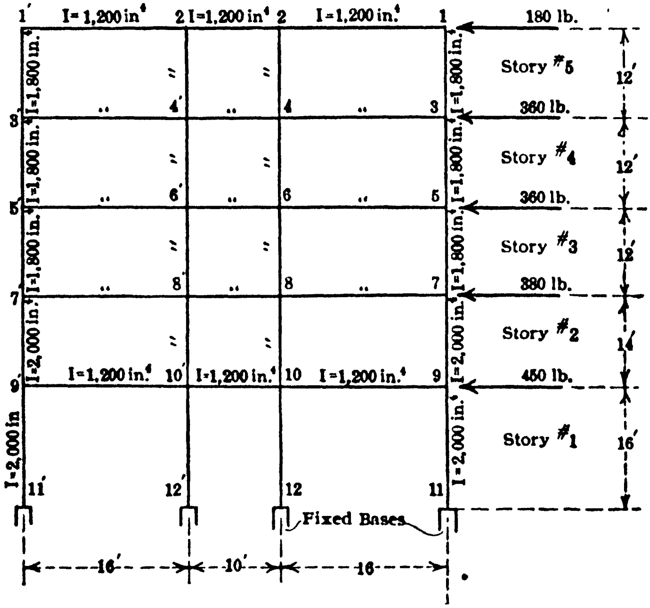


FIG. 36.

Example: Determine the stresses in the frame of Fig. 36 due to a horizontal wind pressure of 30 lb. per sq. ft. Assume that the columns are fixed at the base.

Considering a strip of the building one foot wide in a direction perpendicular to the plane of Fig. 36, the loads applied at each story are proportional to the area tributary to each joint. Thus for joint 1, the area is 6 sq. ft., and the load is $6 \times 30 = 180$ lb. For joint 7, the area is $\frac{1}{2}(12 + 14) = 13$ sq. ft., and the load is $13 \times 30 = 390$ lb. To obtain the full panel loads, multiply these values by the panel length perpendicular to the plane of Fig. 36.

Values of $K = I/l$ are shown in Fig. 37.

Symmetrical Conditions.—As stated in Art. 366, from symmetry, it can be seen that the following relations exist between the twist angles.

$$\begin{array}{lcl}
\theta'_2 = \theta_2 & & \theta'_1 = \theta_1 \\
\theta'_4 = \theta_4 & & \theta'_3 = \theta_3 \\
\theta'_6 = \theta_6 & \text{and} & \theta'_5 = \theta_5 \\
\theta'_8 = \theta_8 & & \theta'_7 = \theta_7 \\
\theta'_{10} = \theta_{10} & & \theta'_9 = \theta_9
\end{array}$$

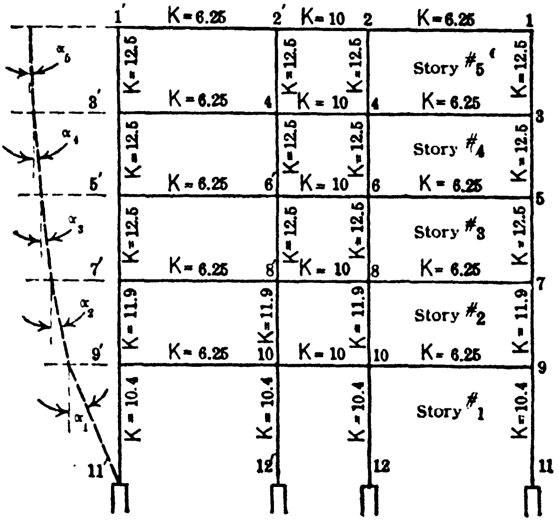


FIG. 37.

Also, since the columns are fixed at the bases, $\theta_{11} = 0$; $\theta'_{11} = 0$; $\theta_{12} = 0$; $\theta'_{12} = 0$.

Unknowns to be Determined.—The unknowns to be determined are the ten twist angles θ_1 to θ_{10} inclusive and five α's for storys 1 to 5, a total of fifteen unknowns.

Formulation of Equations.—Equations for each joint of the frame may be written after eq. (1), Art. 366, giving ten independent equations. Additional equations required for the five α values may be written after eq. (2), Art. 366.

The joint equations after eq. (1) are as follows:

Joint No. 1

$$2(6.25 + 12.5)\theta_1 + 6.25\theta_2 + 12.5\theta_3 - 3(12.5)\alpha_5 = 0$$

$$37.5\theta_1 + 6.25\theta_2 + 12.5\theta_3 - 37.5\alpha_5 = 0 \quad . \quad . \quad (1)$$

Joint No. 2

$$2(10 + 12.5 + 6.25)\theta_2 + 10\theta'_2 + 12.5\theta_4 + 6.25\theta_1 - 3(12.5)\alpha_5 = 0$$

But $\theta_2' = \theta_2$

$$6.25\theta_1 + 67.5\theta_2 + 12.5\theta_4 - 37.5\alpha_5 = 0. \quad . \quad . \quad . \quad (2)$$

Joint No. 3

$$2(12.5 + 6.25 + 12.5)\theta_3 + 12.5\theta_1 + 6.25\theta_4 + 12.5\theta_5 - 3(12.5)\alpha_5 - 3(12.5)\alpha_4 = 0$$

$$12.5\theta_1 + 62.5\theta_3 + 6.25\theta_4 + 12.5\theta_5 - 37.5\alpha_4 - 37.5\alpha_5 = 0. \quad (3)$$

Similar equations may be written for the other joints of the frame. All equations, numbered (1) to (10) are given in Table A.

Equations written for values of α after the manner of eq. (2), Art. 366, are written as follows:

5th Story. The term $P_5h_5 = 180 \times 144 = 25{,}920$ in.-lb. Then

$$6[12.5(\theta_1 + \theta_2 + \theta'_2 + \theta'_1 + \theta_3 + \theta_4 + \theta'_4 + \theta'_3) - (2)(4)(12.5)\alpha_5] = -\frac{25{,}920}{E}.$$

Noting that $\theta'_1 = \theta_1$; $\theta'_2 = \theta_2$; $\theta'_3 = \theta_3$; and $\theta'_4 = \theta_4$, we have

$$75\theta_1 + 75\theta_2 + 75\theta_3 + 75\theta_4 - 300\alpha_5 = -\frac{25{,}920}{2E}. \quad . \quad (11)$$

4th Story. The term $(P_5 + P_4)h_4 = (180 + 360)144 = 77{,}760$

$$6[12.5(\theta_3 + \theta_4 + \theta'_4 + \theta'_3 + \theta_5 + \theta_6 + \theta'_5 + \theta'_6) - (2)(4)(12.5)\alpha_4] = -\frac{77{,}760}{E}.$$

Also

$$\theta'_4 = \theta_4;\ \theta'_3 = \theta_3;\ \theta'_5 = \theta_5;\ \text{and}\ \ \theta'_6 = \theta_6,$$

hence

$$75\theta_3 + 75\theta_4 + 75\theta_5 + 75\theta_6 - 300\alpha_4 = -\frac{77{,}760}{2E}. \quad . \quad (12)$$

Equations for α values, numbered (11) to (15), are given in Table A.

TABLE A

SUMMARY OF EQUATIONS

Eq.	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	α_5	α_4	α_3	α_2	α_1	Absolute Term
1	+37.5	+ 6.25	+12.5								− 37.5					0
2	+ 6.25	+67.5		+12.5							− 37.5					0
3	+12.5		+62.5	+ 6.25	+12.5						− 37.5	− 37.5				0
11	+75.0	+75.0	+75.0	+75.0							−300.0					− 25,920
4		+12.5	+ 6.25	+92.5		+12.5					− 37.5	− 37.5				0
5			+12.5		+62.5	+ 6.25	+12.5					− 37.5	− 37.5			0
12			+75.0	+75.0	+75.0	+75.0						−300.0				− 77,760
6				+12.5	+ 6.25	+92.5		+12.5				− 37.5	− 37.5			0
7					+12.5		+61.3	+ 6.25	+11.9				− 37.5	− 35.7		0
13					+75.0	+75.0	+75.0	+75.0					−300.0			−129,600
8						+12.5	+ 6.25	+91.3		+11.9			− 37.5	− 35.7		0
9							+11.9		+57.1	+ 6.25				− 35.7	− 31.2	0
14							+71.4	+71.4	+71.4	+71.4				−285.6		−216,720
10								+11.9	+ 6.25	+87.1				− 35.7	− 31.2	0
15									+62.4	+62.4					−249.6	−334,080

NOTE.—The term $\frac{1}{2E}$ has been omitted from the right-hand term of eqs. (11) to (15).

Eqs. (1) to (15) are the independent equations from which the fifteen unknowns may be determined. In solving these equations, the term $\frac{1}{2E}$ of the right-hand members of eqs. (11) to (15) will be omitted. All values of θ and α are then $2E$ times their true values. This is done to facilitate the calculation of moments, for it will be noted from Table I, Art. 343, that all equations for moments contain the term $2E$ in the numerator.

Table A gives a summary of eqs. (1) to (15) arranged in the order in which they are encountered in the solution of the equations.

Solution of Equations.—Eqs. (1) to (15) may be solved by the method used in Art. 299, Chap. VII, for Secondary Stresses. The resulting values of θ and α are as follows:

$\theta_1 = +198.68$
$\theta_2 = +126.92$
$\theta_3 = +499.57$
$\theta_4 = +374.35$
$\theta_5 = +978.18$
$\theta_6 = +714.98$
$\theta_7 = +1565.6$
$\theta_8 = +1159.2$
$\theta_9 = +2224.6$
$\theta_{10} = +1459.0$

$\alpha_5 = +386.35$
$\alpha_4 = +900.96$
$\alpha_3 = +1535.9$
$\alpha_2 = +2360.9$
$\alpha_1 = +2259.3$

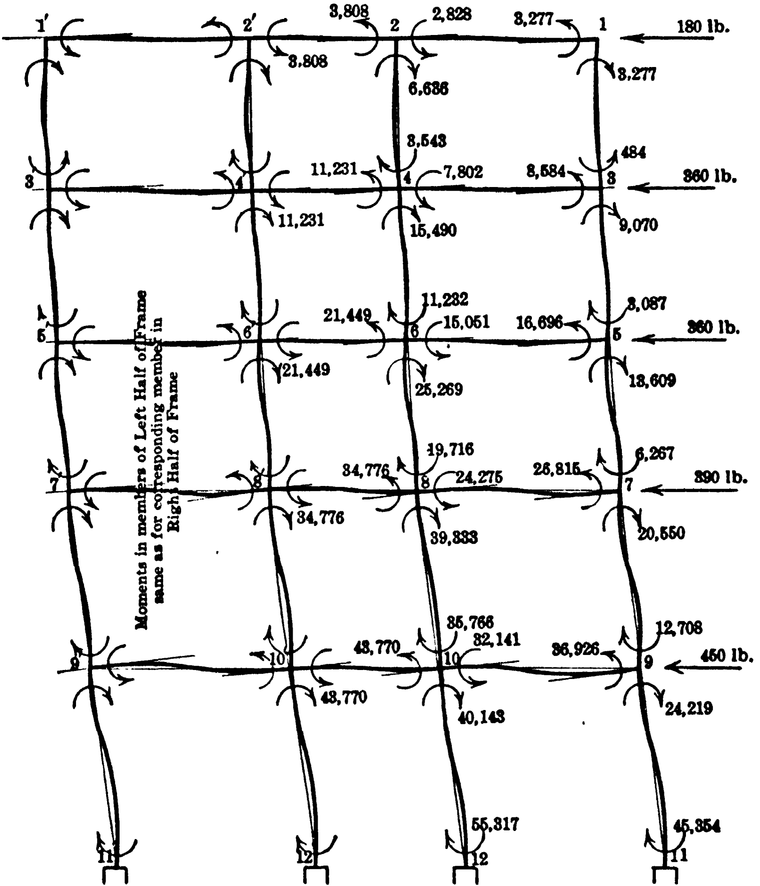
As stated before, these results are all $2E$ times their true values.

Moments in Beams and Columns.—These moments may be determined by substituting values of θ and α in eqs. (2) and (3) of Table I. It is to be noted that for the beams all α values are zero. The general equation for moments is therefore $M_1 = 2EK(2\theta_1 + \theta_2)$. Thus for member 6–5, for which $K = 6.25$ (see Fig. 38), $\theta_6 = 714.98$, $\theta_5 = 978.18$, we have

$$M_{6\text{-}5} = (6.25)[2(714.98) + 978.18] = +15{,}051 \text{ in.-lb.}$$

Moments for other members are calculated in a similar manner. These moments are shown on Fig. 38. Arrows at the ends of each member show the direction of the moment, and the figures give the amount of the moment. Fig. 38 also shows the character of the deformation of the frame.

Shears in Beams and Columns.—Shears at the ends of beams and columns may be determined from eqs. (22) and (23) of Art. 345.



Moments in inch pounds shown thus 40,143

Positive moment shown thus

Bending Moments in Beams and Columns

FIG. 38.

Noting that there are no loads between joints of the frame under consideration, these equations may be written in the form,

$$V_1 = - V_2 = \frac{M_1 + M_2}{l}.$$

Thus for member 5–6,

$$M_{6\text{-}5} = +\ 15{,}050.9 \quad \text{and} \quad M_{5\text{-}6} = +\ 16{,}695.9.$$

Then

$$V_{6\text{-}5} = \frac{15{,}050.9 + 16{,}695.9}{192} = 165.2 \text{ lb.} \quad \text{Also } V_{5\text{-}6} = -\ 165.2 \text{ lb.}$$

Shears for other members are calculated in a similar manner. Fig. 39 shows the calculated shears in position on the frame. Arrows indicate the direction of the shears as acting on the joints and the figures state the amount of the shear.

Direct Stresses in Beams and Columns.—Direct Stresses in the members are shown on Fig. 39 by the figures in parentheses. Thus for members 5–6, 6–6′ and 6′–5′, the stresses may be determined by considering the equilibrium of each joint. For member 5′–6′ at joint 5′, we have:

Stress in $5'\text{–}6' = 138.02 - 84.42 = 53.60$ compression.

For member 6–6′, at joint 6′, we have:

Stress in $6\text{–}6' - 53.60 + 185.57 - 312.40 = 180.43$ compression.

For member 6–5 at joint 6, we have:

Stress in $6\text{–}5 = 180.43 - 185.57 + 312.40 = 307.26$ compression.

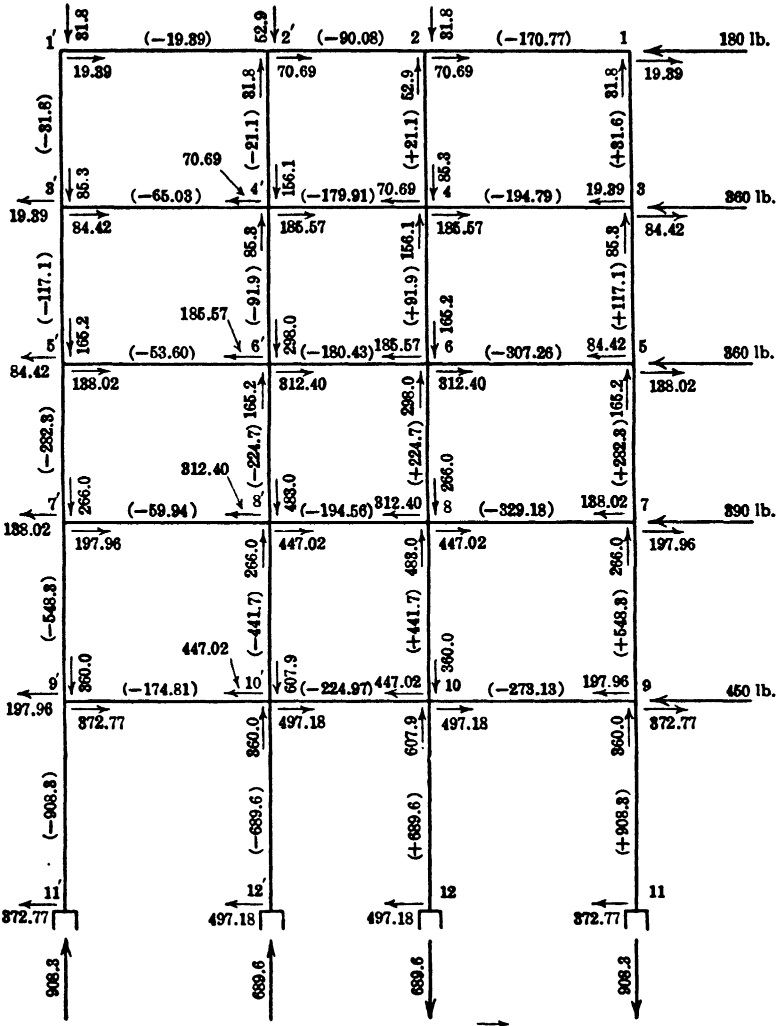
As a check on the calculations, the shears in the columns above and below joint 5 and the stress in member 5–6 must equal the joint load at 5. In the case under consideration we have:

$$307.26 + 138.02 - 84.42 = 360.86,$$

which indicates that there is a slight error in the calculations, for the load at joint 5 is 360 lb. Stresses in other members were determined in a similar manner.

Determination of Reactions.—Reactions at joints 11, 12, 12′ and 11′ are equal to the stresses in the lower story of the columns. On Fig. 39, these reactions are shown in amount and direction at the several joints.

368. Approximate Solutions.—The solution for stresses in a building frame under horizontal or wind loading, presented in Art. 366, is based on the assumption that the steel frame of the building acting



Notes:- Shear in members shown thus $\overrightarrow{372.77}$
Direct Stress in members shown thus (-174.81)
+ = Tension
— = Compression
Shears and Stresses given in pounds

Shears and Direct Stresses in Beams and Columns

FIG. 39.

alone resists the wind forces. No account is taken of the stiffening effect of walls and partitions. This stiffening effect is likely to be very great and therefore the method of Art. 366 does not represent true conditions. In fact, it is impossible to express mathematically the stiffening effect of walls and partitions.

It will be noted on examining the problem given in Art. 367, that the solution is very long and involved. In view of the fact that the results obtained by this method are only approximate, it is usual, in practice, to calculate wind stresses in building frames by approximate methods based on assumptions which will result in an economical design wherein the sections provided have been found by experience to provide for all possible stresses.

Approximate methods for the determination of stresses in building frames under wind loading may be divided into two general cases. The first case will include frames which are formed of beams and columns which are relatively flexible. Bending stresses are to be calculated for both beams and columns. This case will cover buildings with steel frames in which it is assumed that all wind forces are resisted only by the frame, no account being taken of the stiffening effect of the floor system. A second case will include building frames in which the floor system consists of concrete slabs and beams which provide a very rigid system, all of which may reasonably be assumed to act as horizontal beams. In this case, the beams are very rigid as compared to the columns.

a. Frames with Flexible Beams and Columns.—In general, it will be found that the important stresses in a building frame under horizontal loading are those due to the bending moments in the beams and columns. For maximum economy in material, the moments at both ends of all beams in any floor should be equal, and also, the moments in both ends of all columns in any story should be equal. This condition requires that the points of inflection in all beams and columns should be located at their respective mid-points. Again, in order to resist the overturning effect of the wind on the building, the reactions on the outside columns should be large, the greatest resisting effect being obtained when all of the vertical resisting forces are provided by the outside columns.

Conditions approximating those stated above may be secured by assuming a proper division of the horizontal shear between the columns of any story. Then, based on this assumed distribution of shear, all moments and direct stresses may readily be calculated. If proper provision is made for these calculated stresses, the frame will be found to provide adequate resistance to the wind forces. If any portion of the frame is overstrained by the actual loading, these overstrains will result is a slight increase in deflection which will distribute more of the load to nearby portions of the frame which are understressed.

In Proceedings of the American Society of Civil Engineers for January, 1929, pages 189 to 199, Mr. David A. Molitor has proposed an approximate solution of the type mentioned above. This solution gives results which agree fairly well with those given by the more exact method of Art. 366. In applying this approximate solution, it is not necessary that the sizes of all members be known. Also, any portion of the frame may be separated from the rest of the structure and the stresses may be determined for this portion of the frame. Thus the amount of work required for the calculations is greatly reduced.

This approximate method is based on the following assumptions:

(1) Columns have points of inflection at mid-story heights.

(2) Beams have points of inflection at mid-span points.

(3) Wind loads are divided so that interior columns have equal shears of twice the amount taken by each exterior column. All beam moments of the same floor are equal.

In a building frame with bays or panels of equal length, all beam shears on any floor will be equal. Hence only the exterior columns will carry direct loads due to wind. If the bays are unequal in length. the shears in the several beams at any floor level will differ; they will be inversely proportional to the lengths of the bays. In this case the interior columns will also carry direct loads, but in general these loads will be smaller than those on the exterior columns.

Fig. 40 (*a*) shows a portion of a building frame in which the horizontal shears are distributed as assumed above. On a line *a–a* the total shear is $W = W_1 + W_2$. If there are n columns at any story level, the shear on an interior column is $2V = \dfrac{W}{n - 1}$ and the shear on an exterior column is $V = \frac{W}{2(n-1)}$. In Fig. 40 there are shown

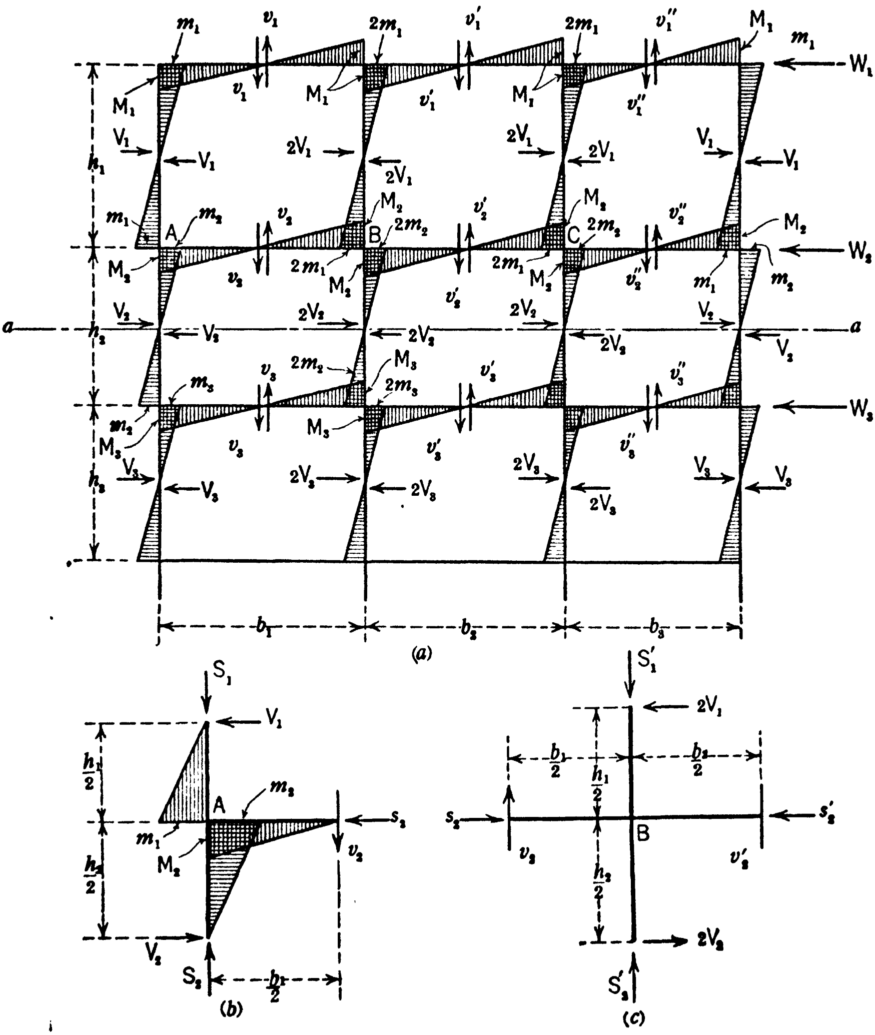


FIG. 40.

four columns at section a–a. Hence the shear for each interior column is $2V_2 = \frac{W}{3}$, and for the exterior columns, the shear is $V_2 = \frac{W}{6}$.

To determine the vertical shear v_2 in beam AB, remove the portion of the frame shown in Fig. 40 (*b*) and take moments about A, from which $v_2 = \frac{V_1h_1 + V_2h_2}{b_1}$. For beam BC, the shear v'_2 is found by taking moments about B of Fig. 40 (*c*), from which

$$v'_2 = \frac{V_1h_1 + V_2h_2}{b_2} = v_2\frac{b_1}{b_2}.$$

From Fig. 40, moments M_2, which are equal for all beams at the floor level A, B, C, D, are found to be $M_2 = v_2\frac{b_1}{2} = \frac{1}{2}(V_1h_1 + V_2h_2)$. For an exterior column $m_1 = V_1\frac{h_1}{2}$; $m_2 = V_2\frac{h_2}{2}$, etc., and for an interior column, the moments are twice the values for the exterior column in the same story.

Direct stresses in exterior columns are equal to the sum of the beams shears for all floors above the column in question. Thus from Fig. 40 (*b*), $S_2 = S_1 + v_2$. But $S_1 = v_1$. Hence $S_2 = v_1 + v_2$. For an interior column, we note from Fig. 42 (*c*) that

$$S_2' = S_1' + v'_2 - v_2 = (v'_1 + v'_2) - (v_1 + v_2) = \left(\frac{b_1}{b_2} - 1\right)(v_1 + v_2)$$

Hence for equal bays, $S'_2 = 0$. Similar values may be obtained for other columns.

Direct stresses in beams are determined in a similar manner. Thus from Fig. 40 (*b*),

$$S_2 = V_2 - V_1 = \frac{W_2}{2(n - 1)}.$$

From Fig. 40 (*c*)

$$S'_2 = \frac{3}{2}\frac{W_2}{(n - 1)}.$$

Values for the other beams may be determined in a similar manner.

Applying this approximate method to the problem solved by the slope-deflection method in Art. 367, we have for the third story columns of Fig. 37,

$$W = (180 + 360 + 360) = 900 \text{ lb.}$$

Then $2V_3 = \frac{900}{4-1} = 300$ lb. shear for interior column 6–8. For exterior column 5–7, the shear is $V_3 = 150$ lb. For the second story columns, $2V_2 = 180$ lb. and $V_2 = 90$ lb. The shear in beam 5′–6′ is

$$v_3 = \frac{V_2h_2 + V_3h_3}{b_1} = \frac{90 \times 12 + 150 \times 12}{16} = 180 \text{ lb.}$$

and the shear in the center panel beam 6′–6 is $v_2\frac{b_1}{b_2} = 180(\frac{16}{12}) = 240$ lb.
The moment in the exterior column 5–7 is

$$m_{5\text{-}7} = \frac{V_3h_3}{2} = 150 \times 72 = 10{,}800 \text{ in.-lb.,}$$

and for the interior column 6–8 the moment is $m_{6\text{-}8} = 2m_{5\text{-}7} = 21{,}600$ in.-lb. All beam moments are equal. For beam 5′–6′ the end moment is

$$M_{5'\text{-}6'} = v_3 \times 96 = 180 \times 96 = 17{,}280 \text{ in.-lb.}$$

On comparing the values calculated above by the approximate method with those calculated by the slope-deflection method and given in Figs. 38 and 39, it will be found that the results agree fairly well. The calculated values are as follows:

	Member	Slope Deflection	Approximate
Column Shears............	3′–5′	84.4 lb.	90 lb.
	5′–7′	138.1 lb.	150 lb.
	4′–6′	185.4 lb.	180 lb.
	6′–8′	312.4 lb.	300 lb.
Column Moments..........	5′–7′	Top 13,600 in.-lb. Base 6,270 in.-lb.	10,800 in.-lb.
	6′–8′	Top 25,300 in.-lb. Base 19,700 in.-lb.	21,600 in.-lb.
Beam Moments...........	5′–6′	16,700 in.-lb.	
	6′–5′	15,100 in.-lb.	17,280 in.-lb.
	6′–6′	21,400 in.-lb.	17,280 in.-lb.

It will be noted that the column shears agree very well. Column moments calculated by the approximate method are very nearly an average of the two end moments calculated by the slope deflection method. The difference in the moments calculated by the two methods is due to the fact that the true points of inflection are not located at the column centers, as assumed in the approximate solution. From the values given above for the slope deflection method, the true position of the inflection point for column 5′–7′ is found be at the 0.683 point, measured from the top of the column, and for column 6′–8′, the true inflection point is at the 0.562 point. Hence the moments at the tops of the columns are respectively 0.683/0.50 = 127 per cent and 0.562/0.50 = 112 per cent of the average of the top and bottom moments. Similar calculations made for columns in other storys will show that these constants vary with the position of the story, being smaller for lower storys and higher for upper storys.

It would therefore seem possible from a study of building frames analyzed by the slope-deflection method, to arrive at some conclusion regarding the probable change in position of inflection points and the corresponding increase of maximum over average moment for the columns in the several storys of typical building frames. Then moment values calculated by the approximate method of this article could be taken as representing average moments which, when modified by the assumed percentages, would give a fairly accurate estimate of true moments.

b. Structures with Rigid Floors.—As stated in Art. 366, many modern buildings are so constructed that the entire floor may be considered as a single member, for the beams and slabs are poured as a unit in concrete buildings, and in a steel frame building, the beams are encased in concrete poured at the same time the floor slabs are formed. Under such conditions, the floors are very rigid as compared to the columns.

Observations made on steel frame buildings * partially wrecked during the hurricane which swept over southern Florida in September, 1926, seem to indicate that the above statement is true. It was noted that the floors were practically uninjured while the steel columns were bent sharply above and below each floor. This distortion seems to indicate that the floor acts as a unit and that it is so rigid, compared to the columns, that the twist angles at the joints may be taken equal to zero.

---

* See article by Mr. F. E. Schmitt in Engineering News-Record for October 14, 1926, pages 624 to 626.

When the twist angles may be taken equal to zero, the α angles in each story are dependent only upon the horizontal shear due to loads above that story. Hence α values may be determined separately for each story.

Placing all θ's = 0 in eq. (2), Art. 366, we have for the second floor columns of Fig. 34,

$$\alpha_2 = \frac{(P_3 + P_2)h_2}{12E(K_{5-9} + K_{6-10} + K_{7-11} + K_{8-12})}.$$

Then from eq. (2), Table I, we have for any column, as 8–12, Fig. 34

$$M_{8-12} = 2EK(-3\alpha_2) = -\frac{(P_3 + P_2)h_2K_{8-12}}{2(K_{5-9} + K_{6-10} + K_{7-11} + K_{8-12})}.$$

From this equation, it can be seen that the top and bottom moments in any column are equal, and the total moment on any column is to the total moment carried by all columns at any story level as the K for the column in question is to the sum of the K's for all columns at that story level. It follows also that the shears in the columns of any story are proportional to the value of K for the column.

Applying this approximate method to the third story of the frame of Fig. 37, we note that the K's for all columns are equal; that the shear above the section is 360 + 360 + 180 = 900 lb., and that the story height is 12 ft. or 144 in. Then

$$M_{6'-8'} = M_{5'-7'} = \tfrac{1}{8}(900)(144) = 16{,}200 \text{ in.-lb.}$$

Note that this moment is an average value for the moments in these two members in the preceding calculation. The moments for all columns in the third story will be the same. This even distribution was not indicated by the approximate solution first stated, nor by the slope deflection method.

Under the assumptions here made the maximum moment in the floor beam will be at its junction with an exterior column and will be equal to $\frac{1}{2}(V_1h_1 + V_2h_2)$ when V_1 and V_2 are the assumed shears in the columns above and below the beam in question.

SECTION VII.—OPEN WEBBED GIRDERS

369. General Methods of Analysis for Open Webbed Girders.—Open webbed girders, or frames without diagonal bracing, as shown in Fig. 41, are often used in building construction and occasionally in concrete highway bridges. Frames of this type are analyzed by an extension of the methods used in Sec. III for Rectangular Frames. It will generally be found best, in solving problems concerning this type of frame, to write out the joint equations, substituting numerical values of K. On solving these equations values of θ and α may be determined. General values for θ and α are not readily determined as the expressions involved become too cumbersome for practical purposes. The following problem illustrates the method of precedure in the case of a simple general problem.

370. Example.—Calculate and draw influence lines for moments in the chords and vertical members of the open webbed frame of Fig. 41. Assume that the panels are square and that all members have the same cross section. Hence, one may write $K = 1$ for all members, as shown on Fig. 41. Since the frame is symmetrical $\theta_2 = \theta_1$; $\theta_3 = \theta_4$; $\theta_6 = \theta_5$; and $\theta_7 = \theta_8$. Neglecting the longitudinal deformation of the members, the α values for all vertical members, which will be denoted by α_1, are equal. Value of α for the chord members will differ for each panel, but the α's for top and bottom members will be equal. Fig. 41 shows the values assigned to these α's. In this frame all values needed for the construction of the influence lines may be determined by placing a one-pound load at Joint 5.

There are eight unknowns to be determined, as follows: θ_1, θ_4, θ_5, θ_8, α_1, α_2, α_3 and α_4. To determine these unknowns, eight equations are required. Four of these equations may be obtained by writing joint equations after the manner of eq. (22) for joints 1, 4, 5, and 8. These equations are as follows:

Joint 1

$$5\theta_1 + \theta_4 - 3\alpha_1 - 3\alpha_2 = 0. \quad \ldots \quad (1)$$

Joint 4

$$\theta_1 + 7\theta_4 + \theta_5 - 3\alpha_1 - 3\alpha_2 - 3\alpha_3 = 0. \quad \ldots \quad (2)$$

Joint 5

$$\theta_4 + 7\theta_5 + \theta_8 - 3\alpha_1 - 3\alpha_3 - 3\alpha_4 = 0. \quad \ldots \quad (3)$$

Joint 8

$$\theta_5 + 5\theta_8 - 3\alpha_1 - 3\alpha_4 = 0. \quad \ldots \quad (4)$$

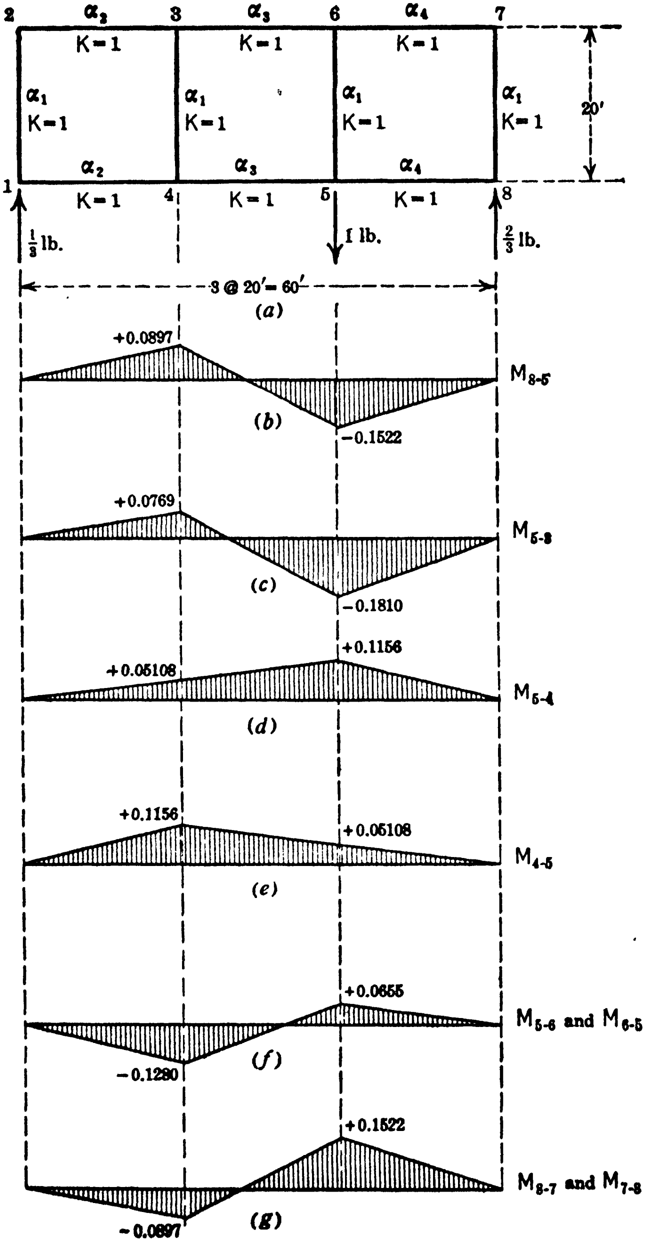

2
α₂
3
α₃
6
α₄
7
K=1
K=1
K=1
α₁
K=1
α₁
K=1
α₁
K=1
α₁
K=1
20'
α₂
α₃
α₄
1
K=1
4
K=1
5
K=1
8
⅓ lb.
1 lb.
⅔ lb.
3 @ 20' = 60'
(a)
+0.0897
−0.1522
M8-5
(b)
+0.0769
−0.1810
M5-8
(c)
+0.05108
+0.1156
M5-4
(d)
+0.1156
+0.05108
M4-5
(e)
+0.0655
−0.1280
M5-6 and M6-5
(f)
+0.1522
−0.0897
M8-7 and M7-8
(g)

FIG. 41.

Four additional equations are required for the determination of the α values. Three of these equations may be obtained in the manner used in deriving eq. (6) of Art. 352. For the case under consideration

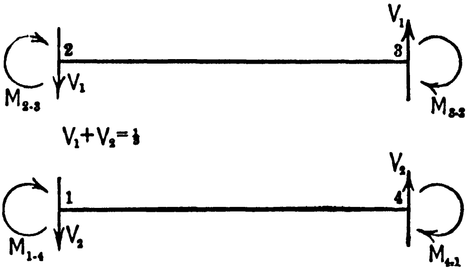


FIG. 42.

vertical sections are cut at each end of the panels. Fig. 42 shows the conditions for panel 1–4. Note that the total shear in panel 1–4 is $\frac{1}{3}$. If the panel length of 20 ft. is taken as unity, the several equations are as follows:

Panel 1–4

$$\theta_1 + \theta_4 - 2\alpha_2 = +\tfrac{1}{36}. \qquad (5)$$

Panel 4–5

$$\theta_4 + \theta_5 - 2\alpha_3 = +\tfrac{1}{36}. \qquad (6)$$

Panel 5–8. Noting that the shear in panel 5–8 is $-\frac{2}{3}$, we have

$$\theta_5 + \theta_8 - 2\alpha_4 = -\tfrac{1}{18}. \qquad (7)$$

A fourth equation may be written by noting that the deflection of joint 8 with respect to joint 1 is zero. This condition may be stated in terms of α values by multiplying each panel length by its α value and adding these products. Again assuming panel lengths equal to unity, we have

$$\alpha_2 + \alpha_3 + \alpha_4 = 0. \qquad (8)$$

On solving these eight equations, the following value of θ and α are obtained.

$$\begin{array}{ll} \theta_1 = -0.013212 & \alpha_1 = +0.0017400 \\ \theta_4 = -0.019598 & \alpha_2 = -0.030293 \\ \theta_5 = +0.012656 & \alpha_3 = -0.017360 \\ \theta_8 = +0.027104 & \alpha_4 = +0.047652 \end{array}$$

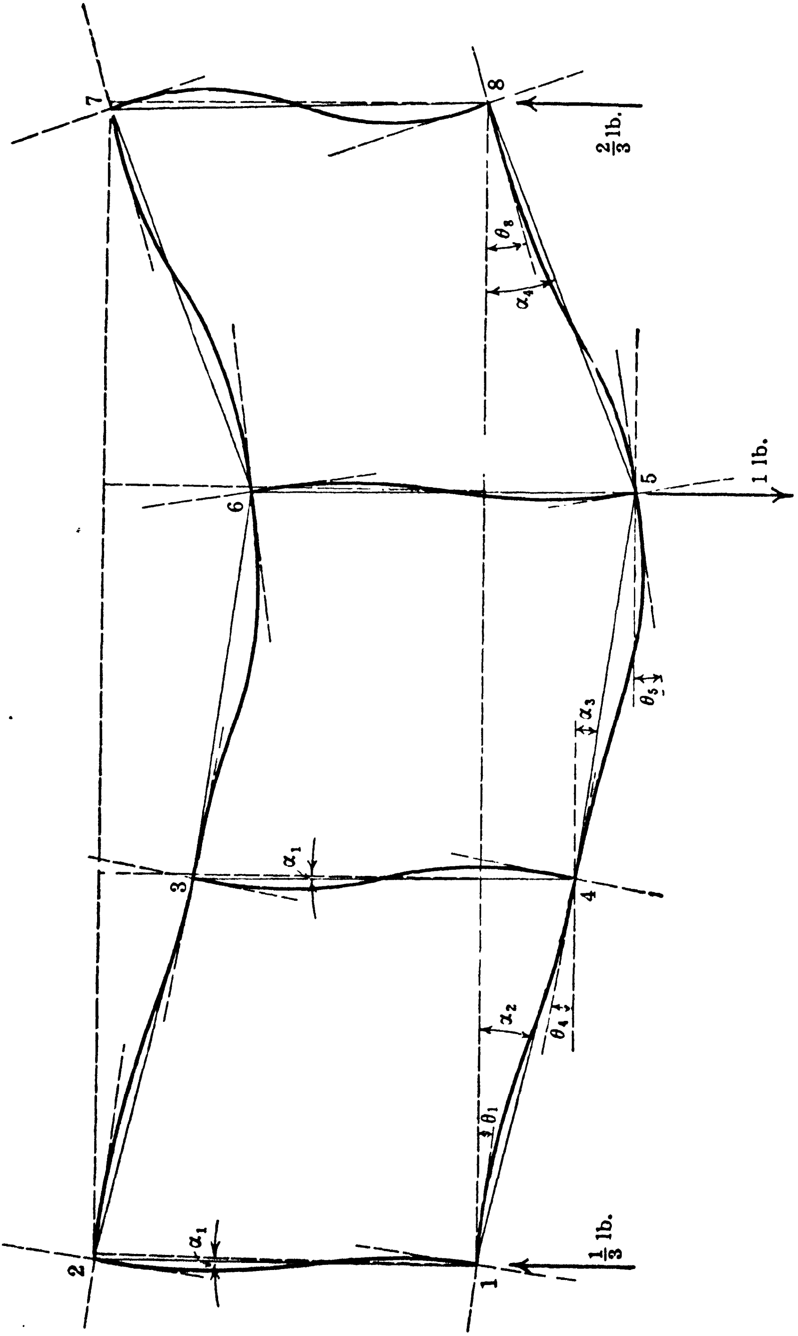

7
8
$\frac{2}{3}$ lb.
θ_8
α_4
6
5
1 lb.
θ_5
α_3
α_1
3
4
α_2
θ_4
θ_1
α_1
2
1
$\frac{1}{3}$ lb.

FIG. 43.

On substituting these values in eqs. (2) and (3) of Table I, the moments at the ends of the several numbers are as follows:

Joint 1

$M_{1\text{-}2} = -0.089712$

$M_{1\text{-}4} = +0.089714$

Joint 5

$M_{5\text{-}4} = +0.115588$

$M_{5\text{-}6} = +0.065496$

$M_{5\text{-}8} = -0.181080$

Joint 4

$M_{4\text{-}3} = -0.128028$

$M_{4\text{-}5} = +0.051080$

$M_{4\text{-}1} = +0.076942$

Joint 8

$M_{8\text{-}5} = -0.152184$

$M_{8\text{-}7} = +0.152184$

Moments at the upper chord joints will be the same as for the corresponding lower chord joints. Fig. 43 shows the deformed frame.

Influence lines for the several members, plotted from the above moments, as shown in Figs. 41 (*b*) to (*g*).

SECTION VIII.—SECONDARY STRESSES

371. General Methods of Analysis for Secondary Stresses.—In applying the general method of analysis of the present chapter to the determination of secondary stresses in trusses, equilibrium equations after eq. (26), Art. 346, may be written for each joint of the frame. These equations will contain as unknowns the twist angle θ for each joint of the truss and the deflection angle for each member. However, by assuming that the deflection of the structure is due only to the axial distortion of the members under their primary stresses and in no part due to the secondary stress bending moments in the members, and assuming further that the primary stresses are unaffected by the secondary bending moments, the deflection angles α for all members may be determined algebraically by the application of the formulas of Art. 279, or graphically by means of Williot Diagrams. On substituting the numerical values of α in the joint equations, there results a system of simultaneous equations equal in number to the unknown θ's. A solution of these equations gives all θ values from which the secondary bending moments or stresses are readily determined.

In the articles which follow, general methods will be given for the determination of α values by algebraic and graphical methods. An example will then be worked out in detail to illustrate the application of the Slope-deflection Method to secondary stress analysis. The determination of α values will first be made by the algebraic method. Then the α values will be determined graphically. In general, it will be found that more precise results can be determined by algebraic methods, as the accuracy of the results obtained by the graphical method depends upon the precision with which the necessary quantities can be scaled from a Williot Diagram. It is recommended that the algebraic method be adopted as the standard.

372. Determination of α Values by Means of Angle Changes.—Fig. 44 shows two consecutive triangles of a truss with the several angle changes $\delta\angle 1$, etc., determined by the methods given in Art. 276. The α values for the several members are as indicated.

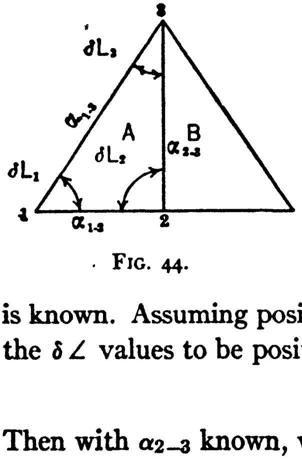


FIG. 44.

If any member, as 1–3, be chosen as a reference member, and its deflection angle α_{1-3} be taken as zero, or any assigned or calculated value, it is evident that the α value for any other member, as 2–3, can be determined as soon as the value of $\delta\angle 3$ is known. Assuming positive α's to be counter-clockwise and assuming the $\delta\angle$ values to be positive, we have

$$\alpha_{2-3} = \alpha_{1-3} + \delta\angle 3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then with α_{2-3} known, we have

$$\alpha_{1-2} = \alpha_{2-3} + \delta\angle 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In order to assist in keeping track of signs it will be found convenient in determining α values to begin with the reference member and take the members in order as they are encountered in going around the triangle in a clockwise direction. This order was adopted in deriving eqs. (1) and (2).

For the next triangle, *B*, Fig. 44, the values may be obtained in a similar manner, using the value of α_{2-3} as determined from triangle *A* as the reference α. The method is fully illustrated in the problem of Art. 375.

By arbitrarily selecting a particular member as a reference member, in the manner explained above, it is obvious that the α values thus determined will not be the true deflection angles of the various truss members, and hence the resulting twist angles θ will not be the true twist angles. They will be, however, correct relative values and for the case of the simple span truss, hinged at supports, the bending moments resulting from the use of these α and θ values will be correct. Where the true α and θ values need to be known, for any reason, as in the case of fixed ends or an arch or continuous girder, the true values may be obtained by applying a constant correction equal to the actual deflection angle of the reference member, and determined by the condition that the deflection angle of a line joining the joints at the supports is equal to zero. This point is illustrated in Art. 380.

From a study of the effect of α values on the twist angles as determined from the joint equations, it will be found that the variation in magnitude between the twist angle values is greatly affected by the choice of the initial α angle. In general, the twist angle values will differ least in magnitude when the member selected as the reference member is one whose true deflection angle is small. While the selection of the reference member has no effect on the resulting moments and stresses, it will be found that the calculations involved can be carried out with greater ease, for any desired degree of precision, if the α and θ values are fairly uniform in magnitude. In the case of a symmetrical structure symmetrically loaded, a chord or web member whose actual rotation is known to be zero should be chosen as the reference member. In a case of unsymmetrical loading conditions, some member near the truss center should be selected as the reference member. For a Pratt truss, select the center vertical and for a Warren truss, select the center chord member.

373. Determination of α Values by Means of Williot Diagrams.—To apply this method, a Williot Diagram is drawn for the truss and the arcs through which the ends of the members move are scaled from the diagram as explained in Art. 221, Part I. On dividing these arcs by the lengths of the corresponding members, the α values are determined in amount and the direction of rotation is determined from the Williot Diagram.

If the bending of the truss is symmetrical about some member, as in the case of a truss symmetrical in form and loading conditions with respect to a member near the truss center, or if it is known that some member moves parallel to itself during the deflection of the truss, this member may be chosen as the reference member and the Williot Diagram drawn as explained in Art. 219, Part I. All α values determined under these conditions represent true values of the rotation of the members. When the bending is unsymmetrical, so that there is no member whose direction is unchanged, any member may be selected as the reference member for the construction of the Williot Diagram. However, the selection of the reference member should be governed by the conditions stated in Art. 372. In Art. 379 this method has been applied to the solution of a problem. If the true values of θ and α are desired in any case, they may be determined by means of a Mohr correction diagram as explained in Art. 221, Part I, or by the method explained in Art. 380.

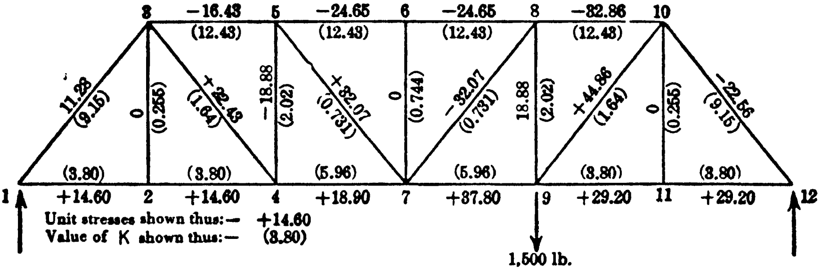


FIG. 45.

374. Determination of Secondary Stresses in a Pratt Truss.—Determine the secondary stresses in the truss of Art. 291 using the loading conditions of Case II. Fig. 45 shows the truss and its loading and also on each member is shown the unit stress in pounds per square inch. These values were taken from Fig. 16, Chapter VII. Values of K for each member, taken from Table A, Art. 291, are also shown on Fig. 45. The quantity E will be omitted from the calculations as usual.

375. *Determination of α Values by Means of Angle Changes.*—Fig. 46 shows the angle changes for the given loading. These values were taken directly from Table B 2 of Art. 292. Following the method out-

lined in Art. 372, taking member 6–7 as a reference member with $\alpha_{6-7} = 0$, we have for Triangle F Fig. 46,

$$\alpha_{6-8} = \alpha_{6-7} + (-43.64) = -43.64$$

$$\alpha_{8-7} = \alpha_{6-8} + 37.26 = -43.64 + 37.26 = -6.38$$

Triangle G:

$$\alpha_{8-9} = \alpha_{7-8} + 60.10 = -6.38 + 60.10 = +53.72$$

$$\alpha_{9-7} = \alpha_{8-9} + (-119.30) = +53.72 - 119.30 = -65.58$$

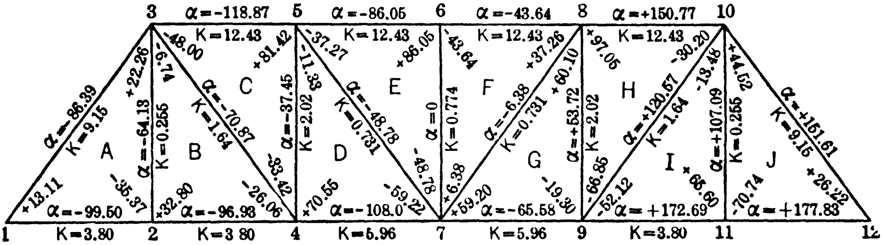


Fig. 46.

Proceeding in a similar manner for all triangles successively, the α values are as shown on Fig. 46. Values of α, K, and $K\,\alpha$ are given in Table A.

376. *Formulation of Equations.*—Writing the general equilibrium equation for any joint as given by eq. (26), Art. 346, in general form for a member m–n entering joint n, we have

$$2\,(\Sigma K)\,\theta_n + \Sigma\,(K\,\theta_m) = 3\,\Sigma\,(K_{nm}\,\alpha_{nm})$$

That is, the general joint equation is composed of a left-hand term consisting of the θ for the joint in question multiplied by twice the sum of the K's for all members entering the joint, and the sum of such terms as the θ at the far end of each member entering the joint multiplied by its K value. The right-hand term of the equation consists of a term equal to three times the sum of such terms as the product of the α angle for each member entering the joint and its K value. Thus for any joint, as for example Joint 9, the general equation is found in the following manner, using the $K\,\alpha$ values given in Table A.

TABLE A

Values of α, K, and $K\,\alpha$

Member	α	K	$K\,\alpha$	Member	α	K	$K\,\alpha$
1–3	− 86.39	9.15	− 790.5	3–5	−118.87	12.43	−1477.6
2–3	− 64.13	0.255	− 16.35	5–6	− 86.05	12.43	−1069.6
3–4	− 70.87	1.64	− 116.2	6–8	− 43.64	12.43	− 542.5
4–5	− 37.45	2.02	− 75.65	8–10	+150.77	12.43	+1874.1
5–7	− 48.78	0.731	− 35.66	1–2	− 99.50	3.80	− 378.1
7–6	0	0.774	0	2–4	− 96.93	3.80	− 368.3
7–8	− 6.38	0.731	− 4.66	4–7	−108.0	5.96	− 643.7
8–9	+ 53.72	2.02	+ 108.5	7–9	− 65.58	5.96	− 390.9
9–10	+120.57	1.64	+ 197.7	9–11	+172.69	3.80	+ 656.2
10–11	+107.09	0.255	+ 27.3	11–12	+177.83	3.80	+ 675.8
10–12	+151.61	9.15	+1387.2				

$$
\begin{aligned}
&2\,(5.96 + 2.02 + 1.64 + 3.80)\,\theta_9 \\
&\qquad + 5.96\,\theta_7 + 2.02\,\theta_8 + 1.64\,\theta_{10} + 3.80\,\theta_{11} \\
&\qquad\quad = 3\,(-\,390.9 + 108.5 + 197.7 + 656.2)
\end{aligned}
$$

Collecting terms:

$$5.96\,\theta_7 + 2.02\,\theta_8 + 26.84\,\theta_9 + 1.64\,\theta_{10} + 3.80\,\theta_{11} = +\,1{,}714.5$$

Equations for other joints, formulated in the same manner, are given in Table B.

377. *Solution of Equations.*—On comparing the equations given in Table B with those given in Table D, Art. 294, it can be seen that they differ only in the values of the absolute terms. Hence a solution of these equations may be made by means of a tabulation similar to that shown in Table E, Art. 295, the only difference being in the column of absolute terms. Values of θ determined in such a manner are given in the lower line of Table B.

An approximate method of solving equations like those of Table B, based on successive approximations, will be found to provide a much shorter and fairly precise method of solution. This approximate solution is carried out by assuming first that all θ values are equal. Based on this assumption, an approximate value of θ_1 may be determined from eq. (1); θ_2 from eq. (2); etc. Having these first approximate values, a more exact value of θ_1 may be obtained from eq. (1) by sub-

TABLE B

Equations and Values of θ

Equation	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	Absolute Term
1	25.90	3.80	9.15	......	......	......	......	......	......	......	......	......	− 3,505.8
2	3.80	15.72	0.255	3.80	......	......	......	......	......	......	......	......	− 2,288.4
3	9.15	0.255	46.96	1.64	12.43	......	......	......	......	......	......	......	− 7,202.1
4	......	3.80	1.64	26.84	2.02	......	5.96	......	......	......	......	......	− 3,611.7
5	......	......	12.43	2.02	55.22	12.43	0.731	......	......	......	......	......	− 7,975.8
6	......	......	......	......	12.43	51.26	0.774	12.43	......	......	......	......	− 4,836.3
7	......	......	......	5.96	0.731	0.774	28.30	0.731	5.96	......	......	......	− 3,225.0
8	......	......	......	......	......	12.43	0.731	55.22	2.02	12.43	......	......	+ 4,306.2
9	......	......	......	......	......	......	5.96	2.02	26.84	1.64	3.80	......	+ 1,714.5
10	......	......	......	......	......	......	......	12.43	1.64	46.96	0.255	9.15	+10,458.9
11	......	......	......	......	......	......	......	......	3.80	0.255	15.72	3.80	+ 4,077.9
12	......	......	......	......	......	......	......	......	......	9.15	3.80	25.90	+ 6,189.0
Values of θ	−81.92	−103.4	−108.4	−83.51	−97.18	−83.07	−101.9	+56.74	+41.57	+176.8	+210.7	+145.3	

stitution of values of θ_2 and θ_3, as determined by the first approximation, and solving for θ_1. A similar procedure is adopted for all other equations. This process may then be repeated for successive approximations until it is found that further substitutions cause little change in the θ values.

It is to be noted that this approximate method can not be applied to any set of simultaneous equations, but that it is applicable to any set of equations where one of the unknowns in each equation has a coefficient much larger than any of the other unknowns. On examining Table B, it will be noted that the coefficient of θ for the joint for which the equation is written is much larger than those of the other θ's (equal to twice the sum of all the others). Hence these equations are readily solved by the proposed method. It will be found that where the θ values are of about the same magnitude, satisfactory results may be obtained after about three successive approximations. Where the θ's vary considerably in magnitude, more than three approximations may be required.

This proposed method of solution will be illustrated by solving the equations of Table B. Assuming first that all θ's are equal, we have from eq. (1), Table B;

$$(25.90 + 3.8 + 9.15)\,\theta_1 = -\,3{,}505.8$$
$$38.85\,\theta_1 = -\,3{,}505.8$$
$$\theta_1 = -\,90.3$$

Values for the other θ's, determined in the same manner are given in Table C in the column headed First Approximation.

To determine the second approximate values, we have from eq. (1), using values of θ_2 and θ_3 from Table C,

$$25.90\,\theta_1 + 3.80\,(-\,97.2) + 9.15\,(-\,102.3) = -\,3{,}505.8,$$

from which

$$\theta_1 = -\,85.1.$$

Values of the other θ's, determined in the same manner, are given in Table D in the column headed Second Approximation.

For the third approximation, we have from eq. (1)

$$25.90\,\theta_1 + 3.80\,(-\,101.5 + 9.15\,(-\,107.8) = -\,3{,}505.8$$
$$\theta_1 = -\,82.4$$

TABLE C

Approximate Solution of Equations

Exact Value	θ	Approximate Values			
		First Approximation	Second Approximation	Third Approximation	Fourth Approximation
− 81.92	θ_1	− 90.3	− 85.1	− 82.4	− 82.4
−103.4	θ_2	− 97.2	−101.5	−102.1	−103.6
−108.4	θ_3	−102.3	−107.8	−107.0	−108.1
− 83.51	θ_4	− 89.7	− 89.5	− 83.6	− 83.1
− 97.18	θ_5	− 96.3	−101.8	− 97.6	− 96.7
− 83.07	θ_6	− 62.9	− 81.2	− 84.2	− 84.2
−101.9	θ_7	− 76.0	−100.6	−103.2	−102.2
+ 56.74	θ_8	+ 52.0	+ 62.5	+ 61.0	+ 57.3
+ 41.57	θ_9	+ 42.5	+ 47.7	+ 43.4	+ 41.9
+176.8	θ_{10}	+148.8	+157.6	+174.2	+176.5
+210.7	θ_{11}	+173.1	+206.5	+209.0	+210.5
+145.3	θ_{12}	+159.2	+153.1	+146.8	+145.5

In Table C four approximations have been made. On comparing the results of the fourth approximation with those given in the first column of Table C, taken from Table B, we find a very close agreement between the two sets of values. In fact, the results obtained on the third approximation are exact enough for all practical purposes.

378. *Moments and Fiber Stresses.*—Moments at the ends of the members may be determined from eq. (2), Table I. Thus, for member 1–2, omitting E,

$$M_{1\text{-}2} = \frac{2\,I}{l}(2\,\theta_1 + \theta_2 - 3\,\alpha_{1\text{-}2})$$

Fiber stresses are given by the equation

$$f = \frac{M\,c}{I} = 2\,\frac{c}{l}(2\,\theta_1 + \theta_2 - 3\,\alpha_{1\text{-}2})$$

For member 1–2 of the terms under consideration, we have, with $\theta_1 = -\,81.92$, $\theta_2 = -103.4$, $\alpha_{1-2} = -\,99.50$, and $K = 3.80$

$$M_{1\text{-}2} = 2\,(3.80)\,[2\,(-\,81.92) - 103.4 - 3\,(-\,99.50)],$$

from which

$$M_{1\text{-}2} = (7.60)(31.31) = +\,238.0 \text{ in.-lb.}$$

With $c = 0.0285$, as given in Table A, Art. 291, we have

$$f = +(2)(0.0285)(31.31) = +1.79.$$

These values check those given in Table F, Art. 296. Other values may be calculated in a similar manner.

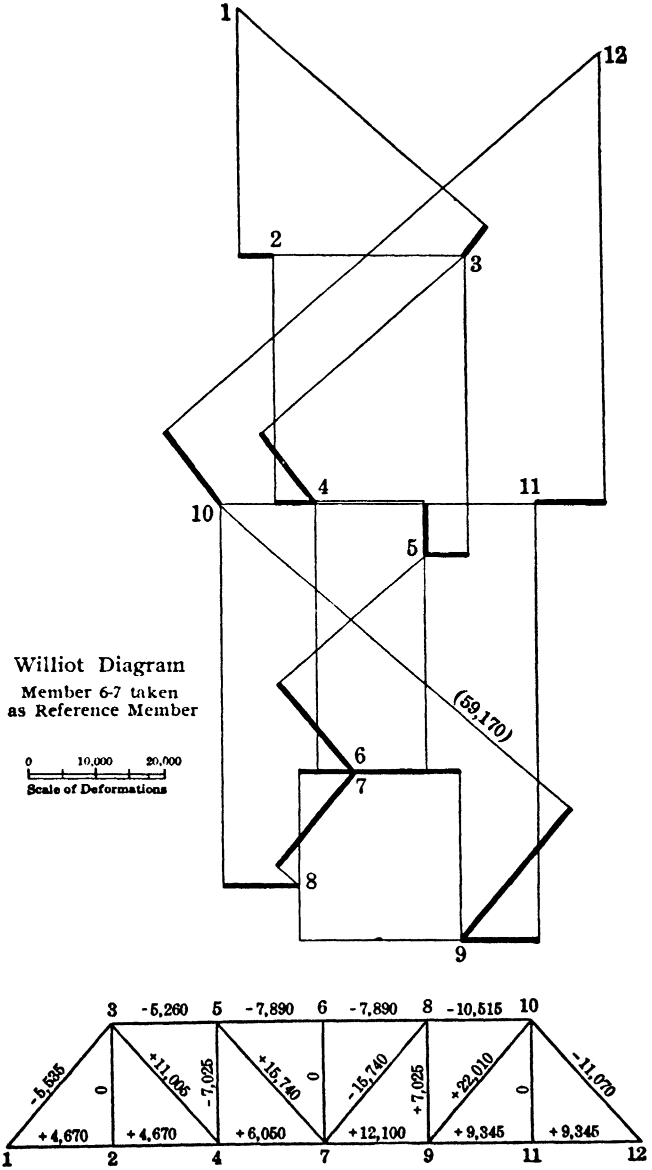
379. *Determination of α Values by Means of a Williot Diagram.*—The graphical method mentioned in Art. 371 differs from the algebraic method, as presented in Arts. 372 to 378, only in the determination of the α values. Hence the discussion relative to the graphical method will be limited to the determination of the α values by means of a Williot Diagram. Fig. 47 shows such a diagram drawn for the truss of Fig. 45. This diagram is drawn for values of $p\,l$ as given in Table D.

TABLE D

Distortion of Members

Member	Length, Inches	*P*, Pounds per Square Inch	*Pl*	Member	Length, Inches	*P*, Pounds per Square Inch	*Pl*
3-5	320	−16.43	− 5,260	1-3	490.7	−11.28	− 5,535
5-6	320	−24.65	− 7,890	3-2	372	0	0
6-8	320	−24.65	− 7,890	3-4	490.7	+22.43	+11,010
8-10	320	−32.86	−10,520	4-5	372	−18.88	− 7,020
1-2	320	+14.60	+ 4,670	5-7	490.7	+32.07	+15,740
2-4	320	+14.60	+ 4,670	7-6	372	0	0
4-7	320	+18.90	+ 6,050	7-8	490.7	−32.07	−15,740
7-9	320	+37.80	+12,100	8-9	372	+18.88	+ 7,020
9-11	320	+29.20	+ 9,345	9-10	490.7	+44.86	+22,010
11-12	320	+29.20	+ 9,345	10-11	372	0	0
				11-12	490.7	−22.56	−11,070

Table E gives all data necessary for the determination of the α values. Column 2 gives the movement of one end of a member with respect to the other end, which is named in Column 3 as the pivot. This information is determined in the manner explained on page 329, Part I. Thus for member 9-10, with joint 9 assumed as a pivot, the Williot Diagram of Fig. 47 shows that joint 10 moves 59,200 units to the left of joint 9. On referring to the truss diagram of Fig. 47,



Williot Diagram
Member 6-7 taken
as Reference Member

Truss Diagram

FIG. 47.

it can be seen that member 9–10 rotates in a positive direction. This information is recorded in Column 4 of Table E. On dividing the movement given in Column 3 by the length of the member given in Column 5, we have the value of α as given in Column 6.

TABLE E

Values of α. Williot Diagram Method

Member 1	Movement 2	Pivot 3	Direction of Rotation 4	Length 5	α 6
1–3	41,900	3	–	490.7	$\alpha_{1-3} = -85.2$
2–3	23,800	3	–	372	$\alpha_{2-3} = -64.0$
3–4	34,500	4	–	490.7	$\alpha_{3-4} = -70.4$
4–5	13,800	5	–	372	$\alpha_{4-5} = -37.1$
5–7	23,600	7	–	490.7	$\alpha_{5-7} = -48.2$
7–6	0	....	....	372	$\alpha_{7-6} = 0$
7–8	3,100	7	+	490.7	$\alpha_{7-8} = +6.32$
8–9	20,000	8	+	372	$\alpha_{8-9} = +53.8$
9–10	59,200	9	+	490.7	$\alpha_{9-10} = +120.6$
10–11	40,000	10	+	372	$\alpha_{10-11} = +107.5$
10–12	74,800	10	+	490.7	$\alpha_{10-12} = +152.6$
3–5	38,100	5	–	320	$\alpha_{3-5} = -119.0$
5–6	27,300	6	–	320	$\alpha_{5-6} = -85.3$
6–8	14,000	6	–	320	$\alpha_{6-8} = -43.8$
8–10	48,200	8	+	320	$\alpha_{8-10} = +150.8$
1–2	31,600	2	–	320	$\alpha_{1-2} = -98.7$
2–4	31,000	4	–	320	$\alpha_{2-4} = -96.9$
4–7	34,400	7	–	320	$\alpha_{4-7} = -107.6$
7–9	21,100	7	–	320	$\alpha_{7-9} = -65.9$
9–11	55,200	9	+	320	$\alpha_{9-11} = +172.6$
11–12	56,800	11	+	320	$\alpha_{11-12} = +177.8$

Note:—Member 7-6 taken as Reference Member.

On comparing these α values with those given in Table A, Art. 375, it will be found that they are in close agreement.

380. Determination of True Values of Deflection and Twist Angles.—The correction to be applied to the calculated values of α and θ in order to determine true values, should they be needed, is found by calculating the rotation of a line joining the joints at supports, as 1 and 12 in the preceding example. As this line does not actually move, the calculated angular movement will be the correction desired and will be equal to the actual angular movement of the reference member 7–6.

Algebraically, this movement is found by summing up the α values for all members of the lower chord and dividing by 6. Or

$$\alpha_{1-12} = (-99.50 - 96.93 - 108.0 - 65.58 + 172.69 + 177.83) \div 6$$
$$= -\frac{19.49}{6} = -3.25.$$

That is, the calculated deflection angle of the line 1–12 is -3.25. Inasmuch as this line actually stands fast, it follows that the true deflection angle of the reference member 7–6 is $+3.25$, and hence this value of $+3.25$ is the correction to be applied to all calculated α and θ values to get their true values.

Graphically this correction is obtained from the Williot Diagram by measuring the movement of point 12 with respect to a horizontal line through point 1 and dividing by span length. A Mohr correction diagram is unnecessary.

If the various members of the truss connecting the joints at the supports (1 and 12 in the problem here considered) do not form a straight line, as in the case of a truss with broken lower chord or an arch truss, the calculated rotation of the line joining the end joints is found, algebraically, by the general expression

$$\alpha_{AB} = \frac{\Sigma\, l\, \alpha \cos \phi}{L},$$

where α_{AB} is the desired rotation of the line joining the end joints A and B; $l\, \alpha \cos \phi$ is the product of length l, deflection angle α, and angle ϕ between the member in question and line AB; $\Sigma\, l\, \alpha \cos \phi$ represents a summation for all members of the chord connecting A and B; and L is the length of line AB.

Graphically, the desired rotation is determined directly from the Williot Diagram in the same manner as in the example given.

381. Comparison of Methods of Secondary Stress Analysis.—On comparing the method of secondary stress analysis given in Chapter VII, which is due to Prof. Winkler, and the method of the present Section, which is due to Prof. Mohr, it will be found that both methods result in the same kind and number of equations. In the Winkler method, the unknowns are the τ angles of Fig. 5, Art. 282, while in the Mohr method, the unknowns are the twist angles at the joints. From a comparison of the equations given in Table D, Art. 298, for the Winkler method and those given in Table B, Art. 376, for the Mohr method, it will be found, that the coefficients of the τ's and θ's for any joint equation are identical but that the absolute terms are different for the two methods.

In the formulation of equations, the Mohr method has certain advantages over the Winkler method. The absolute terms for the Mohr method involve only the $K\ \alpha$ values for the members entering a joint. These values are readily obtained from a truss diagram on which the necessary information is arranged as shown on Fig. 46. In the Winkler method, the absolute terms are formed from the $K\ \Sigma\ \delta\ \angle$ values given in Table C, Art. 293. Considerable time is required for the selection of the proper values from Table C. Hence at this stage of the work, the Mohr method is somewhat shorter.

Since the joint equations are similar for the two methods, differing only in the absolute terms, a solution of equations takes the same amount of time when the solution is made by means of the method used for Table E, Art. 295. However, in the Mohr method, as mentioned in Art. 372, a properly selected reference member will yield equations from which the resulting θ values will be quite uniform in value. A solution of the equations by the approximate method of Art. 377 is then readily made after only three or four approximations, even for cases of unsymmetrical loading, when the greatest variation in θ values will occur. In this respect the Mohr method has a decided advantage over the Winkler method, as the τ values differ in magnitude much more than the θ values. As a rule four or five approximations are required when the method of Art. 377 is used in order to secure a precision equal to that given by the Mohr method for identical conditions.

After the unknown τ's and θ's have been determined, the solution for moments is somewhat more readily carried out by the Winkler method, for the moment equations involve only two τ values. In the Mohr method, the moment equations involve two θ's and an α value.

In general, it can be said that for an experienced computer the total time required for a solution by the Winkler and Mohr methods is about the same, although there is probably some advantage in favor of the Mohr method when the approximate method of Art. 377 is used for the solution of the joint equations. For the beginner the Mohr method is advantageous by reason of the more obvious and simpler process used in the formulation of the joint equations.

382. Secondary Stresses in the Top Chord of a Pratt Truss.—Determine the secondary stresses in the riveted top chord of a pin-connected truss, the effect of the weight of the members being included. Use the truss and loading of the problem of Art. 306.

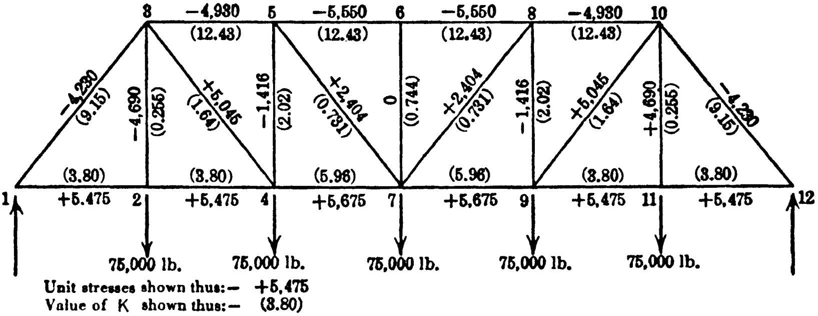


FIG. 48.

Fig. 48 shows the truss and its loading, the unit stresses in the members, and the K values for each member. As in Art. 306, the problem will be solved for direct and eccentric application of the axial stress.

(*a*) *Without Eccentricity.*—Since the top chord is hinged at joints 3 and 10, the chord is essentially a continuous beam on yielding supports, where the settlement of the supports is equal to the deflection of the top chord joints under the applied loads. As soon as the deflection of the top chord joints is known, the α values may be calculated and substituted in the joint equations as known quantities.

As shown in Fig. 48 the truss and its loading is symmetrical about joint 6. Hence, the twist angle at joint 6 is zero. Due to the presence of hinges at joints 3 and 10, the moments at these joints are zero. Hence, the only unknowns are the twist angle θ_5 at joint 5, and θ_8 at joint 8. But, because of symmetry, these may be taken as a single unknown, which will be assumed as θ_5. Therefore only a single joint equation for joint 5 is required for a solution of the problem.

The α values may be determined by the method outlined in Art. 372. Fig. 49 shows the left half of the truss with the angle changes in position. These angle changes were taken direct from Table M of Art. 306. Using the center vertical 6–7 as a reference member because its rotation is known to be zero we have, from Fig. 49 and eq. (1), Art. 372,

$$\alpha_{5-6} = \alpha_{6-7} - 9{,}635 = 0 - 9{,}635 = -9{,}635$$

$$\alpha_{5-3} = \alpha_{5-6} + 2{,}795 - 2{,}815 - 16{,}090 = -25{,}745$$

For equilibrium of joint 5 we may write

$$M_{5-3} + M_{5-6} = 0$$

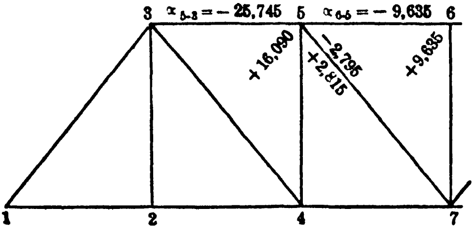


FIG. 49.

Substituting values of these moments in terms of θ and α, as given by eqs. (15) and (10) of Table 1, Art. 343, we have

$$3\,K_{3-5}\,(\theta_5 - \alpha_{3-5}) - D_2 + 2\,K_{5-6}\,(2\,\theta_5 + \theta_6 - 3\,\alpha_{5-6}) + C_1 = 0, \quad (a)$$

From Table II, for uniform loading,

$$D_2 = \tfrac{1}{8}\,w\,l^2 \quad \text{and} \quad C_1 = \tfrac{1}{12}\,w\,l^2.$$

As stated in Art. 306, the top chord member weighs 178 lb. per ft. and the top chord panel length is 320 in. Then

$$D_2 = \tfrac{1}{8}\,w\,l^2 = \tfrac{1}{8}\left(\tfrac{178}{12}\right)(320)^2 = 190{,}000 \text{ in.-lb.}$$

and

$$C_1 = \tfrac{1}{12}\,w\,l^2 = \tfrac{1}{12}\left(\tfrac{178}{12}\right)(320)^2 = 126{,}600 \text{ in.-lb.}$$

With $\alpha_{3-5} = -25{,}745$; $\alpha_{5-6} = -9{,}636$; $K_{3-5} = K_{5-6} = 12.43$; and $\theta_6 = 0$, we have from eq. (a)

$$3\,(12.43)(\theta_5 + 25{,}745) - 190{,}000 + 24.86\,[2\,\theta_5 - 3\,(-9{,}636)] + 126{,}600 = 0$$

from which $\theta_5 = -18{,}570$.

From eq. (15), Table I,

$$
\begin{aligned}
M_{5\text{-}6} &= 2\,K_{5\text{-}6}\,(2\,\theta_5 + \theta_6 - 3\,\alpha_{5\text{-}6}) + C_1 \\
&= 2\,(12.43)\,[2\,(-\,18{,}570) + 0 - 3\,(-\,9{,}636)] + 126{,}600 \\
M_{5\text{-}6} &= -\,78{,}050 \text{ in.-lb.}
\end{aligned}
$$

Since this moment is negative, the top fiber of member 5–6 is compressive, and its fiber stress, using values of *c* and *I* given in Table A, Art. 291, is

$$= 78{,}050\,\frac{9.19}{3{,}897} = 180 \text{ lb. per sq. in.}$$

This checks the value given in Table N, Art. 306.

(*b*) *With Eccentricity*.—Assuming that the axis of the top chord member is one inch above the center line of the truss, Fig. 50 shows the conditions at the top chord joints. Noting that counter-clockwise moments are positive,

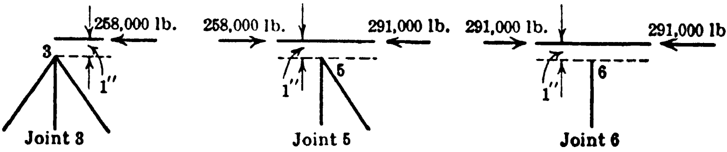


FIG. 50.

$$
\begin{aligned}
M_3 &= +\,258{,}000 \text{ in.-lb.} \\
M_5 &= +\,(291{,}000 - 258{,}000) \times 1 = +\,33{,}000 \text{ in.-lb.} \\
M_6 &= 0
\end{aligned}
$$

Since there is an eccentric moment at joint 3, this joint is partially restrained and an equilibrium equation must be written for joint 3 as well as for joint 5. The equilibrium equation for joint 3 is

$$M_{3\text{-}5} = +\,258{,}000$$

For joint 5, the equilibrium equation is

$$M_{5\text{-}3} + M_{5\text{-}6} = +\,33{,}000.$$

From eq. (10), Table I,

$$2\,K\,(2\,\theta_3 + \theta_5 - 3\,\alpha_{3\text{-}5}) + C_1 = +\,258{,}000,$$

and

$$2\,K\,(2\,\theta_5 + \theta_3 - 3\,\alpha_{3\text{-}5}) - C_1 + 2\,K\,(2\,\theta_5 + \theta_6 - 3\alpha_{5\text{-}6}) + C_1 = +\,33{,}000;$$

with $\alpha_{3-5} = -25{,}745$; $\alpha_{5-6} = -9{,}636$; $K = 12.43$; $C_1 = 126{,}600$; and $\theta_6 = 0$, these equations become

$$49.72\,\theta_3 + 24.86\,\theta_5 = -1{,}787{,}600$$

and

$$24.86\,\theta_3 + 99.44\,\theta_5 = -2{,}605{,}000.$$

Solving these equations

$$\theta_5 = -19{,}650; \quad \theta_3 = -26{,}075.$$

Then

$$M_{5-6} = 2\,(12.43)\,[2\,(-19{,}650) - 3\,(9{,}636)] + 126{,}600$$

$$M_{5-6} = -131{,}900 \text{ in.-lb.}$$

$$\text{Fiber stress} = 131{,}900\,\frac{9.19}{3{,}978} = 305 \text{ lb. per sq. in. compression.}$$

Moments and fiber stresses at other points may be calculated by the same methods.

INDEX

A

Arch bridges, advantages of, 124
  definition of, 116
  deflection of, 126, 140, 179, 193
  kinds of, 116
  loads for, 118
  masonry, 117
  reactions, 118
  secondary stresses in, 316
Arch ribs, general analysis, 120
Arches of three hinges, 133
  dead-load stresses, 133
  deflection of, 140
  equivalent uniform loads for, 140
  influence lines for, 136
  lateral trusses, 145
  live-load stresses, 135
  reaction lines for, 135
  reactions and stresses, 133
  temperature stresses, 144
  wind stresses, 143
Arches of two hinges, 148
  braced arch, 160, 174
  circular arch, 156
  deflection of, 179
  general formulas, 148
  influence lines for, 165
  lateral trusses, 180
  parabolic arch, 153
  plate girder, 169
  reaction lines for, 145
  ribs, 158
  stress calculation, 164
  temperature stresses, 155, 173
  wind stresses, 180
Arches with fixed ends, 183
  advantages of, 199
  braced arch, 189
  deflection of, 193
Arches with fixed ends, formulas for ribs, 183
  influence lines for, 193, 194
  lateral trusses, 196
  parabolic arch, 187, 189
  reaction lines, 190
  temperature stresses, 186, 198
  wind stresses, 198

B

Baltimore truss, counter stresses in, 309
  secondary stresses in, 450
Beams, curved, 120, 126
  deflection angles of, 385
  deflection of, 1, 120, 495
  moments and deflections, 20, 499, 502, 533
  on multiple supports, 371, 360
  restrained, 50
  table of moments, etc, 20
Bridge floors, analysis of, 360, 368, 371
  stresses in I-beam floors, 369
Building frames, 530
  lateral stresses in, 546
  stresses in beams, 533
  stresses in columns, 542
Buildings, lateral bracing in, 348

C

Cable of suspension bridge. *See* Suspension bridges.
Cantilever bridges, 106
  advantages of, 106
  analysis of, 110
  arrangement, 106
  deflection of, 113
  divided support for, 113
  economy of, 108
  lateral trusses, 115

Cantilever bridges, partially continuous, 114
  secondary stresses in, 430
  wind stresses, 115
Catenary cable, 204
Circular arch, 156
Collision strut, effect on secondary stresses, 416
Columns in buildings, 542
Columns, secondary stresses in, 459
Combined stresses, 477
Compression members, effect of lacing, 459
  stresses due to weight, 477
Continuous girders, 29, 507, 508
  effect of settlement of supports, 32, 55
  of several spans, 48
  of three spans, 44
  of two spans, 38
  reactions for, 37
  theorem of three moments, 30
  use of, 55
  variable moment of inertia, 51
Continuous trusses, 54
Counters, stresses in, 308
Curved beams, analysis of, 126
Curved-chord trusses, lateral stresses, 317

D

Deflection due to secondary stresses, 468
  of arch bridges, 126, 179, 193
  of beams, 1, 126, 495
  of cantilever bridges, 113
  of continuous girders, 32
  of curved beams, 126
  of suspension bridges, 235, 239, 252
  of swing bridges, 80, 83, 85
Double intersection trusses, 284
  secondary stresses in, 445
  use of, 307
Double triangular truss with verticals, 297

E

Eccentricity of joints, effect of, 420
Elevated railroad bents, 339
Equivalent uniform loads for Cooper's *E*-50, 67
  for swing bridges, 79
Expansion suspenders, 476

Eye-bars, friction on pins, 422
  secondary stresses in, 477

F

Floorbeams of suspension bridges, 274
  secondary stresses in, 463, 465
Floor members, maximum load on, 367

G

Girders, open webbed, 565
  in buildings, 533

H

Hangers, plate, stresses in, 476

I

Impact formulas, 485, 491, 523, 529
Impact stresses, 484, 522
Influence lines, for arches, 136, 165, 193
  for cantilever bridges, 111
  for continuous girders, 41, 47, 49
  for multiple intersection trusses, 304
  for suspension bridges, 231, 234, 246
  for swing bridges, 73, 84, 07

L

Lateral bracing, deflection of, 354
  effect of portals on, 324
  for arches, 145, 180, 198
  for cantilever bridges, 115
  for curved chord trusses, 317
  for skew bridges, 325
  for suspension bridges, 273
  for swing bridges, 104
  for towers, 315
  for viaducts, 342, 348
  general analysis of, 312
  proportions of, 317
  rigidity of, 317
  secondary stresses in, 453
  stresses due to settlement, 320
  transverse bracing, 326
  transverse bracing, rigidity of, 330
  vertical loads, 325
  See also Quadrangular frames.

M

Manhattan suspension bridge, 241, 268
Moment of inertia of a truss, 55
Moments, in beams, table of, 20
  in continuous girders, 33
Multiple intersection trusses, 280

P

Parabolic arches, 153, 187
Pettit truss, stresses in counters, 309
Pin-connections, friction in, 322
  secondary stresses in, 322
Pins, friction of, 422
Plate-girder arch, 169
Plate-girders. *See* Continuous girders.
Portals, stresses in, 324, 331, 522
  See also Quadrangular frames.
Pratt truss, secondary stresses in, 398, 437, 572
  stresses in counters, 308
Primary and secondary stresses, 423, 498

Q

Quadrangular frames, 331, 512
  deflection of, 354
  inclined posts, 346, 351
  lateral stresses in, 342, 518
  multiple stories, 348
  partially trussed, 344
  posts fixed at one end, 337
    hinged at one end, 338
  symmetrical frames, 334
  temperature stresses in, 340
Queen-post truss without counters, 358

R

Redundant members, stresses in, 280
Restraint factor for beams, 502

S

Secondary stresses, 381
  approximate calculation of, 415
  due to eccentric joints, 420
    fixed supports, 430
    lateral members, 454
    rigid joints, 381
    transverse loads, 426
    weight of members, 428
Secondary stresses, effect of collision strut, 416
    lacing in columns, 459
    pin connections, 422
  effect on primary stresses, 460
  exact method of calculation, 469
  in Baltimore trusses, 450, 455
    columns, 459
    double intersection trusses, 445
    floor systems, 463, 465
    K-truss, 455
Secondary stresses in Pratt trusses, 398, 437, 453, 572
    transverse frames, 463
    types of trusses, 436, 453
    Warren trusses, 443, 454
  influence lines for, 412
  slope-deflection method of analysis, 493
Shear, deflection due to, 19
Skew bridges, stresses in laterals of, 325
Slope-deflection method of analysis, 493
Stringers, secondary stresses due to, 465
Sub-struts, use of, 450
Suspension bridges, 200
  classification of, 200
  cable, analysis of, 200
    deformation of, 211, 214
    form of, 200
    length of, 206
    stresses in, 208, 213, 222
  floorbeam stresses, 274
  lateral stresses, 273
  stiffened bridges, 219
    approximate analysis, 222
    cable length, 206
    cable stress, 222
    deflection of, 235, 239, 252
    deflection of tower, 239
    exact analysis, 250
    examples, 241, 268
    H-component of cable stress, 222, 223, 257
    influence lines, 231, 234, 246
    kinds of, 219
    Manhattan bridge, 241, 268
    methods of analysis, 221
    moments and shears in, 222, 230, 252
    saddle movement, 239

Suspension bridges, stiffened bridges, shears in, 222, 234, 252
    temperature effects, 229, 233, 249, 258
    tower stresses, 276
    towers, deflection of, 239
      stresses in, 276
  unstiffened bridges, 213
    cable stresses in, 213
    deformation of, 214
  wind stresses in, 273
Swing bridges, 57
  centre-bearing bridge, 60
    deflections of, 80, 85
    end conditions, 62
    equivalent uniform loads for, 79
    graphical analysis, 82
    plate-girder, 64
    reactions for, 60, 80
    temperature effects, 86
    true reactions, 80
    truss bridges, 72
    unbroken loads, 72
    uplift required, 85
  continuous girder, formulas for, 59
  double swing bridges, 103
  end support for, 59
  general arrangement, 57
  lateral trusses, 104
  lift swing bridges, 102
  loads for, 59
  partially continuous, 94, 96
  rim-bearing bridge, 87
    centre diagonals of, 93
    continuous over four supports, 87
      three supports, 93
    partially continuous, 94, 96
Swing bridges, rim-bridges, reactions for, 88, 91, 100
    true reactions for, 91, 100

T

Temperature stresses in arches, 144, 155, 173, 186, 198
  in suspension bridges, 229, 249, 258
  in swing bridges, 86
  in viaduct bents, 340
Tension members, secondary stresses in, 477
Three moments, theorem of, 30
Top-chord, secondary stresses in, 424, 477, 583
Towers, stresses in, 276
Transverse bracing, stresses in, 326
    secondary stresses in, 463
Trestles, stresses in, 313, 451, 457
Trussed beams, 356

V

Viaduct bents, 339, 347, 351
  See also Quadrangular frames.
Vibration of trusses, 488

W

Warren truss, secondary stresses in, 443
Whipple truss, 293
Williot diagrams in secondary stress analysis, 571, 578
Wind bracing in tall buildings, 348
Wind stresses in arch bridges, 180, 198
  in buildings, 546
  in cantilever bridges, 115
  in suspension bridges, 273
  in swing bridges, 104

PRESCRIBED AS A
TEXT BOOK
Not to be issued without permission.
LIBRARIAN.